GEOMETRY NOTES

Contents

1. Projective spaces made of vector spaces over skew fields

Definition 1.1. Let $n \geq 1$. Let V be an $(n+1)$-dimensional vector space over a skew field $\mathbb{F}$. The space of its 1-dimensional linear subspaces is called an n-dimensional projective space over $\mathbb{F}$. It is denoted by $P(V)$.

A k-dimensional projective subspace is $P(W)$, where W is a $(k+1)$-dimensional linear subspace of V.

Proposition 1.2. *Intersection of subspaces is a subspace. Also we have*

$$\bigcap_{i \in I} P(W_i) = P(\bigcap_{i \in I} W_i).$$

For every $S \subset P(W)$ there is a unique smallest projective subspace $[S]$ containing S. We have

$$[\cup_{i\in I} P(W_i)] = P(\sum_{i\in I} W_i).$$

Denote $[\cup_{i\in I} P(W_i)]$ by $\sum_{i\in I} P(W_i)$.

Proposition 1.3. *We have*

$$\dim P(W_1) \cap P(W_2) + \dim P(W_1) + P(W_2) = \dim P(W_1) + \dim P(W_2).$$

Note that for affine subspaces this does not hold.

Definition 1.4. Homogenous coordinates: the point $(x_1, \ldots, x_{n+1}) \in \mathbb{F}^{n+1}$ is equivalent to $(\lambda x_1, \ldots, \lambda x_{n+1})$ for every $\lambda \in \mathbb{F}$, $\lambda \neq 0$. Its class is denoted by

$$(x_1 : \ldots : x_{n+1}).$$

Every point $(x_1 : \ldots : x_{n+1})$ can be written as

$$(x_1 : \ldots : x_n : 1) \qquad \text{or} \qquad (x_1 : \ldots : x_n : 0).$$

The point $(x_1 : \ldots : x_n : 0)$ is called the point in the direction of $(x_1, \ldots, x_n)$, this is the ideal point ∞ in the lines parallel to $(x_1, \ldots, x_n)$. Often we imagine that $\mathbb{F}^n$ sits inside $\mathbb{F}^{n+1}$ as

$$(x_1, \ldots, x_n) \mapsto (x_1, \ldots, x_n, 1).$$

Obviously three points P, Q, R are collinear (that is they are in the same projective line) iff all representatives p, q, r are in a 2-dimensional linear subspace, that is p, q, r are linearly dependent. There are such representatives that $p = q + r$.

Definition 1.5. Two triangles ABC and $A'B'C'$ are perspective from the point P if $AA'P$, $BB'P$, $CC'P$ are collinear. The triangles ABC and $A'B'C'$ are perspective from the line e if all the following three lines intersect each other in one-one point:

$$AB, A'B', e \qquad BC, B'C', e \qquad AC, A'C', e$$

Theorem 1.6 (Desargues)**.** *In a projective space two triangles are perspective from a point iff they are perspective from a line.*

Theorem 1.7 (Pappos)**.** *Let e and f be lines in a plane, $e \cap f = p$. Let $A, B, C \neq p$ be distinct points in e and let $A', B', C' \neq p$ be distinct points in f. The following statement is true for a skew field iff the field is commutative: the points $BC' \cap B'C$, $AC' \cap A'C$ and $AB' \cap A'B$ are collinear.*

2. Projective spaces by axioms

Definition 2.1. Let X be a set, $n \geq 0$ and $\mathcal{S}_{-1}, \mathcal{S}_0, \ldots, \mathcal{S}_n \subset \mathcal{P}(X)$, these are the i-dimensional subspaces. Then

$$(X, \mathcal{S}_{-1}, \mathcal{S}_0, \ldots, \mathcal{S}_n)$$

is an n-dimensional projective space if

(1) $\mathcal{S}_{-1} = \{\emptyset\}$,

(2) $\mathcal{S}_0 = \{\{P\} : P \in X\}$,
(3) $\mathcal{S}_n = \{X\}$,
(4) intersection of subspaces is a subspace (so there is generated subspace and "+"),
(5) if $i \neq j$, then $\mathcal{S}_i \cap \mathcal{S}_j = \emptyset$,
(6) for every subspace W_1, W_2 we have

$$\dim W_1 \cap W_2 + \dim W_1 + W_2 = \dim W_1 + \dim W_2,$$

(7) there are $n+2$ points in X such that no $n+1$ among them are in an $(n-1)$-dimensional subspace (these are points in general position).

Proposition 2.2. *A projective space over any (non-commutative) field satisfies these axioms.*

Proposition 2.3. *If $W_1 \subset W_2$, then $\dim W_1 \leq \dim W_2$. $\dim W_1 = \dim W_2$ iff $W_1 = W_2$.*

We denote by pq the subspace $\{p\} + \{q\}$.

Proposition 2.4. *If $W \neq \emptyset$, $p \notin W$, then $W + \{p\} = \bigcup_{q \in W} pq$.*

Proposition 2.5. *If $W \subset X$, then W is a subspace iff for every $p, q \in W, p \neq q$ we have $pq \subset W$.*

Proposition 2.6. *Suppose $(X, \mathcal{S}_{-1}, \mathcal{S}_0, \dots, \mathcal{S}_n)$ and $(X, \mathcal{S}'_{-1}, \mathcal{S}'_0, \dots, \mathcal{S}'_n)$ are two projective space structures on X. If $\mathcal{S}_1 = \mathcal{S}'_1$, then the two structures are the same.*

Definition 2.7. Let $W \in \mathcal{S}_k$. For every $0 \leq i \leq k$ let $\mathcal{S}_i^W = \{V \in \mathcal{S}_i : V \subset W\}$.

Proposition 2.8. *$(W, \mathcal{S}_{-1}^W, \mathcal{S}_0^W, \dots, \mathcal{S}_k^W)$ is a k-dimensional projective space.*

Corollary 2.9. Every projective line has at least three points.

Definition 2.10. Let $(X, \mathcal{S}_{-1}, \mathcal{S}_0, \dots, \mathcal{S}_n)$ an n-dimensional projective space. For every $W \in \mathcal{S}_i$ let

$$W^{\perp} = \{H \in \mathcal{S}_{n-1} : W \subset H\}.$$

Take $X^* = \mathcal{S}_{n-1}$ and $\mathcal{S}_i^* = \{W^{\perp} : W \in \mathcal{S}_{n-i-1}\}$. Then

$$(X^*, \mathcal{S}_{-1}^*, \mathcal{S}_0^*, \dots, \mathcal{S}_n^*)$$

is called the dual projective space.

Proposition 2.11 (no proof)**.** *The space $(X^*, \mathcal{S}_{-1}^*, \mathcal{S}_0^*, \dots, \mathcal{S}_n^*)$ is an n-dimensional projective space.*

The so-called principle of duality is that if we have a statement about a projective space, then by taking the dual space we get a new statement for the dual projective space. Then i-dimensional spaces are replaced by $(n-i-1)$-dimensional spaces, "$\cap$" is replaced by "+", "$\subset$" is by "$\supset$".

Theorem 2.12 (no proof)**.** *Desargues theorem follows for at least 3-dimensional projective spaces. For 2-dimensional projective spaces Desargues theorem does not hold necessarily.*

Definition 2.13. Let $(X, \mathcal{S}_{-1}, \mathcal{S}_0, \dots, \mathcal{S}_n)$ and $(X', \mathcal{S}'_{-1}, \mathcal{S}'_0, \dots, \mathcal{S}'_n)$ be projective spaces. The bijective map $\Phi \colon X \to X'$ is a collineation if we have $A \in \mathcal{S}_k$ iff $\Phi(A) \in \mathcal{S}'_k$.

Proposition 2.14. *Suppose Φ is a bijection. Then Φ is a collineation iff every 3 collinear points have collinear Φ-images.*

Theorem 2.15. *In a projective space every two k-dimensional subspaces are isomorphic.*

Theorem 2.16. *Let X and X' be two n-dimensional projective spaces and let $W \subset X$ and $W' \subset X'$ be k-dimensional subspaces for $2 \leq k \leq n-1$. If W and W' are isomorphic, then X and X' are also isomorphic.*

We prove another theorem which implies this.

Theorem 2.17. *Let H, H' be hyperplanes in X and X', respectively, where $\dim X = \dim X' \geq 3$. Let $\Phi\colon H \to H'$ be a collineation. Let $A, B \notin H$, $A \neq B$ and $A', B' \notin H'$, $A' \neq B'$. Suppose $AB \cap H = C$ and $A'B' \cap H' = C'$ with $\Phi(C) = C'$. Then there is a unique collineation $\tilde{\Phi}\colon X \to X'$ such that it is an extension of Φ and $\tilde{\Phi}(A) = A'$, $\tilde{\Phi}(B) = B'$.*

Note that A', B', C' are collinear.

Definition 2.18. Let X be a projective plane, $\varphi\colon X \to X$ a collineation and $e \in \mathcal{S}_1$ a line. If φ fixes all the points of e, then e is the axis of φ. If $O \in X$ is such that every line going through O is mapped into itself by φ, then O is a center of φ. If φ has axis and center, then it is a central-axial collineation.

Proposition 2.19 (no proof)**.** *On a Desarguesian plane some φ has an axis iff φ has a center.*

Let e be a line on a plane X, let $A, B, A', B' \in X - e$, $A \neq B$ and $A' \neq B'$. Suppose that $AB \cap e = A'B' \cap e$.

Theorem 2.20. *There is a unique central-axial collineation $\varphi\colon X \to X$ with axis e and $\varphi(A) = A'$, $\varphi(B) = B'$ iff Desargues theorem holds on the plane X.*

Theorem 2.21 (no proof)**.** *A Desarguesian plane is isomorphic to a projective plane over a skew-field.*

3. Projective spaces over fields

Definition 3.1. We denote the group of collineations of the n-dimensional projective space $\mathbb{F}P^n$ by $coll(\mathbb{F}P^n)$.

Definition 3.2. For an $A \in GL(n+1, \mathbb{F})$ let $[A]\colon \mathbb{F}P^n \to \mathbb{F}P^n$ denote the map

$$[v] \mapsto [Av].$$

Proposition 3.3. *The map $[A]$ is a collineation because $[A]$ maps k-dimensional projective subspaces to k-dimensional projective subspaces. Also $[A \circ B] = [A] \circ [B]$ and $[A]^{-1} = [A^{-1}]$. The group $PGL(n+1, \mathbb{F})$ of the maps of the form $[A]$ is a subgroup of $coll(\mathbb{F}P^n)$.*

Definition 3.4. Let φ be an automorphism of the field $\mathbb{F}$. Then $[\varphi]\colon \mathbb{F}P^n \to \mathbb{F}P^n$ is defined by

$$(x_1 : \ldots : x_{n+1}) \mapsto (\varphi(x_1) : \ldots : \varphi(x_{n+1})).$$

Proposition 3.5. *For the map*

$$\tilde{\varphi}\colon \mathbb{F}^{n+1} \to \mathbb{F}^{n+1}$$
$$(x_1, \dots, x_{n+1}) \mapsto (\varphi(x_1), \dots, \varphi(x_{n+1}))$$

we have $\tilde{\varphi}(x+y) = \tilde{\varphi}(x)+\tilde{\varphi}(y)$ *and* $\tilde{\varphi}(\lambda x) = \varphi(\lambda)\tilde{\varphi}(x)$. *Then* $\tilde{\varphi}$ *maps* $(k+1)$*-dimensional linear subspaces to* $(k+1)$*-dimensional linear subspaces so* $\tilde{\varphi}$ *induces a collineation*

$$[\varphi]\colon \mathbb{F}P^n \to \mathbb{F}P^n.$$

Also $[\varphi_1 \circ \varphi_2] = [\varphi_1] \circ [\varphi_2]$ *and* $[\varphi]^{-1} = [\varphi^{-1}]$. *This group of the maps of the form* $[\varphi]$ *denoted by* $\widetilde{Aut}(\mathbb{F})$ *is a subgroup of* $coll(\mathbb{F}P^n)$.

Lemma 3.6. *Let* $A_1, \dots, A_{n+2} \in X$ *and* $B_1, \dots, B_{n+2} \in X$ *be points in general position in an* n*-dimensional projective space. Then there is a unique* $\varphi \in PGL(n+1, \mathbb{F})$ *such that for every* $1 \leq i \leq n$ *we have* $\varphi(A_i) = B_i$.

Lemma 3.7 (sketchy proof)**.** *Let*

$$A_1 = (1:0:\dots:0), A_2 = (0:1:\dots:0), \dots, A_n = (0:\dots:1:0),$$
$$O = (0:\dots 0:1), U = (1:\dots 1:1).$$

Let $\varphi \in coll(\mathbb{F}P^n)$. *Then* $\varphi \in \widetilde{Aut}(\mathbb{F})$ *iff* φ *keeps the points* $A_1, \dots, A_n, O, U$ *fixed.*

Theorem 3.8. *We have*

$$coll(\mathbb{F}P^n) = PGL(n+1, \mathbb{F}) \rtimes \widetilde{Aut}(\mathbb{F}).$$

That is every $\varphi \in coll(\mathbb{F}P^n)$ *can be written uniquely as*

$$\varphi = \varphi_2 \circ \varphi_1$$

for $\varphi_1 \in \widetilde{Aut}(\mathbb{F})$ *and* $\varphi_2 \in PGL(n+1, \mathbb{F})$, *moreover* $PGL(n+1, \mathbb{F}) \lhd coll(\mathbb{F}P^n)$ *and* $\widetilde{Aut}(\mathbb{F}) \cap PGL(n+1, \mathbb{F}) = \{id_{\mathbb{F}P^n}\}$.

Some parts of the proof follow from the next statements.

Definition 3.9. Let A, B, C, D be distinct points in a line. Suppose $A = [a]$ and $B = [b]$. Then suppose that $C = [\gamma_1 a + \gamma_2 b]$ and $D = [\delta_1 a + \delta_2 b]$. The cross-ratio $(ABCD)$ is defined to be the field element

$$\frac{\gamma_2}{\gamma_1} : \frac{\delta_2}{\delta_1}.$$

Proposition 3.10. *The cross-ratio has the following properties.*

(1) *The cross-ratio does not depend on the representatives* a *and* b.
(2)
$$(ABCD) = \frac{1}{(BACD)} = \frac{1}{(ABDC)}.$$
(3) $(ABCD)$ *is never equal to* 0 *or* 1.
(4) *If* A, B, C *are collinear distinct points and* $\lambda \in \mathbb{F}$, $\lambda \neq 0, 1$, *then there is a unique* D_λ *with*
$$(ABCD)_\lambda = \lambda.$$
(5) *If* $(ABCD) = (ABCD^*)$, *then* $D = D^*$.
(6) *For a line* $[a]+[b]$ *let* P_λ *be* $[a + \lambda b]$ *for* $\lambda \in \mathbb{F}$. *Then*
$$(P_\lambda P_\mu P_\nu P_\tau) = \frac{\lambda - \nu}{\nu - \mu} : \frac{\lambda - \tau}{\tau - \mu}.$$

(7) *For* $\varphi_1 \in PGL(n+1,\mathbb{F})$, $\varphi_2 \in \widetilde{Aut}(\mathbb{F})$, $\varphi_2 = [\varphi]$ *we have*
   (a)
$$(\varphi_1(A)\varphi_1(B)\varphi_1(C)\varphi_1(D)) = (ABCD),$$
   (b)
$$(\varphi_2(A)\varphi_2(B)\varphi_2(C)\varphi_2(D)) = \varphi((ABCD)).$$

Corollary 3.11. A collineation is in $PGL(n+1,\mathbb{F})$ iff it preserves cross-ratio.

Proposition 3.12. *The group* $PGL(n+1,\mathbb{F})$ *is a normal subgroup of* $coll(\mathbb{F}P^n)$.

Theorem 3.13 (no proof)**.** *For arbitrary lines* e, f *and a map* $\varphi\colon e \to f$ *the following are equivalent.*

(a) φ *can be extended to an element of* $PGL(n+1,\mathbb{F})$,
(b) φ *keeps cross-ratio.*

Proposition 3.14. *Let* $A, B, C \in e$ *and* $A', B', C' \in f$ *be three-three distinct points in the lines* e *and* f. *Then there is a unique cross-ratio keeping* $\varphi\colon e \to f$ *such that* $\varphi(A) = A'$, $\varphi(B) = B'$, $\varphi(C) = C'$.

Proposition 3.15.

(1) *For a line* e *the collineations* $\varphi\colon e \to e$ *keeping the cross-ratio form a group, it is equal to* $PGL(2,\mathbb{F})$.
(2) *The action of this group is simply transitive on the ordered triples of points of the line* e.
(3) *A cross-ratio keeping map* $\varphi\colon e \to e$ *can have* 0, 1 *or* 2 *fixpoints if* $\varphi \neq id$.

Definition 3.16. A cross-ratio keeping map $\varphi\colon e \to e$ with 0 fixpoint is elliptic, with 1 fixpoint it is parabolic, with 2 fixpoints it is hyperbolic.

4. Cross-ratio keeping maps with $\operatorname{char}\mathbb{F} \neq 2$.

Let $e = [a] + [b]$ and then the points of e of the form $[xa + yb]$ correspond to the point $(x : y) \in \mathbb{F}P^1$. The cross-ratio keeping maps of $\mathbb{F}P^1$ are
$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$$
where $ad \neq bc$. Then
$$\left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]\left(\left[\begin{pmatrix} x \\ y \end{pmatrix}\right]\right) = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}\right].$$
If $\varphi = [L]$, then the fixpoints of φ are represented by the eigenvectors of L.

Proposition 4.1. *If* $L = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right]$, *then according to the value of*
$$D = (trL)^2 - 4detL$$
we have that

(1) *if* $D \neq 0$ *and* $\sqrt{D} \in \mathbb{F}$ *exists, then* φ *is hyperbolic,*
(2) *if* $D \neq 0$ *and* $\sqrt{D} \in \mathbb{F}$ *does not exists, then* φ *is elliptic,*

(3) *if $D=0$ and L has only one eigenvector, then φ is parabolic,*
(4) *if $D=0$ and L has two independent eigenvectors, then $\varphi = id$.*

Corollary 4.2. If $\mathbb{F}$ is algebraically closed, then there is no elliptic map in $PGL(2,\mathbb{F})$.

Definition 4.3. The map φ is an involution if $\varphi \neq id$ and $\varphi^2 = id$.

Theorem 4.4. *Suppose φ corresponds to the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The map φ is an involution iff $a+d=0$.*

Proposition 4.5. *Involutions are not parabolic.*

Corollary 4.6. If $\mathbb{F}$ is algebraically closed, then every involution is hyperbolic.

Theorem 4.7. *Suppose $A, A', B, B' \in e$ and $\{A, A'\} \cap \{B, B'\} = \emptyset$. Then there is a unique involution $\varphi\colon e \to e$ such that $\varphi(A) = A'$ and $\varphi(B) = B'$.*

Theorem 4.8. *Every cross-ratio keeping map is a composition of two involutions.*

5. Algebraic hypersurfaces

Definition 5.1. Let P be a degree $\leq d$ polynomial in n variables, that is

$$P(x_1,\dots,x_n) = \sum_{i_1+\cdots+i_n \leq d} a_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n}.$$

Their vector space over $\mathbb{F}$ is denoted by $\mathcal{P}_n^{\leq d}$. The set of roots of $P \in \mathcal{P}_n^{\leq d}$ is the set

$$\Gamma_P = \{(x_1,\dots,x_n) \in \mathbb{F}^n | P(x_1,\dots,x_n) = 0\}.$$

A degree $\leq d$ affine algebraic hypersurface is the equivalence class of non-zero polynomials in $\mathcal{P}_n^{\leq d}$ where two polynomials P_1 and P_2 are equivalent if $P_1 = \lambda P_2$ for some $\lambda \in \mathbb{F}$. The space of such affine algebraic hypersurfaces is

$$P\left(\mathcal{P}_n^{\leq d}\right),$$

which is a projective space of dimension $\binom{n+d}{d} - 1$. The homogenization of $P \in \mathcal{P}_n^{\leq d}$ is the polynomial of degree d in $n+1$ variables defined by

$$\overline{P}(x_1,\dots,x_{n+1}) = \sum_{i_1+\cdots+i_n \leq d} a_{i_1,\dots,i_n} x_1^{i_1}\cdots x_n^{i_n} x_{n+1}^{d-\sum_{j=1}^n i_j}.$$

Such polynomials are denoted by $\overline{\mathcal{P}}_{n+1}^d$. A projective algebraic hypersurface of degree d is the equivalence class of a non-zero homogeneous polynomial in $\overline{\mathcal{P}}_{n+1}^d$ where two polynomials $\overline{P}_1$ and $\overline{P}_2$ are equivalent if $\overline{P}_1 = \lambda \overline{P}_2$ for some $\lambda \in \mathbb{F}$. The space of such projective algebraic hypersurfaces is the projective space

$$P\left(\overline{\mathcal{P}}_{n+1}^d\right).$$

The set of roots of $\overline{P} \in \overline{\mathcal{P}}_{n+1}^d$ is the set

$$\overline{\Gamma}_{\overline{P}} = \{(x_1 : \ldots : x_{n+1}) \in \mathbb{F}P^n | \overline{P}(x_1,\dots,x_{n+1}) = 0\}.$$

If

$$\mathbb{F}^n \subset \mathbb{F}P^n$$
$$(x_1, \dots, x_n) \mapsto (x_1 : \dots : x_n : 1)$$

is the embedding of the affine space $\mathbb{F}^n$, then we have

$$\overline{\Gamma}_{\overline{P}} \cap \mathbb{F}^n = \Gamma_P.$$

Definition 5.2. A pencil of projective algebraic hypersurfces is a line in the projective space $P\left(\overline{\mathcal{P}}^d_{n+1}\right)$. For two elements $[\overline{P}_1]$ and $[\overline{P}_2]$ in $P\left(\overline{\mathcal{P}}^d_{n+1}\right)$ the elements $[\alpha\overline{P}_1 + \beta\overline{P}_2]$ form the pencil.

Theorem 5.3. *For a given pencil a point $A \in \mathbb{F}P^n$ is a point of every set of roots of the elements in the pencil or A is in exactly only one such set of roots.*

If $d = 2$, then the elements of the vector space $\overline{\mathcal{P}}^2_{n+1}$ are in one-to-one correspondence with the symmetric matrices made of their coefficients.

Definition 5.4. If A is such matrix, then the symmetric bilinear map $\{x, y\}$ is defined to be $x^T Ay$.

Notice that $\overline{\mathcal{P}}^2_{n+1}$ is the quadratic form of $\{\cdot, \cdot\}$.

Definition 5.5. The points $[x]$ and $[y]$ are conjugate with respect to the hypersurface $[A]$ if $\{x, y\} = 0$. In notation $[x] \sim [y]$.

Of course $[x] \sim [x]$ iff $[x]$ is in the hypersurface.

If $[x] \in \mathbb{F}P^n$, then the set of conjugate points is

$$\{[y] | x^T Ay = 0\}.$$

This is equal to the entire space $\mathbb{F}P^n$ or to a hyperplane.

Definition 5.6. This hyperplane is called the polar of the point $[x]$. The surface $[A]$ is regular if $detA \neq 0$. The point $[x]$ is the pole of the hyperplane.

Proposition 5.7. *Every hyperplane is the polar of exactly one point.*

Theorem 5.8 (no proof)**.** *The relation between pole and polar for a regular 2-curve is an isomorphism (collineation) between the projective space and its dual space. Such isomorphism is called a correlation.*

Definition 5.9. A tangent line is a line e which intersects a hypersurface in one single point or it is fully contained in the hypersurface.

Theorem 5.10. *Let $[p]$ be a point in the regular hypersurface $[A]$ and let e be a line which goes through $[p]$. Then e is tangent to $[A]$ iff e is contained in the polar of $[p]$.*

Definition 5.11. The tangent hyperplane of the regular $[A]$ at $[p]$ is the polar of $[p]$.

Theorem 5.12. *Let P be a point in the projective space and let $[A]$ be a regular hypersurface. The set of tangency points in $[A]$ of the tangent lines of $[A]$ going through P is equal to the intersection of $[A]$ and the polar of P.*

Definition 5.13. Two hypersurfaces $[A]$ and $[B]$ are projective equivalent if for some $L \in GL(n+1, \mathbb{F})$ and for some $\lambda \in \mathbb{F}$, $\lambda \neq 0$, we have

$$B = \lambda L^T A L.$$

Theorem 5.14 (From linear algebra)**.** *For every symmetric bilinear form there is a basis such that the matrix of the form is diagonal.*

Theorem 5.15. *If A is a symmetric matrix, then the hypersurface $[A]$ is projective equivalent to a hypersurface given by a diagonal matrix. If $\mathbb{F}$ is algebraically closed, then we can suppose that in the diagonal there are only 1 and 0. So then every hypersurface is projective equivalent to*

$$x_1^2 + \cdots + x_r^2 = 0$$

for some $r \leq n+1$ and two such hypersurfaces are equivalent iff the corresponding value r is the same.

Corollary 5.16. For example, for $n = 2$ and $\mathbb{F} = \mathbb{C}$ the possible hypersurfaces are $x_1^2 = 0$, $x_1^2 + x_2^2 = 0$ and the regular $x_1^2 + x_2^2 + x_3^2 = 0$.

Proposition 5.17. *If $\mathbb{F} = \mathbb{R}$, then there is a basis such that in the diagonal of the corresponding $L^T A L$ there are only p copies of 1, r copies of -1 and some 0. The value $p + r$ is invariant under projective equivalence. Every hypersurface is projective equivalent to*

$$x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+r}^2 = 0$$

for some $p + r \leq n+1$ and two such hypersurfaces are equivalent iff the corresponding set $\{p, r\}$ is the same.

Corollary 5.18. For example, for $n = 2$ and $\mathbb{F} = \mathbb{R}$ the possible hypersurfaces are $x_1^2 = 0$, $x_1^2 + x_2^2 = 0$, the regular $x_1^2 + x_2^2 + x_3^2 = 0$ and the regular $x_1^2 + x_2^2 - x_3^2 = 0$. At the embedding $\mathbb{R}^2 \hookrightarrow \mathbb{R}P^2$, $(x_1, x_2) \mapsto (x_1 : x_2 : 1)$, the last one is a unit circle.

6. Spherical geometry

Let G be a sphere in $\mathbb{R}^3$ with radius $r > 0$ and center 0.

Definition 6.1. The straight lines in G are the circles whose planes go through 0.

If $A, B \in G$, then there is a unique line through them if $-A \neq B$, that is A and B are not antipodal. The intersection of two lines is a set of two antipodal points.

Definition 6.2. If $A, B \in G$, then their spherical distance $d(A, B)$ is equal to the length of the shorter arc connecting them on a connecting line.

Then $d(A, B) = r\alpha$, where α is the angle $\leq \pi$ between the vectors OA and OB. Three points on G which are not on one line determine a spherical triangle. Then there are no antipodal points among them so the shortest connecting arcs exist and they are the sides of the spherical triangle.

Definition 6.3. The angle between two arcs in G intersecting each other at a point is equal to the angle between the two tangent vectors to these arcs at that point.

Let $A, B, C \in G$ be the vertices of a spherical triangle. Then α at A is the same as the angle between the two half-planes determined by the triples A, $-A$, B and A, $-A$, C.

Definition 6.4. For a spherical triangle A, B, C there is a spherical triangle A^*, B^*, C^* defined by

(1) $OA^* \perp OB, OC$,
(2) $OB^* \perp OA, OC$ and
(3) $OC^* \perp OA, OB$.

This is called the polar triangle $A^*B^*C^*$ of the triangle ABC. The vertices of the polar triangle are determined by the condition that $d(A, A^*) < r\pi/2$, $d(B, B^*) < r\pi/2$ and $d(C, C^*) < r\pi/2$.

Proposition 6.5. *The polar triangle of a polar triangle is the original triangle.*

Proposition 6.6.

$$\alpha^* + a/r = \beta^* + b/r = \gamma^* + c/r = \alpha + a^*/r = \beta + b^*/r = \gamma + c^*/r.$$

In the following suppose that $r = 1$.

Proposition 6.7. *If $\underline{a}$, $\underline{b}$, $\underline{c}$ denote the vectors OA, OB, OC, then the vectors pointing to the vertices of the polar triangle are*

$$\underline{a}^* = \frac{\underline{b} \times \underline{c}}{|\underline{b} \times \underline{c}|}, \qquad \underline{b}^* = \frac{\underline{c} \times \underline{a}}{|\underline{c} \times \underline{a}|}, \qquad \underline{c}^* = \frac{\underline{a} \times \underline{b}}{|\underline{a} \times \underline{b}|}.$$

Theorem 6.8. *For a triangle we have*

$$\frac{\sin a}{\sin \alpha} = \frac{\sin b}{\sin \beta} = \frac{\sin c}{\sin \gamma}.$$

Theorem 6.9. *For a triangle we have*

$$\cos c = \cos a \cos b + \sin a \sin b \cos \gamma.$$

Corollary 6.10. By applying this to the polar triangle we get

$$\cos c^* = \cos a^* \cos b^* + \sin a^* \sin b^* \cos \gamma^*$$

which implies

$$-\cos \gamma = \cos \alpha \cos \beta - \sin \alpha \sin \beta \cos c$$

so the angles determine the side lengths a, b, c.

Theorem 6.11. *For the side lengths of a triangle we have $a + b > c$.*

Corollary 6.12. We have $a + b + c < 2\pi$ and $\alpha + \beta + \gamma > \pi$.

Definition 6.13. The value $\alpha + \beta + \gamma - \pi$ is the spherical excess of the triangle.

Theorem 6.14. *The area of a spherical triangle is equal to the spherical excess*

$$\alpha + \beta + \gamma - \pi.$$

7. Hyperbolic Geometry

In $\mathbb{R}^n \oplus \mathbb{R}$ we have the vectors (x,t) with $x \in \mathbb{R}^n$, $t \in \mathbb{R}$. The last coordinate will help in computations though it is not a "time" coordinate in special relativity. Let $A \in \mathbb{R}^{(n+1)\times(n+1)}$ be the diagonal matrix

$$\begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots \\ & & \vdots & \\ & & & 0 \\ 0 & \cdots & 0 & -1 \end{pmatrix}.$$

We define the scalar product

$$\{\xi,\eta\} = \xi^T A\eta,$$
$$\{\xi,\eta\} = \langle x,y\rangle - tt'$$

if $\xi = (x,t)$ and $\eta = (y,t')$. The subgroup of $GL(n+1,\mathbb{R})$ keeping $\{\cdot,\cdot\}$ fixed is called Lorentz group. The group of the affin isomorphisms of the form $v \to Av + b$, where A is in the Lorentz group, is called the Poincaré group. The structure $(\mathbb{R}^{n+1}, \{\cdot,\cdot\})$ is called Minkowski spacetime.

Definition 7.1. If $\xi \in \mathbb{R}^{n+1}$, then

(1) if $\{\xi,\xi\} < 0$, then ξ is time-like,
(2) if $\{\xi,\xi\} > 0$, then ξ is space-like,
(3) if $\{\xi,\xi\} = 0$, then ξ is light-like.

In special relativity a point (observer) can move only along paths $\gamma(s)$ with time-like tangent vectors $\gamma'(s)$, their simultaneous world (existing at the same moment) is infinitesimally the hyperplane $\{\gamma'(s), z\} = 0$.

For the points $\xi = (x,t)$ and $\eta = (y,t')$ the equation

$$\{\eta-\xi,\eta-\xi\} = 0, \quad \text{that is} \quad |y-x|^2 = (t'-t)^2,$$

where η varies and ξ is fixed, defines a cone called the light cone at ξ.

The path $\gamma\colon [a,b] \to \mathbb{R}^{n+1}$ is time-like (space-like) if γ' is time-like (space-like, respectively) all along.

Definition 7.2. The length of a space-like path is

$$\int_a^b \sqrt{\{\gamma'(s),\gamma'(s)\}}ds,$$

this gives the length of objects in the 3-dimensional world at a moment. The so-called proper time of a time-like path is

$$\int_a^b \sqrt{\{-\gamma'(s),-\gamma'(s)\}}ds,$$

this gives the time passed along the path in the spacetime in the relative world of the moving point.

Theorem 7.3. *The orbits of the Lorentz group are the hypersurfaces* $\{\xi,\xi\} = C$ *for the constant numbers* $C \neq 0$. *If* $C < 0$, *then the orbit has two connected components, if* $C > 0$, *then the orbit has one connected component. The light-cone* $\{\xi,\xi\} = 0$ *consists of two orbits: the set* $\{0\}$ *and the other points.*

Definition 7.4. The n-dimensional hyperbolic space H^n is the quotient of the orbit

$$\tilde{H}^n = \{\xi \in \mathbb{R}^{n+1} | \{\xi,\xi\} = -1\}$$

by the relation $(x,t) \sim -(x,t)$. So H^n is connected and homeomorphic to $\mathbb{R}^n$. Often we consider H^n to be

$$\tilde{H}^n_+ = \{(x,t) \in \tilde{H}^n | t > 0\}.$$

Recall that a tangent line to an algebraic hypersurface of order 2 in $\mathbb{R}^{n+1}$ is a line in the affine space $\mathbb{R}^{n+1}$ which is intersecting the surface in one point or contained in the surface. The direction vectors of the tangent line (considered as having starting point in the surface) are called tangent vectors.

Proposition 7.5. *At a point* $\xi \in \tilde{H}^n$ *some* $\eta \in \mathbb{R}^{n+1}$ *is a tangent vector to* $\tilde{H}^n$ *iff* $\{\xi,\eta\} = 0$.

Another definition for tangent vectors is the following.

Definition 7.6. The speed vectors of the curves going through a point in a hypersurface form a vector space which is called the tangent space of the surface at the point.

Proposition 7.7. *The previous statement holds with this definition of the tangent vectors too.*

Proposition 7.8. *The vector space* V_ξ *of the tangent vectors at a point* $\xi \in \tilde{H}^n$ *is a Euclidean space with the restriction of the Lorentz metric* $\{\cdot,\cdot\}$, *which is positive definite on* V_ξ.

Definition 7.9. The k-dimensional subspaces of H^n are the sets of the form

$$\tilde{H}^n \cap \Sigma$$

factored out by the equivalence relation $(x,t) \sim -(x,t)$, where Σ is a $(k+1)$-dimensional linear subspace of $\mathbb{R}^{n+1}$ intersecting $\tilde{H}^n$. The lines in the hyperbolic space H^n are the 1-dimensional subspaces.

Proposition 7.10. *Through every two points* $\xi,\eta \in \tilde{H}^n_+$ *there is a unique line. The distance between* ξ *and* η *is defined to be the length of this arc (as we defined the length of arcs which are space-like, in the integral we use* $\{\cdot,\cdot\}$*). Let* v *be the tangent vector at* ξ *to this arc, where* $\{v,v\} = 1$. *Then*

$$\gamma(s) = (\operatorname{ch} s)\xi + (\operatorname{sh} s)v$$

is a parametrization of this arc by arc-length, that is the distance between $\gamma(a)$ *and* $\gamma(b)$ *is equal to* $b - a$.

Corollary 7.11. The distance between $\gamma(0) = \xi$ and $\gamma(d) = \eta$ is equal to $d - 0 = d$. So

$$\eta = (chd)\xi + (\operatorname{sh} d)v$$

and by applying $\{\cdot,\xi\}$ for both sides we get

$$\{\xi,\eta\} = -chd.$$

Definition 7.12. Suppose two differentiable oriented arcs in $\tilde{H}^n_+$ intersect each other at $p \in \tilde{H}^n_+$. Then the angle between them is α for which $0 \le \alpha \le \pi$ and

$$\cos\alpha = \{v_1, v_2\},$$

where $v_{1,2}$ are the unit length tangent vectors to the two arcs (note that the tangent space at p is a Euclidean space with $\{\cdot,\cdot\}$ as scalar product).

Theorem 7.13. *For a triangle in the hyperbolic space we have*

$$\operatorname{ch} c = \operatorname{ch} b cha - \operatorname{sh} b \operatorname{sh} a \cos\gamma.$$

Corollary 7.14. For a triangle we have $a + b > c$.

Proposition 7.15. *For a triangle with an angle $\pi/2$ (opposite to its side c) we have*

(1) $\operatorname{ch} c = \operatorname{ch} a \operatorname{ch} b$,
(2) $\cos\alpha = \frac{\operatorname{th} b}{\operatorname{th} c} = \frac{\operatorname{sh} b}{\operatorname{ch} b}\frac{\operatorname{ch} c}{\operatorname{sh} c}$,
(3) $\sin\alpha = \frac{\operatorname{sh} a}{\operatorname{sh} c}$,
(4) $\cos\beta = \sin\alpha \operatorname{ch} b$.

Theorem 7.16. *Let $e \subset H^n$ be a line and $p \notin e$ be a point. Then there is a unique point $q \in e$ such that the distance bewteen p and q is minimal. Also there is a unique point $q' \in e$ such that the line pq' is perpendicular to e. Moreover we have*

$$q = q'.$$

Theorem 7.17. *In a triangle*

$$\sin\alpha : \sin\beta : \sin\gamma = \operatorname{sh} a : \operatorname{sh} b : \operatorname{sh} c.$$

Theorem 7.18. *In a triangle*

$$\cos\gamma = -\cos\alpha\cos\beta + \sin\alpha\sin\beta \operatorname{ch} c.$$

Theorem 7.19. *In a triangle $\alpha + \beta + \gamma < \pi$.*

Definition 7.20. The hyperbolic space $\tilde{H}^n$ has a model in $\mathbb{R}P^n$ called the Beltrami-Cayley-Klein model. For a point $\xi = (x, t) \in \tilde{H}^n$ we take the point $[\xi] \in \mathbb{R}P^n$ and its representative x/t in the affine hyperplane with last coordinate $= 1$.

Lemma 7.21. *In this way $\tilde{H}^n$ is mapped bijectively onto a usual open unit disk in $\mathbb{R}^n$. The k-dimensional hyperbolic subspaces correspond to k-dimensional projective subspaces intersecting this unit disk.*

Theorem 7.22. *If we have a hyperbolic line and two points x', y' on it and we have the corresponding two points x, y on the corresponding projective line e in the unit disk, then for the distance d between x' and y' we have*

$$d = \frac{|\ln(xyuv)|}{2},$$

where $(\cdots)$ denotes cross ratio and the points u and v are the intersections of the line e with the boundary circle of the unit disk in the order $uxyv$.

8. Topology

Definition 8.1. Let X be a set and $\Omega \subset \mathcal{P}(X)$ such that

(1) $\emptyset \in \Omega$ and $X \in \Omega$,
(2) if $A_\alpha \in \Omega$ for arbitrary indices α, then $\cup_\alpha A_\alpha \in \Omega$,
(3) if $A_i \in \Omega$ for finitely many indices $1 \leq i \leq n$, then $\cap_{1 \leq i \leq n} A_i \in \Omega$.

The elements of Ω are called the *open sets* of the *topological space* (X, Ω). A set $F \subset X$ is called *closed* if $X - F$ is open.

Of course instead of these conditions we have the corresponding conditions for closed sets too (finite union and arbitrary intersection instead of finite intersection and arbitrary union, etc).

Definition 8.2. A *neighborhood* of a point $x \in X$ is an open set U_x such that $x \in U_x$. (In another version of these definitions a neighborhood is just an arbitrary set which contains such an open set U_x. The provable interesting statements are the same.) If $A \subset X$, then the *interior* of A is

$$\mathrm{int} A = \cup\{B \in \Omega : B \subset A\},$$

that is the largest open set in A. An *interior point* of A is an $x \in X$ such that for some neighborhood U_x of x we have $U_x \subset A$. An *exterior point* of A is a point in X having a neighborhood disjoint from A. The *frontier* of A is the set

$$\mathrm{fr} A = \{x \in X : \text{for every neighborhood } U_x \text{ we have } U_x \cap A \neq \emptyset \text{ and } U_x \cap (X - A) \neq \emptyset\}.$$

The *closure* of a set A is the set

$$\overline{A} = \cap\{F \subset X : A \subset F \text{ and } F \text{ is closed}\},$$

that is the smallest closed set containing A. We denote the set of exterior points of A by $\mathrm{ext} A$.

Proposition 8.3. *Let X be a topological space.*

(1) *For every $A \subset X$ the set of interior points of A is equal to* $\mathrm{int} A$.
(2) *We have*
$$\exp A = \mathrm{int}(X - A).$$
(3) *For the space X we have*
$$X = \mathrm{int} A \sqcup \mathrm{fr} A \sqcup \mathrm{ext} A,$$
(4)
$$\overline{A} = A \cup \mathrm{fr} A,$$
(5)
$$\overline{A} = \mathrm{int} A \sqcup \mathrm{fr} A,$$
(6) $\overline{A}$ *consists of the points p such that every neighborhood U_p intersects A.*
(7) *If $A \subset X$, then the collection of all the sets of the form $A \cap U$, where $U \in \Omega$, gives a topology on A denoted by Ω_A and called the subspace topology. The space (A, Ω_A) is called a topological subspace of X.*

Definition 8.4. A *basis* of a topological space (X, Ω) is a collection Σ of open sets such that every $U \in \Omega$ is a union of some elements in Σ.

For example for a metric space (X,d) the collection of open balls with all the possible radii around all the possible centers in X form a basis, to prove this the only thing which is not obvious is that a finite intersection of unions of open balls is a union of open balls too.

Definition 8.5. The topology on X obtained in this way is called the *topology induced by the metric* d and it is denoted by Ω_d.

Proposition 8.6. *If (X,d) is a metric space and $A \subset X$, then we have*

$$(\Omega_d)_A = \Omega_{(d|_A)}.$$

Definition 8.7. A map $f\colon X \to Y$ between two topological spaces (X,Ω) and (Y,τ) is *continuous* if for every $U \in \tau$ we have $f^{-1}(U) \in \Omega$, that is the f-preimage of every open set is open. The map f is *continuous at a given point* $x \in X$ if for every neighborhood $V_{f(x)}$ of $f(x)$ there is a neighborhood U_x of x such that $f(U_x) \subset V_{f(x)}$.

Proposition 8.8. *For a map $f\colon X \to Y$ the following are equivalent.*

(1) *The map f is continuous.*
(2) *Every closed set in Y has closed f-preimage.*
(3) *Every basis open set in Y has open f-preimage.*
(4) *The map f is continuous at every point.*

Proposition 8.9. *The composition of two continuous maps is continuous.*

Definition 8.10. A sequence x_n *converges* to a point in a topological space X if for every neighborhood U_x of x there is an $N \in \mathbb{N}$ such that for $n > N$ we have $x_n \in U_x$. The map $f\colon X \to Y$ between two topological spaces X and Y is *sequentially continuous* if for every sequence x_n converging to some $x \in X$ the sequence $f(x_n)$ converges to $f(x)$.

Proposition 8.11 (no proof)**.** *If a map $f\colon X \to Y$ is continuous, then f is sequentially continuous too. In a metric space a sequentially continuous map is also continuous.*

Definition 8.12. A map $f\colon X \to Y$ between topological spaces is a *homeomorphism* if it is a continuous bijection and f^{-1} is also continuous. In other words a set $A \subset X$ is open iff $f(A)$ is open. If f is a homeomorphism, then X and Y are *homeomorphic.*

Clearly being homeomorphic is an equivalence relation.

Definition 8.13. A *path* between two points $p,q \in X$ is a continuous map $s\colon [0,1] \to X$ such that $s(0) = p$ and $s(1) = q$. The space X is *path-connected* if all points $p,q \in X$ can be connected by a path. The *concatenation* s_1s_2 of two paths $s_1\colon [0,1] \to X$ and $s_2\colon [0,1] \to X$ is the map defined by

$$s_1s_2(t) = \begin{cases} s_1(2t) & 0 \le t \le 1/2 \\ s_2(2t-1) & 1/2 \le t \le 1. \end{cases}$$

Proposition 8.14. *Suppose that $X = \cup_{i=1}^n F_i$, where all F_i are closed. Let $f\colon X \to Y$ be a map such that all restrictions $f|_{F_i}$ are continuous. Then f is continuous.*

Proposition 8.15. *Let $f\colon X \to Y$ be a continuous map. If X is path-connected, then the subspace $f(X)$ is path-connected.*

Definition 8.16. If for all points $p,q \in X$, where $p \ne q$, there exist disjoint neighborhoods U_p and U_q, then the space is called a *Hausdorff space.* Another name for these spaces is T_2 *space.*

Proposition 8.17. *A metric space is a Hausdorff space.*

Definition 8.18. The distance between a point x and a set $A \subset X$ in a metric space (X, d) is defined as

$$d(x, A) = \inf\{d(x, a) : a \in A\}.$$

Proposition 8.19.

(1) *We have $d(x, A) = 0$ iff $x \in \overline{A}$.*
(2) *The map $d(\cdot, A)\colon X \to [0, \infty)$ is continuous.*

So for non-empty disjoint closed subsets $A, B \subset X$ the map

$$f_{A,B} = \frac{d(\cdot, A)}{d(\cdot, A) + d(\cdot, B)}$$

is continuous and takes the values 0 and 1 on A and B, respectively.

Proposition 8.20. *A metric space is normal, that is for all closed subsets $A, B \subset X$, where $A \cap B = \emptyset$, there exist disjoint neighborhoods U_A and U_B of A and B, respectively.*

Definition 8.21.

(1) A topological space X is *compact* if for every open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X there is a finite subcovering $\{U_{\alpha_i}\}_{1\le i\le k}$.
(2) A space is a *Lindelöf space* if for every open covering $\{U_\alpha\}_{\alpha\in\mathcal{A}}$ of X there is an at most countable subcovering $\{U_{\alpha_i}\}_{i\in\mathbb{N}}$.
(3) A space is called M_2 if it has a countable collection of basis open sets.
(4) A space X is called *sequentially compact* if every sequence a_n in X has a convergent subsequence a_{n_k} with a limit point $x_0 \in X$, that is for every neighborhood U_{x_0} there is an N such that $k \ge N$ implies $a_{n_k} \in U_{x_0}$.

Proposition 8.22. *Let $f\colon X \to Y$ be a continuous map. If X is compact, then the subspace $f(X)$ is compact.*

Proposition 8.23.

(1) *A sequentially compact space X satisfies the Cantor property: if*

$$F_1 \supset F_2 \supset \cdots \supset \cdots$$

is a decreasing sequence of non-empty closed subsets in X, then $\cap_{i=1}^\infty F_i \ne \emptyset$.
(2) *Let X be a space which has the Cantor property. If there are countably many closed subsets of X such that the intersections of any finitely many sets among them are non-empty, then the intersection of all of these sets is also non-empty.*
(3) *If the space X has the Cantor property, then every countable open covering of X has a finite subcovering.*
(4) *An M_2 space is also Lindelöf.*
(5) *If a space satisfies the Cantor property and it is M_2, then it is compact.*
(6) *A sequentially compact metric space X is separable, that is, it has an at most countable subset whose closure is equal to X.*
(7) *A separable metric space is M_2.*

Definition 8.24. A subset $A \subset X$ of a topological space is *dense* if its closure is equal to X. This is equivalent to that A intersects every open subset of X as we can see easily.

So in fact we proved that a sequentially compact metric space X has a countable dense subset.

Theorem 8.25. *A sequentially compact metric space is compact.*

Definition 8.26. Let (X,d) be a metric space and $A \subset X$. Let $\varepsilon > 0$. The set A is an *ε-net* if the union of the open balls centered at the points of A and having radius ε covers X. The metric space (X,d) is *totally bounded* if for every $\varepsilon > 0$ it has an ε-net consisting of finitely many points.

Proposition 8.27. *A complete subspace of a metric space is closed. A closed subspace of a complete metric space is complete.*

Proposition 8.28 (no proof)**.** *If a metric space is compact, then it is complete.*

Theorem 8.29. *A metric space is compact iff it is complete and totally bounded. In a complete metric space X a subset A is compact iff A is totally bounded and closed.*

Later we will study the space of compact non-empty subsets of a metric space X and show that it is a metric space too.

9. Quaternions

A complex number is a 2×2 real matrix of the form

$$\begin{pmatrix} x & y \\ -y & x \end{pmatrix}$$

that we denote also by $x + iy$.

Definition 9.1. A quaternion is a 2×2 complex matrix of the form

$$\begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix}$$

and the set of quaternions is denoted by $\mathbb{H}$.

The quaternions $\mathbb{H}$ is a skew-field (non-commutative field), the multiplication is the matrix multiplication. If $q \in \mathbb{H}$, $q \neq 0$, then there exists a multiplicative inverse q^{-1} of q. We have $\det q = |\alpha|^2 + |\beta|^2 = 0$ iff $q = 0$. Then

$$q^{-1} = \frac{1}{\det q} \begin{pmatrix} \cdots & \\ \cdots & \end{pmatrix}$$

for some 2×2 complex matrix. We use the following notations.

$$1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

In terms of 4×4 real matrices we have

$$1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad i = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$j = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad k = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix}.$$

Then $1, i, j, k$ is a basis in $\mathbb{H}$ as in a vector space over $\mathbb{R}$. As a vector space over $\mathbb{R}$ the quaternions $\mathbb{H}$ is isomorphic to $\mathbb{R}^4$. If $q = x + iy + jz + kw$, then we have $x + iy = \alpha$ and $z + iw = \beta$. We have

$$|q|^2 = x^2 + y^2 + z^2 + w^2 = |\alpha|^2 + |\beta|^2 = \det q.$$

Then

$$|q_1 q_2| = |q_1||q_2|$$

so the unit length quaternions form a subgroup of the multiplicative group $\mathbb{H}^*$ of $\mathbb{H}$.

The real numbers $\mathbb{R}$ is a linear subspace of $\mathbb{H}$ by the inclusion

$$x \mapsto \begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}.$$

The orthogonal complement of this $\mathbb{R}$ is denoted by $\mathbb{R}^\perp$ and it is equal to $\mathbb{R}^3$, that is the space of purely imaginary quaternions generated by i, j, k. If $q = x + iy + jz + kw$, then $Re(q) = x$ is the real part of q and $Im(q) = iy + jz + kw$ is the imaginary part of q. Then

$$|q|^2 = x^2 - Im(q)^2$$

since $Im(q)^2 = -(y^2 + z^2 + w^2)$. We orient $\mathbb{R}^3$ so that i, j, k is the positive orientation.

Proposition 9.2. *If $a, b \in \mathbb{R}^3$, then for the multiplication of purely imaginary quaternions we have*

$$ab = -\langle a, b\rangle + a \times b.$$

The real part of ab is equal to $-\langle a, b\rangle$ and the imaginary part of ab is equal to $a \times b$.

Proposition 9.3. *Let $q \in \mathbb{H}$. Then $q \in \mathbb{R}$ iff for every $p \in \mathbb{H}$ the equation $qp = pq$ holds. So the center of $\mathbb{H}$ is $\mathbb{R}$.*

For a $q \in \mathbb{H}$, $q \neq 0$, we define the map

$$\varrho_q \colon \mathbb{H} \to \mathbb{H}$$

by

$$x \mapsto qxq^{-1}.$$

Then ϱ is a linear map over $\mathbb{R}$ and it is orthogonal:

$$|\varrho_q(x)| = |q||x||q^{-1}| = |x|$$

so $\varrho_q \in O(4)$.

Proposition 9.4. *We have* $\det \varrho_q = 1$ *so* $\varrho_q \in SO(4)$.

The restriction $\varrho_q|_{\mathbb{R}}$ is equal to the identity map $\mathrm{id}_{\mathbb{R}}$ so ϱ_q keeps fixed $\mathbb{R}^{\perp}$ too. We denote $\varrho_q|_{\mathbb{R}^3}$ by $\widehat{\varrho}_q$. The matrix of ϱ_q in the basis $1, i, j, k$ is equal to

$$\begin{pmatrix} 1 & 0 \quad 0 \quad 0 \\ 0 & \\ 0 & [\widehat{\varrho}_q]_{i,j,k} \\ 0 & \end{pmatrix}$$

which implies that $\widehat{\varrho}_q \in SO(3)$.

Theorem 9.5. *The map*

$$\widehat{\varrho}\colon \mathbb{H}^* \to SO(3)$$

$$q \mapsto \widehat{\varrho}_q$$

is a group homomorphism. Its kernel is isomorphic to the non-zero real numbers $\mathbb{R}^*$. The map $\widehat{\varrho}$ is surjective. So we have the group isomorphism

$$\mathbb{H}^*/\mathbb{R}^* \cong SO(3).$$

The restriction $\widehat{\varrho}$ to the group S^3 of quaternions of length 1 is also surjective. So we have

$$S^3/\{1, -1\} \cong SO(3).$$

It's easy to show that this implies that $SO(3)$ is homeomorphic to $\mathbb{R}P^3$.

Note that the unit sphere S^3 is equal to the set

$$\left\{ \begin{pmatrix} \alpha & \beta \\ -\overline{\beta} & \overline{\alpha} \end{pmatrix} \in \mathbb{C}^{2\times 2} : |\alpha|^2 + |\beta|^2 = 1 \right\}.$$

Recall that $U(n) = \{A \in \mathbb{C}^{n\times n} : AA^* = I\}$ and $SU(n)$ denotes the subgroup of $U(n)$ consisting of the matrices of determinant equal to 1.

Proposition 9.6. *We have $S^3 = SU(2)$.*

For $q_1, q_2 \in S^3$ we define the map $\Phi_{q_1,q_2}(x) = q_1 x q_2^{-1}$. Then we have

$$\Phi\colon S^3 \times S^3 \to O(4)$$

$$(q_1, q_2) \mapsto \Phi_{q_1,q_2}$$

since $|q_1 x q_2^{-1}| = |q_1||x||q_2^{-1}| = |x|$.

Proposition 9.7. *Φ maps into $SO(4)$.*

Proposition 9.8. *The map Φ is surjective and it is a group homomorphism between the groups $S^3 \times S^3$ and $SO(4)$. The kernel of Φ is the group $\{(-1,-1),(1,1)\}$ having two elements. We have the group isomorphism*

$$SO(4) \cong S^3 \times S^3/\{(-1,-1),(1,1)\}.$$

10. Compact subsets in a metric space and Hausdorff metric

For a metric space (X,d) and a subset $A \subset X$ and for an $\varepsilon > 0$ we have the notation
$$B(A,\varepsilon) = \{x \in X \mid \text{there is a } y \in A \text{ such that } d(x,y) < \varepsilon\}.$$
Sometimes we write $B^X(A,\varepsilon)$ instead of this to show the metric d. Of course $B(a,\varepsilon)$ for a point $a \in X$ is equal to the open ball around a with radius ε. Recall that in a metric space a compact subset is also closed and bounded.

Definition 10.1. Let $K_1, K_2 \subset X$ be compact subsets. Then there is an $\varepsilon > 0$ such that
$$K_1 \subset B(K_2,\varepsilon) \qquad \text{and} \qquad K_2 \subset B(K_1,\varepsilon).$$
The infimum of these ε is denoted by $d_H(K_1,K_2)$ and called the *Hausdorff distance* of K_1 and K_2.

Denote the collection of non-empty compact subsets of X by $\mathcal{K}$.

Proposition 10.2. *The map $d_H\colon \mathcal{K} \times \mathcal{K} \to [0,\infty)$ is a metric on the set $\mathcal{K}$.*

Theorem 10.3. *Suppose the space (X,d) is complete and suppose that in X all the closed balls are compact. Then the space $(\mathcal{K}, d_H)$ is complete.*

Theorem 10.4 (no proof)**.** *Suppose that (X,d) is compact. Then $(\mathcal{K}, d_H)$ is compact.*

Since now let X be the Euclidean space $\mathbb{R}^n$. Then $(\mathcal{K}, d_H)$ is complete. Denote by $\mathcal{C}$ the collection of convex sets in $\mathcal{K}$.

Proposition 10.5 (no proof)**.** *The space $\mathcal{C}$ is closed in $\mathcal{K}$.*

A simple corollary is that the metric space $(\mathcal{C}, d_H|_{\mathcal{C}})$ is complete.

Theorem 10.6. *Let $K_i \in \mathcal{K}$ be a sequence. Suppose that all K_i is in a given closed ball. Then there is a subsequence K_{n_k} which converges to some $K \in \mathcal{K}$. If $K_i \in \mathcal{C}$, then $K \in \mathcal{C}$ too.*

Proposition 10.7. *Let $A, B, C \in \mathcal{C}$. Suppose $C \subset \operatorname{int} B$ and $B \subset \operatorname{int} A$. Then the set*
$$\{K \in \mathcal{C} : C \subset K \subset A\}$$
contains an open neighborhood of B in the metric space $(\mathcal{C}, d_H|_{\mathcal{C}})$.

Recall that in an affine space a set is called a polytope if it is a convex hull of finitely many points. Let $\mathcal{P}$ denote the set of convex polytopes in $\mathbb{R}^n$.

Proposition 10.8. *The subset $\mathcal{P}$ of $\mathcal{C}$ is dense in $\mathcal{C}$.*

Proposition 10.9. *Let $\eta > 1$, $K \in \mathcal{C}$, $a \in \operatorname{int} K$ and let $\Phi_{a,\eta}$ be the homothety with ratio η and center a. Then there is a polytope $P \in \mathcal{P}$ such that $P \subset K \subset \Phi_{a,\eta}(P)$.*

With a more careful proof it is possible to get a polytope $P \in \mathcal{P}$ such that $P \subset \operatorname{int} K$ and $K \subset \operatorname{int} \Phi_{a,\eta}(P)$, for this in the proof of the previous Proposition 10.8 we need to reach $P \subset \operatorname{int} K$ instead of $P \subset K$.

Proposition 10.10. *The n-dimensional Lebesgue measure $\lambda_n\colon \mathcal{C} \to \mathbb{R}$ is continuous on the space $(\mathcal{C}, d_H|_{\mathcal{C}})$.*

10.1. **Blaschke theorem, isodiametric inequality and Minkowski theorem.** Let $K \subset \mathbb{R}^n$ be a compact subset and $H \subset \mathbb{R}^n$ be a hyperplane. Let $x \in H$ and denote by e_x the line perpendicular to H such that $e_x \cap H = x$. Let $I_x \subset e_x$ be a compact interval such that for the Lebesgue measure λ_1 we have $\lambda_1(I_x) = \lambda_1(e_x \cap K)$ and the middle point of I_x is equal to x. Let $I_x = \emptyset$ if $e_x \cap K = \emptyset$.

Definition 10.11. The set $\bigcup_{x \in H} I_x$ is called the *Steiner symmetrization* of K with respect to H and it is denoted by $St_H(K)$.

Proposition 10.12 (no proof)**.** *The Steiner symmetrization $St_H(K)$ of a compact set K is compact. If K is convex, then $St_H(K)$ is also convex. By Fubini theorem $\lambda_n(K) = \lambda_n(St_H(K))$.*

Proposition 10.13. *For the diameters of K and $St_H(K)$ we have*

$$\mathrm{diam} K \geq \mathrm{diam} St_H(K).$$

Theorem 10.14 (Blaschke)**.** *Suppose $\mathcal{A}$ is a non-empty closed subset of $\mathcal{K}$. Also suppose that there is a $p \in \mathbb{R}^n$ such that for every hyperplane containing p and for every $K \in \mathcal{A}$ we have $St_H(K) \in \mathcal{A}$. Then $\mathcal{A}$ contains a compact ball centered at p with some radius ≥ 0.*

Theorem 10.15 (Isodiametric inequality of Bieberbach)**.** *Let $\beta(n)$ denote the volume of the ball of unit radius in $\mathbb{R}^n$. If K is a compact subset of $\mathbb{R}^n$, then*

$$\lambda_n(K) \leq \beta(n) \left(\frac{\mathrm{diam} K}{2} \right)^n .$$

In other words among compact sets with a given diameter the ball has the largest volume.

In the following the symbols λ_n denote variables of polynomials so we denote the Lebesgue measure (volume) of a Lebesgue measurable set $A \subset \mathbb{R}^n$ by $V_n(A)$.

Theorem 10.16 (Minkowski)**.** *Let $K_1, \ldots, K_m \in \mathcal{C}$ and let $\lambda_1, \ldots, \lambda_m \geq 0$. Then the volume $V_n(\sum_{i=1}^m \lambda_i K_i)$ is an at most degree n homogeneous polynomial of the variables $\lambda_1, \ldots, \lambda_m$.*

Definition 10.17. Let $K_1, \ldots, K_m \in \mathcal{C}$ and let $\lambda_1, \ldots, \lambda_m \geq 0$. Then

$$V_n(\sum_{i=1}^m \lambda_i K_i) = \sum_{i_1=1}^m \sum_{i_2=1}^m \cdots \sum_{i_n=1}^m w_{i_1, i_2, \ldots, i_n} \lambda_{i_1} \cdots \lambda_{i_n},$$

where the coefficients $w_{i_1, i_2, \ldots, i_n}$ are uniquely determined if we suppose that they are symmetric:

$$w_{i_1, i_2, \ldots, i_n} = w_{i_{\sigma(1)}, i_{\sigma(2)}, \ldots, i_{\sigma(n)}}$$

for every permutation $\sigma \in \mathcal{S}_n$. The numbers

$$w_{i_1, i_2, \ldots, i_n}$$

are called the *mixed volumes* of the convex sets $K_1, \ldots, K_m$.

Corollary 10.18. For a subset $A \subset \mathbb{R}^n$ let

$$\mathbb{B}(A, \lambda) = \{x \in \mathbb{R}^n | \text{ there is a } y \in A \text{ such that } d(x, y) \leq \lambda\}.$$

Of course $\mathbb{B}(0, \lambda) \in \mathcal{C}$. If $K \in \mathcal{C}$ and $\lambda, \mu \geq 0$, then $V_n(\mathbb{B}(\mu K, \lambda)) = V_n(\mu K + \lambda \mathbb{B}(0, 1)) = v_n \mu^0 \lambda^n + \cdots + v_0 \mu^n \lambda^0$, where $v_1, \ldots, v_n \in \mathbb{R}$. For $\mu = 1, \lambda = 0$ we get $V_n(K) = v_0$ and for $\mu = 0, \lambda = 1$ we get $V_n(\mathbb{B}(0, 1)) = \beta(n) = v_n$.

Corollary 10.19. If $K \in \mathcal{C}$ and $\lambda \geq 0$, then
$$V_n(\mathbb{B}(K,\lambda)) = v_n\lambda^n + \cdots + v_1\lambda + v_0,$$
where $v_0 = V_n(K)$ and $v_n = V_n(\mathbb{B}(0,1))$.

10.2. **Isoperimetric inequality.** For a $K \in \mathcal{C}$ in this way we have the numbers $v_i(K) \in \mathbb{R}$, $i = 0, \ldots n$, so we have the functions
$$v_i \colon \mathcal{C} \to \mathbb{R}$$
for $i = 0, \ldots n$.

Lemma 10.20. *Let $K, L \in \mathcal{K}$. If we define*
$$d'_H(K,L) = \inf\{r > 0 | K \subset \mathbb{B}(L,r) \text{ and } L \subset \mathbb{B}(K,r)\},$$
then we have
$$d'_H(K,L) = d_H(K,L).$$

Lemma 10.21. *Let $\lambda \geq 0$. Then the map $\Phi \colon \mathcal{C} \to \mathcal{C}$, $K \mapsto \mathbb{B}(K,\lambda)$, is continuous with respect to the Hausdorff metric.*

Proof. Of course $\mathbb{B}(K,\lambda) = K + \lambda\mathbb{B}(0,1)$. Let $\eta > 0$ and $\lambda \geq 0$. Suppose that $L \subset \mathbb{B}(K,\eta)$. Then
$$\begin{aligned}\mathbb{B}(L,\lambda) \subset \mathbb{B}(\mathbb{B}(K,\eta),\lambda) = \mathbb{B}(K,\eta) + \lambda\mathbb{B}(0,1) = K + \eta\mathbb{B}(0,1) + \lambda\mathbb{B}(0,1) = \\ K + \lambda\mathbb{B}(0,1) + \eta\mathbb{B}(0,1) = \mathbb{B}(K,\lambda) + \eta\mathbb{B}(0,1) = \mathbb{B}(\mathbb{B}(K,\lambda),\eta).\end{aligned}$$
So if also $K \subset \mathbb{B}(L,\eta)$, then analogously
$$\mathbb{B}(K,\lambda) \subset \mathbb{B}(\mathbb{B}(L,\lambda),\eta).$$
This means that if we find the infimum of all $\eta > 0$ such that
$$L \subset \mathbb{B}(K,\eta) \qquad \text{and} \qquad K \subset \mathbb{B}(L,\eta),$$
then it is greater or equal than the infimum of all $\eta > 0$ such that
$$\mathbb{B}(L,\lambda) \subset \mathbb{B}(\mathbb{B}(K,\lambda),\eta) \qquad \text{and} \qquad \mathbb{B}(K,\lambda) \subset \mathbb{B}(\mathbb{B}(L,\lambda),\eta).$$
So
$$d_H(\mathbb{B}(K,\lambda),\mathbb{B}(L,\lambda)) \leq d_H(K,L).$$
This means that if an upper limit is given for the distance between $\mathbb{B}(L,\lambda)$ and $\mathbb{B}(K,\lambda)$, then by choosing $d_H(K,L)$ small enough, the distance $d_H(\mathbb{B}(L,\lambda),\mathbb{B}(K,\lambda))$ is also small enough. In other words, the map Φ is continuous. □

Proposition 10.22. *All functions $v_k \colon \mathcal{C} \to \mathbb{R}$, $k = 0, \ldots, n$, are continuous.*

Proof. For every $i \geq 0$ let $g_i \colon \mathcal{C} \to \mathbb{R}$ be the function defined by
$$g_i(K) = V_n(\mathbb{B}(K,i))$$
for $K \in \mathcal{C}$. Of course the function g_i is continuous because it is the composition of the continuous maps Φ (for $\lambda = i$) and V_n. We are able to write the values $v_j(K)$ as some algebraic expressions of all $g_i(K)$, this proves the statement. To see this let us consider the system of linear equations
$$x_0 = g_0(K)$$

$$x_0 + \cdots + x_n = g_1(K)$$
$$x_0 + 2x_1 + \cdots + 2^n x_n = g_2(K)$$
$$\vdots$$
$$x_0 + nx_1 + \cdots + n^n x_n = g_n(K)$$

with coefficient matrix

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 1 & \cdots & 1 \\ 1 & 2 & 2^2 & \cdots & 2^n \\ & & \vdots & & \\ 1 & n & n^2 & \cdots & n^n \end{pmatrix}$$

whose determinant is non-zero (it is a Vandermonde matrix). So we can write the unique solutions

$$x_0, \ldots, x_n$$

as some algebraic expressions of the numbers $g_0(K), \ldots, g_n(K)$ by Cramer's rule. Hence $x_0, \ldots, x_n$ depend continuously on $K \in \mathcal{C}$. On the other hand, if we consider the polynomial

$$V_n(\mathbb{B}(K, \lambda))$$

at $\lambda = i$, then we get

$$V_n(\mathbb{B}(K, i)) = v_0 + v_1 i + \cdots + v_n i^n$$

but since

$$g_i(K) = V_n(\mathbb{B}(K, i))$$

and the solution to the system of linear equations is unique, we get that

$$x_k = v_k$$

for $k = 1, \ldots, n$. So every v_k is a continuous function on $\mathcal{C}$. □

Proposition 10.23. *The Steiner symmetrization does not increase v_1. So if $K \in \mathcal{C}$, then $v_1(St_H(K)) \leq v_1(K)$ for all hyperplanes H.*

Proof. Note that $St_H(K)$ is also compact and convex (we did not prove these). Let $\varepsilon > 0$ and let H be an arbitrary hyperplane. By using Lemma 10.24 we have

$$\mathbb{B}(St_H(K), \varepsilon) = St_H(K) + \varepsilon\mathbb{B}(0,1) = St_H(K) + \varepsilon St_H(\mathbb{B}(0,1)) \subset St_H(K + \varepsilon\mathbb{B}(0,1)) = St_H(\mathbb{B}(K, \varepsilon)).$$

Since the volume V_n is monotone (larger set has larger volume), we have that

$$V_n(\mathbb{B}(St_H(K), \varepsilon)) \leq V_n(St_H(\mathbb{B}(K, \varepsilon))).$$

Also $V_n(St_H(\mathbb{B}(K, \varepsilon))) = V_n(\mathbb{B}(K, \varepsilon))$ because Steiner symmetrization keeps the volume (Lebesgue measure). We have that

$$V_n(\mathbb{B}(St_H(K), \varepsilon)) = V_n(St_H(K)) + \varepsilon v_1(St_H(K)) + \varepsilon^2 N_\varepsilon$$

and

$$V_n(\mathbb{B}(K, \varepsilon)) = V_n(K) + \varepsilon v_1(K) + \varepsilon^2 M_\varepsilon,$$

where N_ε and M_ε are bounded as $\varepsilon \to 0$.

Of course $V_n(St_H(K)) = V_n(K)$ so the inequality
$$V_n(\mathbb{B}(St_H(K),\varepsilon)) \le V_n(\mathbb{B}(K,\varepsilon))$$
yields
$$\varepsilon v_1(St_H(K)) + \varepsilon^2 N_\varepsilon \le \varepsilon v_1(K) + \varepsilon^2 M_\varepsilon$$
and by dividing by ε and taking the limit $\varepsilon \to 0$ finally we get
$$v_1(St_H(K)) \le v_1(K).$$
□

Lemma 10.24. *Let $K, L \in \mathcal{C}$ and H be a hyperplane. Then*

(1) *for every $0 \le \mu \le 1$ we have*
$$\mu St_H(K) + (1-\mu) St_H(L) \subset St_H(\mu K + (1-\mu)L)$$
and

(2) *for every $\mu, \lambda > 0$ we have*
$$\mu St_H(K) + \lambda St_H(L) \subset St_H(\mu K + \lambda L).$$

Proof. The statement follows from an easy geometrical argument. □

Theorem 10.25 (Isoperimetric inequality)**.** *Let $K \in \mathcal{C}$ and denote by $\beta(n)$ the volume of $\mathbb{B}(0,1)$. Then we have*
$$\frac{v_1(K)}{n\beta(n)} \ge \left(\frac{V_n(K)}{\beta(n)}\right)^{\frac{n-1}{n}}.$$

Proof. Take the set
$$\mathcal{A} = \{L \in \mathcal{C} \mid V_n(K) = V_n(L) \text{ and } v_1(L) \le v_1(K)\}.$$
Then $\mathcal{A} \ne \emptyset$ because $K \in \mathcal{A}$. Also $\mathcal{A}$ is a closed subset of the metric space $(\mathcal{K}, d_H)$ because the limit set in $\mathcal{K}$ of a convergent sequence in $\mathcal{A}$ is in $\mathcal{C}$ because $\mathcal{C}$ is a closed subspace of $\mathcal{K}$ and the conditions defining the set $\mathcal{A}$ hold also for the limit set because the functions V_n and v_1 are continuous (hence sequentially continuous too) on $\mathcal{C}$.

Finally, the set $\mathcal{A}$ is closed under Steiner symmetrization by the proved statement that v_1 is not increased then and by the unproved statement that the Steiner symmetrization of a convex set is convex.

Hence we can apply Blaschke theorem and we get that $\mathcal{A}$ contains a closed ball $\mathbb{B}(a,r)$ of radius $r \ge 0$ and with a center $a \in \mathbb{R}^n$.

Then we have
$$\frac{v_1(K)}{n\beta(n)} \ge \frac{v_1(\mathbb{B}(a,r))}{n\beta(n)}$$
because of the condition about v_1 of the sets in $\mathcal{A}$. We know that in general
$$\frac{V_n(\mathbb{B}(K,\varepsilon)) - V_n(K)}{\varepsilon} = v_1(K) + \varepsilon N_\varepsilon$$
with bounded N_ε as $\varepsilon \to 0$. For $K = \mathbb{B}(a,r)$
$$V_n(\mathbb{B}(\mathbb{B}(a,r),\varepsilon)) = V_n(\mathbb{B}(a, r+\varepsilon)) = \beta(n)(r+\varepsilon)^n$$

so

$$\frac{V_n(\mathbb{B}(\mathbb{B}(a,r),\varepsilon)) - V_n(\mathbb{B}(a,r))}{\varepsilon} = \frac{\beta(n)(r+\varepsilon)^n - \beta(n)r^n}{\varepsilon} = \beta(n)nr^{n-1} + \varepsilon M_\varepsilon$$

with bounded M_ε as $\varepsilon \to 0$ which all together by $\varepsilon \to 0$ imply that

$$v_1(\mathbb{B}(a,r)) = \beta(n)nr^{n-1}.$$

So

$$\frac{v_1(\mathbb{B}(a,r))}{n\beta(n)} = \frac{\beta(n)nr^{n-1}}{n\beta(n)} = r^{n-1} = \left(\frac{\beta(n)r^n}{\beta(n)}\right)^{\frac{n-1}{n}} = \left(\frac{V_n(K)}{\beta(n)}\right)^{\frac{n-1}{n}}$$

where the last equation follows from $\beta(n)r^n = V_n(\mathbb{B}(a,r)) = V_n(K)$ as the conditions for elements in $\mathcal{A}$ require. Hence we obtain

$$\frac{v_1(K)}{n\beta(n)} \geq \left(\frac{V_n(K)}{\beta(n)}\right)^{\frac{n-1}{n}}.$$

$\square$