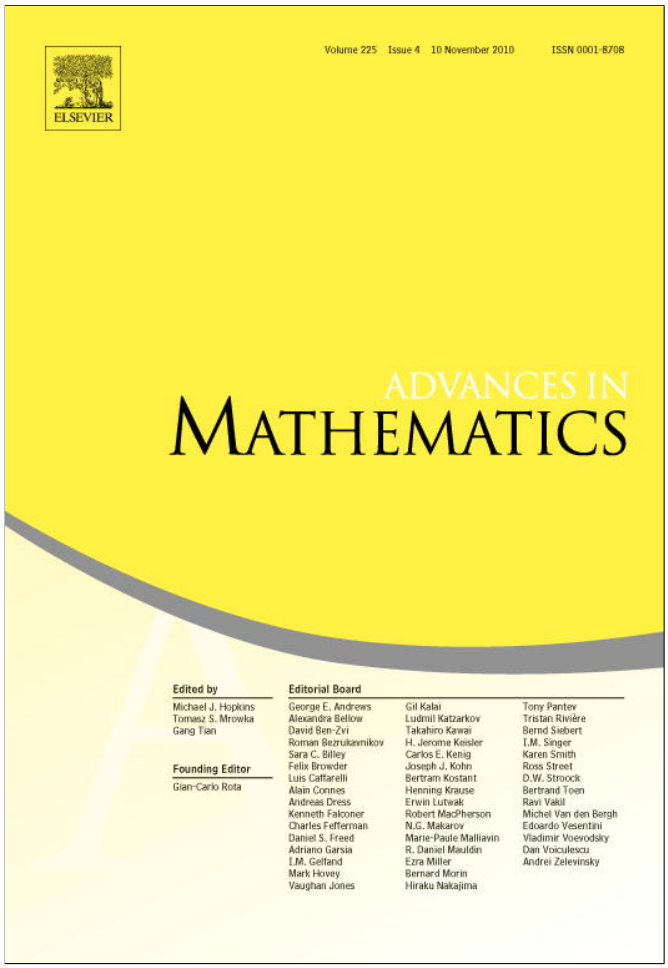

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Stability of the Blaschke–Santaló and the affine isoperimetric inequality

Károly J. Böröczky [1]

Hungarian Academy of Sciences, Alfred Renyi Institute of Mathematics, Budapest, Hungary

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Abstract

A stability version of the Blaschke–Santaló inequality and the affine isoperimetric inequality for convex bodies of dimension $n \geqslant 3$ is proved. The first step is the reduction to the case when the convex body is o-symmetric and has axial rotational symmetry. This step works for related inequalities compatible with Steiner symmetrization. Secondly, for these convex bodies, a stability version of the characterization of ellipsoids by the fact that each hyperplane section is centrally symmetric is established.

MSC: 52A40

Keywords: Affine invariant inequalities; Stability

1. Introduction

Stability versions of geometric inequalities have been investigated since the days of H. Minkowski, see the beautiful survey of H. Groemer [19], or K.J. Böröczky [10] for some more recent results. Here we prove stability versions of two classical inequalities originating from the beginning of the 20th century, the Blaschke–Santaló inequality and the affine isoperimetric inequality. For all the basic affine invariant notions, consult the thorough monograph of

E-mail address: carlos@renyi.hu.

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K. Leichtweiß [32], and for notions of convexity in general, see P.M. Gruber [22] and R. Schneider [50].

We write o to denote the origin of $\mathbb{R}^n$, $\langle\cdot,\cdot\rangle$ to denote the standard scalar product, and $V(\cdot)$ to denote volume. Let B^n be the unit Euclidean ball with volume $\kappa_n = V(B^n)$, and let $S^{n-1} = \partial B^n$. A convex body K in $\mathbb{R}^n$ is a compact convex set with non-empty interior. If $z \in \operatorname{int} K$, then the polar of K with respect to z is the convex body

$$K^z = \{x \in \mathbb{R}^n\colon\ \langle x - z, y - z\rangle \leqslant 1 \text{ for any } y \in K\}.$$

It is easy to see that $(K^z)^z = K$. According to L.A. Santaló [49] (see also M. Meyer and A. Pajor [38]), there exists a unique $z \in \operatorname{int} K$ minimizing the volume product $V(K)V(K^z)$, which is called the Santaló point of K. In this case z is the centroid of K^z. The well-known Blaschke–Santaló inequality states that if z is the Santaló point (or centroid) of K, then

$$V(K)V\left(K^z\right) \leqslant \kappa_n^2, \tag{1}$$

with equality if and only if K is an ellipsoid. The inequality was proved by W. Blaschke [7] for $n \leqslant 3$, and by L.A. Santaló [49] for all n. The case of equality was characterized by J. Saint-Raymond [48] among o-symmetric convex bodies, and by C.M. Petty [44] among all convex bodies (see also M. Meyer and A. Pajor [38], D. Hug [24], and M. Meyer and S. Reisner [39] for simpler proofs).

Our main task is to provide a stability version of this inequality. A natural tool is the Banach–Mazur distance $\delta_{\mathrm{BM}}(K, M)$ of the convex bodies K and M, which is defined by

$$\delta_{\mathrm{BM}}(K, M) = \min\{\lambda \geqslant 1\colon\ K - x \subset \Phi(M - y) \subset \lambda(K - x) \text{ for } \Phi \in \mathrm{GL}(n),\ x, y \in \mathbb{R}^n\}.$$

In particular, if K and M are o-symmetric, then $x = y = o$ can be assumed. It follows from a theorem of F. John [25] that $\delta_{\mathrm{BM}}(K, B^n) \leqslant n$ for any convex body K in $\mathbb{R}^n$ (see also K.M. Ball [4]).

Theorem 1.1. *If K is a convex body in $\mathbb{R}^n$, $n \geqslant 3$, with Santaló point z, and*

$$V(K)V\left(K^z\right) > (1 - \varepsilon)\kappa_n^2 \quad \text{for } \varepsilon \in \left(0, \frac{1}{2}\right),$$

then for some $\gamma > 0$ depending only on n, we have

$$\delta_{\mathrm{BM}}\left(K, B^n\right) < 1 + \gamma\varepsilon^{\frac{1}{6n}}|\log\varepsilon|^{\frac{1}{6}}.$$

Taking K to be the convex body resulting from B^n by cutting off two opposite caps of volume ε shows that the exponent $1/(6n)$ cannot be replaced by anything larger than $2/(n+1)$. Therefore the exponent of ε is of the correct order. Since $1/(6n)$ is most probably not the optimal exponent of ε in Theorem 1.1, no attempt was made to find an explicit γ in Theorem 1.1. In principle, this would be possible following the argument in this paper if the exponent $1/(6n)$ is replaced by $1/(6n+6)$ (see the discussion after (20)). We note that a stability version of the Blaschke–Santaló inequality in the planar case is proved in K.J. Böröczky, E. Makai, M. Meyer, S. Reisner [11], using a quite different method.

The literature about the Blaschke–Santaló inequality is so extensive that only just a small portion can be discussed here. The PhD thesis of K.M. Ball [3] started off the quest for suitable functional versions. This point of view is for example pursued in M. Fradelizi and M. Meyer [15] and S. Artstein, B. Klartag and V.D. Milman [2]. Stability questions on a related problem are discussed in M. Meyer and E. Werner [40].

We note that the minimum of the volume product $V(K)V(K^z)$ is not known for convex bodies K in $\mathbb{R}^n$ and $z \in K$ for $n \geqslant 3$. According to the well-known conjecture of K. Mahler [36], the volume product is minimized by simplices, and among o-symmetric convex bodies by cubes. The planar case was actually proved in [36], and simpler arguments are provided by M. Meyer [37] and M. Meyer and S. Reisner [39]. For $n \geqslant 3$, the Mahler conjecture for o-symmetric convex bodies has been verified among unconditional bodies by J. Saint-Raymond [48] (see also S. Reisner [46]), and among zonoids by S. Reisner [45] (see also Y. Gordon, M. Meyer and S. Reisner [17]). The best lower bound for the volume product of an o-symmetric convex body K in $\mathbb{R}^n$ is

$$V(K)V\left(K^o\right) > 2^{-n}\kappa_n^2, \tag{2}$$

due to G. Kuperberg [29]. With a non-explicit exponential factor instead of 2^{-n}, it was proved by J. Bourgain and V.D. Milman [12].

The Mahler conjecture for general convex bodies was verified by M. Meyer and S. Reisner [39] among polytopes of at most $n+3$ vertices. In a yet unpublished revision of [29], G. Kuperberg also showed, based on (2) and the Rogers–Shephard inequality [47], that if $z \in \operatorname{int} K$ for a convex body K in $\mathbb{R}^n$, then

$$V(K)V\left(K^z\right) > 4^{-n}\kappa_n^2. \tag{3}$$

It was probably W. Blaschke who first noticed that the Blaschke–Santaló inequality is equivalent to the affine isoperimetric inequality. This and other equivalent formulations are discussed in depth in E. Lutwak [35] and K. Leichtweiß [32, Section 2]. To define the affine surface area of a convex body K in $\mathbb{R}^n$, we always consider its boundary endowed with the $(n-1)$-dimensional Hausdorff measure. According to Alexandrov's theorem (see P.M. Gruber [22, p. 74]), ∂K is twice differentiable in a generalized sense at almost every point, hence the generalized Gauß–Kronecker curvature $\kappa(x)$ can be defined at these $x \in \partial K$ (see K. Leichtweiß [32, Section 1.2]). The affine surface area of K is defined by

$$\Omega(K) = \int_{\partial K} \kappa(x)^{\frac{1}{n+1}}\, dx.$$

If ∂K is C^2, then this definition is due to W. Blaschke [6]. Since then various equivalent definitions were given for general convex bodies (including the above one) by K. Leichtweiß [31], C. Schütt and E. Werner [52] and E. Lutwak [34], which were shown to be equivalent by C. Schütt [51], and G. Dolzmann and D. Hug [14] (see K. Leichtweiß [32, Section 2]). The affine surface area is a valuation invariant under volume preserving affine transformations, and it is upper semi-continuous. These properties are characteristic, as any upper semi-continuous valuation on the space of convex bodies which is invariant under volume preserving affine transformations is a linear combination of affine surface area, volume, and the Euler characteristic by M. Ludwig and

M. Reitzner [33]. We note that affine surface area comes up e.g. in polytopal approximation (see P.M. Gruber [22, Section 11.2]), in limit shape of lattice polygons (see I. Bárány [5]), and many other applications (see K. Leichtweiß [32, Section 2]).

The affine isoperimetric inequality states that

$$\Omega(K)^{n+1} \leqslant \kappa_n^2 n^{n+1} V(K)^{n-1}, \tag{4}$$

with equality if and only if K is an ellipsoid. The inequality itself is due to W. Blaschke [6], whose proof in $\mathbb{R}^3$ for convex bodies with C^2 boundaries readily extends to general dimension and to general convex bodies. W. Blaschke characterized the equality case among convex bodies with C^2 boundary, and this characterization was extended to all convex bodies by C.M. Petty [44]. We note that W. Blaschke and L.A. Santaló deduced the Blaschke–Santaló inequality from the affine isoperimetric inequality. Here we take a reverse path.

An inequality on p. 59 of E. Lutwak [34] (see also Lemma 3.7 in D. Hug [24], or (1106) in K. Leichtweiß [32]) says that if $z \in \operatorname{int} K$, then

$$\Omega(K)^{n+1} \leqslant n^{n+1} V(K)^n V\big(K^z\big).$$

Therefore Theorem 1.1 yields

Theorem 1.2. *If K is a convex body in $\mathbb{R}^n$, $n \geqslant 3$, and*

$$\Omega(K)^{n+1} > (1-\varepsilon)\kappa_n^2 n^{n+1} V(K)^{n-1} \quad \text{for } \varepsilon \in \left(0, \frac{1}{2}\right),$$

then for some $\gamma > 0$ depending only on n, we have

$$\delta_{\mathrm{BM}}\big(K, B^n\big) < 1 + \gamma\, \varepsilon^{\frac{1}{6n}} |\log \varepsilon|^{\frac{1}{6}}.$$

For convex bodies K and M, we write $V_1(K, M)$ to denote the mixed volume

$$V_1(K, M) = \lim_{t \to 0} \frac{V(K + tM) - V(K)}{n \cdot t}$$

(see T. Bonnesen and W. Fenchel [9, Section 29], or P.M. Gruber [22, Section 6]). It satisfies $V_1(K, K) = V(K)$. We write $\mathcal{K}_o^n$ to denote the family of convex bodies whose centroid is o. C.M. Petty [42] defined the geominimal surface area by

$$G(K) = \kappa_n^{-\frac{1}{n}} n \inf\big\{ V_1\big(K, M^o\big) V(M)^{\frac{1}{n}} \colon\ M \in \mathcal{K}_o^n \big\}.$$

It is also invariant under volume preserving affine transformations. Positioning K in a way such that o is the Santaló point of K and taking $M = K^o$, yields the so-called geominimal surface area inequality of C.M. Petty [43]

$$G(K) \leqslant \kappa_n^{1/n} n V(K)^{\frac{n-1}{n}}, \tag{5}$$

with equality if and only if K is an ellipsoid. From Theorem 1.1 we directly obtain

Theorem 1.3. *If K is a convex body in $\mathbb{R}^n$, $n \geqslant 3$, and*

$$G(K) > (1-\varepsilon)\kappa_n^{\frac{1}{n}} n V(K)^{\frac{n-1}{n}} \quad \text{for } \varepsilon \in \left(0, \frac{1}{2}\right),$$

then for some $\gamma > 0$ depending only on n, we have

$$\delta_{\mathrm{BM}}\big(K, B^n\big) < 1 + \gamma\, \varepsilon^{\frac{1}{6n}} |\log \varepsilon|^{\frac{1}{6}}.$$

One of our main tools is to reduce the proof of Theorem 1.1 to o-symmetric convex bodies with axial rotational symmetry.

Theorem 1.4. *For any convex body K in $\mathbb{R}^n$, $n \geqslant 2$, with $\delta_{\mathrm{BM}}(K, B^n) > 1 + \varepsilon$ for $\varepsilon > 0$, there exists an o-symmetric convex body C with axial rotational symmetry and a constant $\gamma > 0$ depending on n such that $\delta_{\mathrm{BM}}(C, B^n) > 1 + \gamma\varepsilon^2$ and C results from K as a limit of subsequent Steiner symmetrizations and affine transformations.*

Remark. If K is o-symmetric, then $1 + \gamma\varepsilon^2$ can be replaced by $1 + \gamma\varepsilon$. In particular, if K is o-symmetric, then wherever the factor $1/6$ occurs in Theorems 1.1, 1.2 and 1.3, it can be replaced by $1/3$.

Theorem 1.4 shows that it is possible to use Steiner-symmetrization to obtain a convex body that is highly symmetric but still far from being an ellipsoid. On the other hand, B. Klartag [27] proved that any convex body K in $\mathbb{R}^n$ gets ε close to some ball after suitable chosen $cn^4|\log\varepsilon|^2$ Steiner symmetrizations where $c > 0$ is an absolute constant.

After discussing the basic tools such as the isotropic position of convex bodies and Steiner symmetrization in Section 2, we prove Theorem 1.4 in Section 3. A stability version of the False Centre theorem in a special case is presented in Section 4, which combined with Theorem 1.4 leads to the proof of Theorem 1.1 in Section 5. For stability versions of some other classical geometric characterizations of ellipsoids, see, e.g., H. Groemer [20] and P.M. Gruber [21].

2. Some tools

2.1. Isotropic position

In this paper, we use the term isotropic position in a weak sense. More precisely, we say that a convex body K in $\mathbb{R}^n$ is in weak isotropic position if its centroid is o, and $\int_K \langle u, x\rangle^2\, dx$ is independent of $u \in S^{n-1}$. In particular, in this case

$$\int_K \langle u, x\rangle^2\, dx = L_K^2 V(K)^{\frac{n+2}{n}}$$

for any $u \in S^{n-1}$ (see, e.g., A.A. Giannopoulos and V.D. Milman [16]), and the Legendre ellipsoid (the ellipsoid of inertia) is a ball. For any convex body C there is a volume preserving affine transformation T such that TC is in weak isotropic position. In the literature, two different normalizations are used. Either $V(K) = 1$ (see, e.g., A.A. Giannopoulos and V.D. Milman [16]), or

$\|v\|^2 = \int_K \langle v, x\rangle^2\,dx$ for any $v \in \mathbb{R}^n$ (see, e.g., R. Kannan, L. Lovász and M. Simonovits [26]). In this paper, if K is in weak isotropic position, then we compare it to balls, therefore we frequently assume $V(K) = \kappa_n$.

It is known that L_K is minimized by ellipsoids (see F. John [25] or A.A. Giannopoulos and V.D. Milman [16]). It follows by Gy. Sonnevend [53] (see also R. Kannan, L. Lovász and M. Simonovits [26]) that if K is in weak isotropic position, then

$$K \subset L_K^{\frac{2}{n+2}} V(K)^{\frac{1}{n}} \sqrt{n(n+2)}\, B^n.$$

Now $L_K \leqslant c_0 \sqrt[4]{n}$ for some absolute constant c_0 according to B. Klartag [28]. Therefore, if $V(K) = \kappa_n$ and K is in weak isotropic position, then

$$K \subset c\sqrt{n}\, B^n \tag{6}$$

for some absolute constant $c \geqslant 1$.

For properties of o-symmetric convex bodies in isotropic position, see the discussion in V.D. Milman and A. Pajor [41].

2.2. Steiner symmetrization

Given a convex body K in $\mathbb{R}^n$ and a hyperplane H, for any l orthogonal to H and intersecting K, translate $l \cap K$ along l in a way such that the midpoint of the image lies in H. The union of these images is the Steiner symmetrial K_H of K with respect to H. It follows that K_H is convex, $V(K_H) = V(K)$, and, if the centroid of K lies in H, then it coincides with the centroid of K_H.

We write $|\cdot|$ to denote the $(n-1)$-dimensional Lebesgue measure, where the measure of the empty set is defined to be zero. For $u \in S^{n-1}$ and $t \in \mathbb{R}$, let $u^\perp$ denote the linear $(n-1)$-space orthogonal to u, let $h_K(u) = \max_{x\in K}\langle u, x\rangle$ be the support function of K, and let

$$K(u,t) = K \cap \big(tu + u^\perp\big).$$

If M is a compact convex set of dimension $n-1$, then the classical Brunn–Minkowski inequality (see, e.g., T. Bonnesen and W. Fenchel [9, p. 94], P.M. Gruber [22, Section 8.1], or the monograph R. Schneider [50], which is solely dedicated to the Brunn–Minkowski theory) yields

$$\left|\frac{1}{2}(M - M)\right| \geqslant |M|. \tag{7}$$

K.M. Ball proved in his PhD thesis [3] that Steiner symmetrization through $u^\perp$ for $u \in S^{n-1}$ increases $V(K^o)$ if K is o-symmetric. The basis of his argument is the observation that for $\widetilde{K} = K_{u^\perp}$, we have

$$\frac{1}{2}\big(K^o(u,t) - K^o(u,t)\big) \subset \widetilde{K}^o(u,t) - tu \tag{8}$$

(see also M. Meyer and A. Pajor [38]). Here the $(n-1)$-measure of the left-hand side is at least $|K^o(u,t)|$ according to the Brunn–Minkowski inequality, hence the Fubini Theorem yields

$V(\widetilde{K}^o) \geqslant V(K^o)$. K.M. Ball's result was further exploited by M. Meyer and A. Pajor [38]. The ideas and statements in [38] yield the following.

Lemma 2.1 *(Meyer, Pajor). Let K be a convex body in $\mathbb{R}^n$, and let H be a hyperplane. If z and z' denote the Santaló points of K and K_H, respectively, then $z' \in H$, and $V(K^z) \leqslant V((K_H)^{z'})$.*

This statement is more explicit in Theorem 1 of M. Meyer and S. Reisner [39] (see the proof of Theorem 13 in [39]).

3. Proof of Theorem 1.4

The following lemma is the basis of the proof of Theorem 1.4.

Lemma 3.1. *Let K be a convex body in $\mathbb{R}^n$. If $\delta_{\mathrm{BM}}(K, B^n) > 1 + \varepsilon$ for $\varepsilon > 0$, then there exists a convex body C with axial rotational symmetry that results from K as a limit of subsequent Steiner symmetrizations and affine transformations, and satisfies $\delta_{\mathrm{BM}}(C, B^n) > 1 + \gamma\varepsilon$, where $\gamma > 0$ depends only on n. Moreover if K is o-symmetric, then so is C.*

Proof. We may assume that $V(K) = \kappa_n$ and K is in weak isotropic position. Using the $c \geqslant 1$ from (6), we claim that there exists some $u \in S^{n-1}$ such that

(i) either $h_K(u) \geqslant 1 + \frac{\varepsilon}{4}$ and $V(K \setminus B^n) \leqslant \tilde{\gamma}\varepsilon$ for $\tilde{\gamma} = \frac{1}{4c^2 n} \int_{B^n} \langle u, x\rangle^2 \, dx$, or
(ii) $h_K(u) \leqslant 1 - \frac{\tilde{\gamma}}{n\kappa_n}\varepsilon$.

To prove this statement, let h_K attain its maximum on S^{n-1} at $v \in S^{n-1}$, and its minimum at $w \in S^{n-1}$. If $h_K(w) \leqslant 1 - \frac{\varepsilon}{4}$, then $u = w$ works, thus we may assume $h_K(w) \geqslant 1 - \frac{\varepsilon}{4}$. Since $\delta_{\mathrm{BM}}(K, B^n) > 1 + \varepsilon$, it follows that $h_K(v) \geqslant 1 + \frac{\varepsilon}{4}$. Now if $V(K \setminus B^n) \leqslant \tilde{\gamma}\varepsilon$, then we are done again, hence we may assume $V(B^n \setminus K) = V(K \setminus B^n) \geqslant \tilde{\gamma}\varepsilon$. We conclude that $h_K(w) \leqslant 1 - \frac{\tilde{\gamma}}{n\kappa_n}\varepsilon$, which completes the proof of (i) and (ii).

Let C be the image of K after applying first Schwarz rounding (see P.M. Gruber [22, p. 178]) in the direction of u, and secondly the linear transformation that dilates by the factor $h_K(u)^{-1}$ in the direction of u, and by the factor $h_K(u)^{\frac{1}{n-1}}$ orthogonal to u. Since Schwarz rounding can be obtained as the limit of repeated applications of Steiner symmetrizations through hyperplanes containing the line $\mathbb{R}u$, we have $V(C) = V(K)$ and o is the centroid of C (see Section 2.2). The linear transformation following the Schwarz rounding ensures that $u \in \partial C$.

Let $h = h_K(u)$ and $\tilde{h} = h_K(-u)$. In the case of (ii), $L_K \geqslant L_{B^n}$ yields

$$\int_C \langle u, x\rangle^2 \, dx = \int_0^1 r^2 \big|C(u, r)\big| \, dr + \int_0^{\tilde{h}/h} r^2 \big|C(-u, r)\big| \, dr$$

$$= \int_0^1 r^2 h \big|K(u, hr)\big| \, dr + \int_0^{\tilde{h}/h} r^2 h \big|K(-u, hr)\big| \, dr$$

$$
\begin{aligned}
&= \frac{1}{h^2}\left(\int_0^h s^2\big|K(u,s)\big|\,ds + \int_0^{\tilde{h}} s^2\big|K(u,s)\big|\,ds\right)\\
&= \frac{1}{h^2}\int_K \langle u,x\rangle^2\,dx\\
&= \frac{1}{h^2}L_K^2\kappa_n^{\frac{n+2}{n}}\\
&\geqslant \frac{1}{h^2}\int_{B^n}\langle u,x\rangle^2\,dx\\
&> \left(1+\frac{\tilde{\gamma}}{n\kappa_n}\varepsilon\right)\int_{B^n}\langle u,x\rangle^2\,dx. \qquad (9)
\end{aligned}
$$

In the case of (i), we have $K\subset c\sqrt{n}B^n$ according to (6). It follows that

$$
\begin{aligned}
\int_C \langle u,x\rangle^2\,dx &= \frac{1}{h^2}\int_K \langle u,x\rangle^2\,dx\\
&< \frac{1}{h^2}\left(\int_{B^n}\langle u,x\rangle^2\,dx + c^2nV\big(K\setminus B^n\big)\right)\\
&\leqslant \frac{1+\frac{\varepsilon}{4}}{h^2}\int_{B^n}\langle u,x\rangle^2\,dx\\
&< \left(1-\frac{\varepsilon}{8}\right)\int_{B^n}\langle u,x\rangle^2\,dx. \qquad (10)
\end{aligned}
$$

Let $\delta_{\mathrm{BM}}(C,B^n)=1+\eta$, where we may assume that $\eta\in(0,1)$. Since C has axial rotational symmetry around $\mathbb{R}u$, and o is the centroid of C, there exists $\gamma_1>0$ depending only on n, and an o-symmetric ellipsoid E with axial rotational symmetry around $\mathbb{R}u$ such that $E\subset C\subset(1+\gamma_1\eta)E$. It follows by $V(C)=V(B^n)$ and $u\in\partial C$ that there exists $\gamma_2>0$ depending only on n such that

$$(1+\gamma_2\eta)^{-1}B^n\subset C\subset(1+\gamma_2\eta)B^n.$$

Therefore, we conclude Lemma 3.1 by (10) in the case of (i), and by (9) in the case of (ii). □

Let us write $W(M)$ to denote the mean width of a planar compact convex set M. In particular $\pi W(M)$ is the perimeter of M. Writing $R(M)$ and $r(M)$ to denote the circum- and the inradius of M, and $A(M)$ to denote the area of M, the Bonnesen inequality (appearing first in W. Blaschke [8], see H. Groemer [19] for more references) states

$$W(M)^2-\frac{4}{\pi}A(M)\geqslant\big(R(M)-r(M)\big)^2. \qquad (11)$$

To prove Theorem 1.4 for convex bodies in $\mathbb{R}^n$, we need the following statement.

Proposition 3.2. *If M is a planar compact convex set in $\mathbb{R}^2$ with an axis of symmetry satisfying $\delta_{\mathrm{BM}}(K, B^2) > 1 + \varepsilon$ for $\varepsilon \in (0, 1)$, then there exist orthogonal lines l_1 and l_2 such that $\delta_{\mathrm{BM}}((K_{l_1})_{l_2}, B^2) > 1 + c'\varepsilon^2$ for $c' = 0.001$.*

Proof. Let l be the line of symmetry of K. We may assume that $A(K) = \pi$, and that l intersects K in a segment of length 2 whose midpoint is o.

First we try Steiner symmetrization through l, and the line l' that is orthogonal to l through o. If $\delta_{\mathrm{BM}}((K_l)_{l'}, B^2) > 1 + c'\varepsilon^2$, then we are done. Otherwise there is an ellipse E whose principal axis is contained in l and l' such that

$$E \subset (K_l)_{l'} \subset \left(1 + c'\varepsilon^2\right)E.$$

We deduce that

$$\left(1 + c'\varepsilon^2\right)^{-3} B^2 \subset (K_l)_{l'}. \tag{12}$$

Since $\delta_{\mathrm{BM}}(K, B^2) > 1 + \varepsilon$, it follows that $R(K) - r(K) \geqslant \varepsilon/2$. Therefore, the Bonnesen inequality (11) yields

$$W(K) \geqslant 2 \cdot \left(1 + \frac{\varepsilon^2}{16}\right)^{\frac{1}{2}}.$$

In particular, if the distance of $x_1, x_2 \in \partial K$ is the diameter of K, then

$$\|x_1 - x_2\| > 2 \cdot \left(1 + c'\varepsilon^2\right)^5.$$

Next let s be the segment orthogonal to $x_1 - x_2$ and of length $2(1 + c'\varepsilon^2)^{-3}$. Since K is symmetric through l, (12) yields that $s' \subset K$ for a translate s' of s. We deduce that the convex hull Q of x_1, x_2 and s' satisfies

$$A(Q) > 2 \cdot \left(1 + c'\varepsilon^2\right)^2.$$

Let l_1 be the line determined by x_1 and x_2, let l_2 be an orthogonal line, and let $K' = (K_{l_1})_{l_2}$. Then K' contains a quadrilateral of area larger than $(1 + c'\varepsilon^2)^2 \cdot \frac{2}{\pi} A(K')$, which in turn yields that $\delta_{\mathrm{BM}}(K', B^2) > 1 + c'\varepsilon^2$. □

Proof of Theorem 1.4. If K is o-symmetric, then Lemma 3.1 yields Theorem 1.4. Even if K is not o-symmetric, we may assume that K has rotational symmetry around $\mathbb{R}u$ for some $u \in S^{n-1}$ and satisfies $\delta_{\mathrm{BM}}(K, B^n) > 1 + \gamma\varepsilon$ for the γ in Lemma 3.1. We deduce by Proposition 3.2 that there exist orthogonal hyperplanes H_1 and H_2 containing o such that $H_1 \cap H_2$ is orthogonal to u, and $\delta_{\mathrm{BM}}(\widetilde{K}, B^n) > 1 + c'\gamma^2\varepsilon^2$ for $\widetilde{K} = (K_{H_1})_{H_2}$ and the absolute constant c' of Proposition 3.2. Since $\widetilde{K}$ is o-symmetric, the o-symmetric case of Lemma 3.1 applied to $\widetilde{K}$ yields Theorem 1.4 for K. □

4. Stability of the False Centre Theorem in a special case

For any convex body K in $\mathbb{R}^n$, P.W. Aitchison, C.M. Petty, C.A. Rogers [1] and D.G. Larman [30] proved the False Centre Theorem, which states that if there exists a point p such that all hyperplane sections of K by hyperplanes passing through p are centrally symmetric, then K is either symmetric through p or an ellipsoid. An important part of their proof is concerned with the case when K is o-symmetric and has axial rotational symmetry. We will deal with this special case in Lemma 4.2.

We measure how close a compact convex set M is to be centrally symmetric by the so-called Minkowski measure of symmetry $q(M)$. It is defined by (see, e.g., B. Grünbaum [23])

$$q(M) = \min\bigl\{\lambda \geqslant 1\colon \exists x \in M, -(M-x) \subset \lambda(M-x)\bigr\}.$$

Obviously, $q(M) = 1$ if and only if M is centrally symmetric. Moreover, it is known essentially since the time of H. Minkowski that $q(M) \leqslant n$ for $M \subset \mathbb{R}^n$, where equality holds only for n-dimensional simplices. To prove Lemma 4.2, we need the following estimate:

Proposition 4.1. *Let g be a positive concave function on $(-\varrho, \varrho)$ for $\varrho > 0$, and let M be the compact convex set that is the closure of the convex hull of the graphs of g and $-g$. If $q(M) \leqslant 1+\varepsilon$ for $\varepsilon > 0$, then for any $t \in (0, \varrho)$, we have*

$$\left(1 + \frac{2\varrho\varepsilon}{\varrho - t}\right)^{-1} g(t) \leqslant g(-t) \leqslant \left(1 + \frac{2\varrho\varepsilon}{\varrho - t}\right) g(t).$$

Proof. We may assume that $\varrho = 1$. Writing u to denote the first coordinate unit vector, the condition $q(M) \leqslant 1+\varepsilon$ yields that $M \subset -(1+\varepsilon)M + pu$, where $|p| \leqslant \varepsilon$. In particular, for any $t \in (0,1)$, we have

$$g(-t) \leqslant (1+\varepsilon) g\left(\frac{t+p}{1+\varepsilon}\right).$$

If $\frac{t+p}{1+\varepsilon} \geqslant t$, then, considering the points $(-1,0)$, $(t, g(t))$ and $(\frac{t+p}{1+\varepsilon}, g(\frac{t+p}{1+\varepsilon}))$ of ∂M, leads to

$$g\left(\frac{t+p}{1+\varepsilon}\right) \leqslant \frac{1 + \frac{t+p}{1+\varepsilon}}{1+t} \cdot g(t) \leqslant \frac{1}{1+\varepsilon} \cdot \left(1 + \frac{2\varepsilon}{1+t}\right) \cdot g(t),$$

and if $\frac{t+p}{1+\varepsilon} \leqslant t$, then

$$g\left(\frac{t+p}{1+\varepsilon}\right) \leqslant \frac{1 - \frac{t+p}{1+\varepsilon}}{1-t} \cdot g(t) \leqslant \frac{1}{1+\varepsilon} \cdot \left(1 + \frac{2\varepsilon}{1-t}\right) \cdot g(t).$$

In turn, we conclude the required upper bound for $g(-t)$. To get the lower bound, one applies the same argument for $h(t) = g(-t)$. □

Lemma 4.2. *Let K be an o-symmetric convex body in $\mathbb{R}^n$, $n \geqslant 3$, with axial rotational symmetry. If $\delta_{\mathrm{BM}}(K, B^n) > 1 + \varepsilon$ for $\varepsilon > 0$, then there exists a hyperplane H intersecting $\frac{2}{3}K$ such that $q(H \cap K) \geqslant 1 + c_1\varepsilon^3|\log\varepsilon|^{-1}$, where $c_1 > 0$ is an absolute constant.*

Remark. In the proof we only use hyperplanes that pass through one of the endpoints of the axis of K, therefore we do have a stability version of the False Centre Theorem in this special case. We believe that in Lemma 4.2, the term $\varepsilon^3|\log\varepsilon|^{-1}$ can be improved to ε.

Proof. The proof is based on ideas of P.W. Aitchison, C.M. Petty, C.A. Rogers [1]. We may assume that $u, -u \in \partial K$ where $u \in S^{n-1}$ and $\mathbb{R}u$ is the axis of symmetry of K. We prove the lemma in the following form. There exists a positive absolute constant $\tilde{c}$ such that if $\varepsilon \in (0, 4^{-4})$ and $q(H \cap K) \leqslant 1 + \varepsilon$ holds for any hyperplane H intersecting $\frac{2}{3}K$ and containing $-u$, then $\delta_{\mathrm{BM}}(K, B^n) \leqslant 1 + \tilde{c}\varepsilon^{\frac{1}{3}}|\log\varepsilon|$. To prove this statement, we may assume that ∂K is C^1.

By the symmetry of K, we may assume that $n = 3$. Let $v \in S^2$ be orthogonal to u, and let L be the linear plane spanned by u and v. There exists a non-negative even concave function r on $[-1, 1]$ such that $tu + r(t)v \in \partial K$ for $t \in [-1, 1]$ and $r(0) = 1$. This r is differentiable on $(-1, 1)$ because ∂K is C^1. To prove that K is close to some ellipsoid is equivalent to showing that the function

$$f(t) = \frac{1 - r(t)^2}{t^2}$$

is essentially the constant one function on $(0, 1)$. In this proof, the implied constant in $O(\cdot)$ is always some absolute constant.

For $m \in (0, \frac{1}{4}]$, let H be the plane containing $-u$ and $(1-m)u + r(1-m)v$, whose normal vectors are contained in L, and let $\eta = \frac{r(1-m)}{2-m}$ be the "slope" of $H \cap L$. In particular, if $l \subset H$ is a line orthogonal to L and passing through the point $tu + \eta(1+t)v$, $t \in (-1, 1-m)$, then it intersects K in a segment of length $2\sqrt{r(t)^2 - \eta^2(1+t)^2}$. Since $q(H \cap K) \leqslant 1 + \varepsilon$, Proposition 4.1 yields for any $t \in [0, 1-m)$ that

$$r(-t-m)^2 - \eta^2(1-t-m)^2 \begin{cases} \leqslant \left(1 + \frac{(2-m)\varepsilon}{1-m-t}\right)^2 (r(t)^2 - \eta^2(1+t)^2), \\ \geqslant \left(1 + \frac{(2-m)\varepsilon}{1-m-t}\right)^{-2} (r(t)^2 - \eta^2(1+t)^2). \end{cases}$$

In particular, if $t \in [0, 1-2m]$, then

$$r(t)^2 - r(t+m)^2 = \eta^2(2t+m)(2-m) + O\left(\frac{\varepsilon}{1-t}\right). \tag{13}$$

For $t = 0$, we have

$$m^2 f(m) = 1 - r(m)^2 = \eta^2 m(2-m) + O(\varepsilon) = \frac{m \cdot r(1-m)^2}{2-m} + O(\varepsilon). \tag{14}$$

If $t \in [m, 1-2m]$, then (13) can be written in the form

$$(t+m)^2 f(t+m) - t^2 f(t) = \eta^2(2t+m)(2-m) + O\left(\frac{\varepsilon}{1-t}\right)$$
$$= \left(2tm+m^2\right) f(m) + O\left(\frac{(2t+m)\varepsilon}{m}\right),$$

therefore,

$$f(t+m) = \frac{t^2}{(t+m)^2} f(t) + \frac{2tm+m^2}{(t+m)^2} f(m) + O\left(\frac{\varepsilon}{mt}\right). \tag{15}$$

We deduce by (15) and induction that if $i = 2, \dots, \lfloor \frac{1}{m} - 1 \rfloor$, then

$$f(im) = f(m) + O\left(\sum_{j=1}^{i-1} \frac{\varepsilon}{jm^2}\right) = f(m) + O\left(\frac{\varepsilon |\log m|}{m^2}\right). \tag{16}$$

We define

$$\tilde{m} = \frac{1}{4\lfloor \varepsilon^{-\frac{1}{3}} |\log \varepsilon|^{-\frac{1}{3}} \rfloor}.$$

By definition, $\tilde{m}$ satisfies

$$\frac{\varepsilon |\log \tilde{m}|}{\tilde{m}^2} = O(\tilde{m}) \quad \text{and} \quad \tilde{m} \leqslant \frac{1}{8}. \tag{17}$$

We claim that

$$f(i\tilde{m}) = 1 + O(\tilde{m}) \quad \text{for } i = 1, \dots, \frac{1}{\tilde{m}} - 1. \tag{18}$$

First we observe that according to (16), (17), and the definition of f, we have

$$f(i\tilde{m}) = f(1-\tilde{m}) + O(\tilde{m}) \leqslant (1-\tilde{m})^{-2} + O(\tilde{m}) = 1 + O(\tilde{m})$$

for $i = 1, \dots, \frac{1}{\tilde{m}} - 1$. On the other hand, it follows by (14) that

$$r(1-\tilde{m})^2 = (2-\tilde{m})\tilde{m}\, f(\tilde{m}) + O\left(\frac{\varepsilon}{\tilde{m}}\right) = O(\tilde{m}).$$

In particular,

$$f(1-\tilde{m}) = \frac{1 - r(1-\tilde{m})^2}{(1-\tilde{m})^2} \geqslant \frac{1 - O(\tilde{m})}{(1-\tilde{m})^2} \geqslant 1 - O(\tilde{m}),$$

which in turn yields (18) by (16) and (17).

Finally we verify that if $\tilde{m} \leqslant t \leqslant 1-\tilde{m}$, then

$$f(t) = 1 + O(\tilde{m}) \quad \text{for } t \in [\tilde{m}, 1-\tilde{m}]. \tag{19}$$

First let $t \in [\frac{1}{2}, 1-\tilde{m}]$. In this case,

$$f'(t) = \frac{-2r(t)r'(t)t - 2(1-r(t)^2)}{t^3} \geqslant -16$$

as $r'(t) \leqslant 0$. Since there exists an integer $i \leqslant \frac{1}{\tilde{m}} - 2$ such that $\frac{1}{2} \leqslant i\tilde{m} \leqslant t \leqslant (i+1)\tilde{m}$, we deduce (19) from (16) and (17).

Next let $t \in [\tilde{m}, \frac{1}{2}]$. There exist integers j and i such that $m \in [\tilde{m}, 2\tilde{m}]$ holds for $m = t/j$, and $im \in [\frac{1}{2}, 1-m]$, thus (16) and the previous case of (19) yield

$$f(t) = f(jm) = f(m) + O\left(\frac{\varepsilon|\log m|}{m^2}\right) = f(im) + O\left(\frac{\varepsilon|\log m|}{m^2}\right) = 1 + O(\tilde{m}).$$

With this, we have proved (19), which in turn yields Lemma 4.2. □

From Lemma 4.2 we immediately obtain:

Corollary 4.3. *Let K be an o-symmetric convex body in $\mathbb{R}^n$ with axial rotational symmetry. If $\delta_{\mathrm{BM}}(K, B^n) > 1+\varepsilon$ for $\varepsilon > 0$, then there exist $u \in S^{n-1}$ and $a \in (0, \frac{2}{3})$ such that*

$$q\big(K\big(u, h_K(u)t\big)\big) \geqslant 1 + c_2\varepsilon^3|\log\varepsilon|^{-1} \quad \text{for } t \in \big(a, a + c_2\varepsilon^3|\log\varepsilon|^{-1}\big),$$

where $c_2 > 0$ is an absolute constant.

5. Proof of Theorem 1.1

We will need a stability version of the Brunn–Minkowski inequality. According to V.I. Diskant [13], if M is a compact convex set of dimension $n-1$ with $q(M) \geqslant 1+\tau$, then

$$\left|\frac{1}{2}(M-M)\right| \geqslant \big(1 + \gamma\tau^{n-1}\big)|M|, \tag{20}$$

for $\gamma > 0$ depending on n (see also H. Groemer [19]). Here no explicit γ is known. Actually H. Groemer [18] proved a stability estimate with explicit γ but with the exponent n instead of $n-1$.

In this section, $\gamma_1, \gamma_2, \ldots$ denote positive constants depending only on n. We prove Theorem 1.1 in the following equivalent form: If K is a convex body in $\mathbb{R}^n$ with Santaló point z and $\delta_{\mathrm{BM}}(K, B^n) > 1+\varepsilon$ for $\varepsilon > 0$, then (21) holds.

It follows from Theorem 1.4 and Lemma 2.1 that there exists an o-symmetric convex body C with axial rotational symmetry such that $\delta_{\mathrm{BM}}(C, B^n) > 1+\gamma_1\varepsilon^2$ and $V(K)V(K^z) \leqslant V(C)V(C^o)$. In particular, C^o is an o-symmetric convex body with axial rotational symmetry and satisfies $\delta_{\mathrm{BM}}(C^o, B^n) > 1+\gamma_1\varepsilon^2$. According to Corollary 4.3, there exist $u \in S^{n-1}$ and $a \in (0, \frac{2}{3})$ such that

$$q\big(C^o\big(u, h_{C^o}(u)t\big)\big) \geqslant 1 + \gamma_2\varepsilon^6|\log\varepsilon|^{-1} \quad \text{for } t \in \big(a, a + \gamma_2\varepsilon^6|\log\varepsilon|^{-1}\big).$$

We may assume that $h_{C^o}(u) = 1$.

Let $\widetilde{C} = C_{u^\perp}$. Since the convexity of C^o yields $|C^o(u,t)| \geqslant 4^{-(n-1)}|C^o(u,0)|$ if $t \leqslant \frac{3}{4}$, we deduce from (7), (8) and (20) that

$$V\big(\widetilde{C}^o\big) \geqslant 2\int_0^1 \big|\widetilde{C}^o(u,t)\big|\,dt \geqslant 2\int_0^1 \big|C^o(u,t)\big|\,dt + \gamma_3\big|C^o(u,0)\big|\varepsilon^{6n}|\log\varepsilon|^{-n}.$$

On the one hand, $V(C^o) \leqslant 2|C^o(u,0)|$ by the Fubini Theorem and the Brunn–Minkowski inequality (7). Therefore,

$$\begin{aligned} V(K)V\big(K^z\big) &\leqslant V(C)V\big(C^o\big) \leqslant \big(1 - \gamma_4\varepsilon^{6n}|\log\varepsilon|^{-n}\big)V(\widetilde{C})V\big(\widetilde{C}^o\big) \\ &\leqslant \big(1 - \gamma_4\varepsilon^{6n}|\log\varepsilon|^{-n}\big)\kappa_n^2, \end{aligned} \tag{21}$$

which concludes the proof of Theorem 1.1.

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