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POISSON POLYTOPES

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We prove the central limit theorem for the volume and the f-vector of the Poisson random polytope Π_η in a fixed convex polytope $P \subset \mathbb{R}^d$. Here Π_η is the convex hull of the intersection of a Poisson process X of intensity η with P.

1. Introduction and main results. Let $K \subset \mathbb{R}^d$ be a convex set of volume one. Assume $X = X(\eta)$ is a Poisson point process in $\mathbb{R}^d$ of intensity η. The intersection of K with $X(\eta)$ consists of uniformly distributed random points $X_1, \ldots, X_N$ (where N is a random variable). Define the *Poisson polytope* Π_η, as the convex hull $[X_1, \ldots, X_N] = [K \cap X(\eta)]$.

Studying properties of random convex hulls is a classical subject in stochastic geometry and dates back until 1864. Due to the geometric nature of the available methods, for more then one hundred years investigations mainly concentrated on the expectation of functionals of random convex hulls such as volume or number of vertices, see e.g. the survey of Weil and Wieacker [24].

First distributional results were only proved twenty years ago. In 1988 Groeneboom [14] obtained the central limit theorem, CLT for short, for the number of vertices of the Poisson polytope, when the convex body K is the planar disc. And in 1994 a CLT for the area of a random polygon in the planar disc was proved by Hsing [16]. Recently this was generalized to arbitrary dimensions by Reitzner [19], who established a CLT for $V(\Pi_\eta)$, the volume of the Poisson polytope, and for $f_\ell(\Pi_\eta)$, the number of ℓ-dimensional faces of the Poisson polytope, when the body $K \subset \mathbb{R}^d$ has smooth boundary.

The situation seems to be much more involved when the underlying convex set is a polytope P. In the planar case, when P is a convex polygon, a CLT for the number of vertices $f_0(\Pi_\eta)$ was proved by Groeneboom [14], and a CLT for the area of Π_η by Cabo and Groeneboom [12], but it seems that the stated variances are incorrect (see the discussion in Buchta [11]).

The main result of the present paper is the central limit theorem for the

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Poisson polytope Π_η for all dimensions $d \geq 2$, when the mother body is a polytope in $\mathbb{R}^d$.

Theorem 1.1 *There is function $\varepsilon(\eta)$, tending to zero as $\eta \to \infty$, such that for every polytope $P \subset \mathbb{R}^d$, of volume one,*

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{V(\Pi_\eta) - \mathbb{E}V(\Pi_\eta)}{\sqrt{\operatorname{Var} V(\Pi_\eta)}} \leq x \right) - \Phi(x) \right| \leq c(P)\varepsilon(\eta),$$

and for all $\ell = 0, \ldots, d-1$

$$\sup_{x \in \mathbb{R}} \left| \mathbb{P}\left(\frac{f_\ell(\Pi_\eta) - \mathbb{E}f_\ell(\Pi_\eta)}{\sqrt{\operatorname{Var} f_\ell(\Pi_\eta)}} \leq x \right) - \Phi(x) \right| \leq c(P)\varepsilon(\eta),$$

where $c(P)$ is a constant depending only on P.

Remark. It will turn out that the error term in Theorem 1.1 is

$$\varepsilon(\eta) = (\ln \eta)^{-\frac{d-1}{2} + o(1)}.$$

The constant $c(P)$ depends on the dimension and a power of $F(P)$, the number of flags of the polytope P. A *flag* is a sequence of faces $F_0, F_1, \ldots, F_{d-1}$ of P such that, for all i, $\dim F_i = i$ and $F_i \subset F_{i+1}$.

The Poisson polytope Π_η is closely related to the random polytope P_n defined in the following way: Fix $n \in \mathbb{N}$ and choose n random points $X_1, \ldots, X_n$ independently and uniformly from K. The *random polytope* P_n is just the convex hull of these points: $P_n = [X_1, \ldots, X_n]$. Clearly, P_n equals in distribution the Poisson polytope Π_η given that the (Poisson distributed) number of points of $X \cap K$ is precisely n.

Starting with Rényi and Sulanke [17] in 1963, there have been many results concerning various properties of P_n as $n \to \infty$. For instance, the asymptotic behaviour of the expectation of the volume $V(P_n)$, and of the number, $f_\ell(P_n)$, of ℓ-dimensional faces of P_n ($\ell = 0, \ldots, d-1$) have been determined as $n \to \infty$, see [24] for an extensive survey, and also [4], [5], and [18] for more recent results. These results on P_n imply immediately analogous results for the Poisson polytope Π_η. For the sake of completeness we state here the results concerning the expected volume and number of faces.

Theorem 1.2 *Assume P is a polytope of volume one. Then*

$$1 - \mathbb{E}V(\Pi_\eta) = \frac{F(P)}{(d+1)^{d-1}(d-1)!} \eta^{-1} \ln^{d-1} \eta \, (1 + o(1)),$$

$$\mathbb{E} f_\ell(\Pi_\eta) = c(d,\ell) F(P) \ln^{d-1}\eta\,(1+o(1)),$$

where $c(d,\ell) > 0$ *is a constant depending on* d *and* ℓ*.*

Somewhat surprisingly, the value of these expectations is not needed for the proof of our main theorems.

The proof of Theorem 1.1 is not simple. It uses a combination of ideas from probability theory and convex geometry. Section 3 contains a short sketch of this proof. First we have to introduce some notation and background, and more importantly, the economic cap covering theorem that will be used repeatedly. This is the content of the next section.

2. Notation and background. The unit sphere is S^{d-1}. As usual, $h_K(u)$ denotes the support function of K in direction $u \in S^{d-1}$:

$$h_K(u) = \max\{u \cdot x : \; x \in K\}.$$

A cap C of K is the intersection of K with a closed halfspace. This halfspace can be written as $\{x \in \mathbb{R}^d \,|\, u \cdot x \geq h_K(u) - t\}$ with $u \in S^{d-1}$. Thus

$$C = K \cap \{x \in \mathbb{R}^d \,|\, u \cdot x \geq h_K(u) - t\}.$$

The *bounding hyperplane* of C is the one with equation $u \cdot x = h_K(u) - t$. We define, for $\lambda > 0$, C^λ by

$$C^\lambda = K \cap \{x \in \mathbb{R}^d \,|\, u \cdot x \geq h_K(u) - \lambda t\}.$$

An important role throughout plays the function $v : \; K \to \mathbb{R}$ defined as

$$v(z) = \min\{V(K \cap H) : \; H \text{ is a halfspace and } z \in H\}.$$

The *floating body* with parameter t is just the level set $K(v \geq t) = \{z \in K : \; v(z) \geq t\}$, which is clearly convex. The *wet part* is $K(v \leq t)$, that is, where v is at most t. The name comes from the 3-dimensional picture when K is a container containing t units of water.

The *minimal cap* of $z \in K$ is a cap $C(z) = C_K(z)$ containing z such that $v(z) = V(C(z))$. It need not be unique. The *centre* of the cap $C = K \cap \{x \in \mathbb{R}^d : \; u \cdot x \geq h_K(u) - t\}$ is a point $x \in \partial K$ with $u \cdot x = h_K(u)$. The centre, again, need not be unique, but this will cause no harm. Assuming that x is the centre of C, observe that for $\lambda \geq 1$,

$$C^\lambda \subset x + \lambda(C - x)$$

and thus $V(C^\lambda) \leq \lambda^d V(C)$ always holds. Also, $\frac{\lambda}{d} V(C) \leq V(C^\lambda)$ holds as long as λt is smaller than the width of K in direction u. The proof is simple:

let L be the section that has maximal $(d-1)$-dimensional volume among all sections of the form

$$K \cap \{x \in \mathbb{R}^d : \ u \cdot x \geq h_K(u) - \tau\} \text{ when } \tau \in [0, t].$$

Then $V(C) \leq tV_{d-1}(L)$. Here V_{d-1} stands for $(d-1)$-dimensional volume. On the other hand, the double cone with base L and apices x and a point in $K \cap \{x \in \mathbb{R}^d : \ u \cdot x = h_K(u) - \lambda t\}$ is contained in C^λ and its volume is at least $\frac{\lambda t}{d} V_{d-1}(L)$. So $\frac{\lambda}{d} V(C) \leq \frac{\lambda t}{d} V_{d-1}(L) \leq V(C^\lambda)$ which is the inequality we wanted to prove.

Analogously for $0 < \mu < 1$ we have $\mu^d V(C) \leq V(C^\mu) \leq d\mu V(C)$. For the proof define $D = C^\mu$ and $\lambda = 1/\mu > 1$. Then D is a cap of K and $D^\lambda = C$. The inequalities $\frac{1}{d}\lambda V(D) \leq V(D^\lambda) \leq \lambda^d V(D)$ translate directly to $\mu^d V(C) \leq V(C^\mu) \leq d\mu V(C)$. These inequalities will be used often. We call them the *trivial volume estimates*:

$$\frac{\lambda}{d} V(C) \leq V(C^\lambda) \leq \lambda^d V(C) \quad \text{for} \quad \lambda \geq 1,$$
$$\mu^d V(C) \leq V(C^\mu) \leq d\mu V(C) \quad \text{for} \quad 0 \leq \mu \leq 1$$

where the left hand side of the first inequality only holds for $C^\lambda \neq K$ and the right hand side of the second inequality only for $C \neq K$. The *Macbeath region, or M-region,* for short, with centre z and factor $\lambda > 0$ is

$$M(z, \lambda) = M_K(z, \lambda) = z + \lambda[(K - z) \cap (z - K)].$$

The M-region with $\lambda = 1$ is just the intersection of K and K reflected with respect to z. Thus $M(z, 1)$ is convex and centrally symmetric with centre z, and $M(z, \lambda)$ is a homothetic copy of $M(z, 1)$ with centre z and factor of homothety λ. We define the function $u : \ K \to \mathbb{R}$ by

$$u(z) = V(M(z, 1)).$$

These definitions are from [13], [8], and [3]. The following results come from the same sources. We will use them extensively. We assume $K \subset \mathbb{R}^d$ is a convex body of volume one. Set

$$s_0 = (2d)^{-2d}. \tag{2.1}$$

Lemma 2.1 *If $M(x, \frac{1}{2}) \cap M(y, \frac{1}{2}) \neq \emptyset$, then $M(x, 1) \subset M(y, 5)$.*

Lemma 2.2 *If C is a cap and $z \in C$ and $\lambda > 0$, then $K \cap M(z, \lambda) \subset C^{\lambda+1}$.*

Lemma 2.3 *If the cap C is contained in the M-region $M(z, \mu)$, and $\lambda > 0$, then $C^\lambda \subset M(z, \lambda\mu)$.*

Lemma 2.4 *If the bounding hyperplane of a cap C is tangent to $K(v \geq s)$, then $s \leq V(C) \leq ds$.*

Let $K(v = s) = \partial K(v \geq s)$. Assume $s \leq s_0$ and choose a maximal system of points $Z = \{z_1, \ldots, z_m\}$ on $K(v = s)$ having pairwise disjoint Macbeath regions $M(z_i, \frac{1}{2})$. Such a system will be called *saturated*. Note that Z (and even m) is not defined uniquely. However, for each K and s we fix a saturated system Z. We write $Z(s)$ and $m(s) = |Z(s)|$ when we want to emphasize that our fixed saturated system comes from the level set $K(v = s)$. Evidently, $V(C(z_i)) = s$. Set

$$K_i'(s) = M(z_i, \frac{1}{2}) \cap C(z_i) \text{ and } K_i(s) = C^6(z_i).$$

Note that $K_i(s)$ is a cap of K and so for $\lambda > 0$ the set $K_i^\lambda(s) = C^{6\lambda}(z_i)$ is another cap of K.

The sets $K_i'(s)$ and $K_i(s)$ for $i = 1, \ldots, m(s)$ form what is called an *economic cap covering* in the paper of Bárány and Larman [8]. The following result, the *economic cap covering theorem*, comes from Theorem 6 in [8] and Theorem 7 in [3].

Theorem 2.5 *For all $s \in (0, s_0]$ and for all convex bodies $K \subset \mathbb{R}^d$ with $V(K) = 1$ we have*

(i) $\bigcup_1^{m(s)} K_i'(s) \subset K(v \leq s) \subset \bigcup_1^{m(s)} K_i(s)$,
(ii) $s \leq V(K_i(s)) \leq 6^d s$, $i = 1, \ldots, m(s)$,
(iii) $(6d)^{-d} s \leq V(K_i'(s)) \leq 2^{-d} s$, $i = 1, \ldots, m(s)$,
(iv) *every C with $V(C) \leq s$ is contained in $M(z_i, 15d) \subset K_i^{3d}(s)$ for some i.*

The sets $K_i'(s)$ are pairwise disjoint, all of them have volume $\geq (6d)^{-d} s$, and are all contained in $K(v \leq s)$. This gives an upper bound for $m(s)$. Similarly, the sets $K_i(s)$ cover $K(v \leq s)$, all of them have volume $\leq 6^d s$. This gives a lower bound for $m(s)$. These simple arguments will be used repeatedly, and we call them the *usual volume arguments*. Summarizing, we have

$$\frac{1}{6^d s} V(K(v \leq s)) \leq m(s) \leq \frac{1}{(6d)^{-d} s} V(K(v \leq s)) \tag{2.2}$$

for $s \leq s_0$.

The economic cap covering theorem has the following direct consequence.

Claim 2.6 *For $s \leq s_0$ and $\lambda > 1$*

$$K(v \leq \lambda s) \subset \bigcup K_i^{3d^2\lambda}(s).$$

Proof. It is clear that $K(v \leq \lambda s)$ is contained in the union of all caps C with $V(C) = \lambda s$. Let C be a cap with $V(C) = \lambda s$. The trivial volume estimates show that the cap $C^{1/(d\lambda)}$ has volume at most s and thus is contained in some set $M(z_i, 15d)$. Then by Lemma 2.3, C is contained in $M(z_i, 15d^2\lambda)$ which is, by Lemma 2.2, a subset of $C^{15d^2\lambda+1}(z_i) \subset (C^6(z_i))^{3d^2\lambda} = K_i^{3d^2\lambda}(s)$. □

When P is a polytope of volume $V(P)$, the volume of the wet part $P(v \leq s)$ was determined by Schütt [21], and by Bárány and Buchta [6]. As $s \to 0$

$$\frac{V\left(P(v \leq sV(P))\right)}{V(P)} = \frac{F(P)}{d!d^{d-1}} s \ln^{d-1}\left(\frac{1}{s}\right) (1 + o(1)).$$

Later we need an estimate for $m(s)$ and $V(P(v \leq s))$, depending on P only via $F(P)$. Such an estimate follows from results in Bárány [3], see also [4], formula (4).

Theorem 2.7 *If $P \subset \mathbb{R}^d$ is a polytope with $V(P) > 0$, then*

$$\underline{c}(d) s \ln^{d-1}\left(\frac{1}{s}\right) \leq \frac{V\left(P(v \leq sV(P))\right)}{V(P)} \leq \overline{c}(d) F(P) s \ln^{d-1}\left(\frac{1}{s}\right)$$

and

$$\underline{c}(d) \ln^{d-1}\left(\frac{1}{s}\right) \leq m(sV(P)) \leq \overline{c}(d) F(P) \ln^{d-1}\left(\frac{1}{s}\right).$$

for $s \leq s_0$, where $\underline{c}(d), \overline{c}(d) > 0$ are constants depending on d.

The second estimate concerning the number of caps, $m(s)$, follows from (2.2).

3. Plan of proof.

This section explains the basic steps of the proof of Theorem 1.1.

Step 1. Our proof relies on a precise description of the boundary of a convex polytope. The essential ingredients are good bounds on how many sets $K_i'(s)$ meet a given cap C of P, and on the size of the set visible from z within $P(v \leq T)$. This is done in Section 4.

Step 2. In what follows α, β are positive constants, to be specified later, that depend only on dimension. Also, we use $\operatorname{lln} x$ as a shorthand for $\ln(\ln x)$. Define

$$T = T_\eta = \frac{\alpha \operatorname{lln} \eta}{\eta} \quad \text{and} \quad s = s_\eta = \frac{1}{\eta \ln^\beta \eta}. \tag{3.1}$$

We wish to show that, with high probability, Π_η is sandwiched between $P(v \geq T)$ and $P(v \geq s)$, that is,

$$P(v \geq T) \subset \Pi_\eta \subset P(v \geq s).$$

(For technical reasons we will have to replace T by the slightly larger $T^* = d6^dT$.) A convenient way to do so is to define a certain event A, which implies sandwiching, and whose complement, $\overline{A}$, has very small probability, namely, $\mathbb{P}(\overline{A}) \ll F(P) \ln^{-4d^2} \eta$. This will be achieved in Section 5.

The basic tool for proving our main result is a central limit theorem with weakly dependent random variables. Such an approach has already been used in geometric probability by Avram and Bertsimas [1] who also suggested its use in the study of random convex hulls. For the CLT we are going to use the weak dependence of random variables is given by the so-called *dependency graph* which is defined as follows: Let ζ_i, $i \in \mathcal{V}$, be a finite collection of random variables. The graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is said to be a dependency graph for ζ_i if for any pair of disjoint sets $\mathcal{W}_1, \mathcal{W}_2 \subset \mathcal{V}$ such that no edge in $\mathcal{E}$ goes between $\mathcal{W}_1$ and $\mathcal{W}_2$, the sets of random variables $\{\zeta_i : i \in \mathcal{W}_1\}$ and $\{\zeta_i : i \in \mathcal{W}_2\}$ are independent. The following central limit theorem with weak dependence is due to Rinott [20]. A slightly weaker version (that would also do here) had been proved earlier by Baldi and Rinott [2].

Theorem 3.1 (Rinott) *Let ζ_i, $i \in \mathcal{V}$, be random variables having a dependency graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. Set $\zeta = \sum_{i \in \mathcal{V}} \zeta_i$ and $\sigma^2(\zeta) = \operatorname{Var} \zeta$. Denote the maximal degree of $\mathcal{G}$ by D and suppose that $|\zeta_i - \mathbb{E}\zeta_i| \leq M$ almost surely. Then, for every x*

$$\left| \mathbb{P}\left(\frac{\zeta - \mathbb{E}\zeta}{\sqrt{\operatorname{Var} \zeta}} \leq x \right) - \Phi(x) \right| \leq \frac{1}{\sqrt{2\pi}} \frac{DM}{\sigma(\zeta)} + 16 \, \frac{|\mathcal{V}|^{\frac{1}{2}} D^{\frac{3}{2}} M^2}{\sigma^2(\zeta)} + 10 \, \frac{|\mathcal{V}| \, D^2 M^3}{\sigma^3(\zeta)}.$$

When using this theorem one has to define the dependency graph and prove the necessary properties. Also, we need a lower bound on $\operatorname{Var} \zeta$ (see Theorem 3.3 below) which comes from the companion paper [9].

Step 3. Define a graph $\mathcal{G}$ whose vertex set $\mathcal{V}$ is $\{1, 2, \dots, m(T)\}$ where $m(T)$ is the size of the fixed saturated system of points on $P(v = T)$ as explained just before the cap covering theorem. The corresponding cap covering $K_1(T), \dots, K_{m(T)}(T)$ is indexed by the vertices of $\mathcal{G}$. Two vertices $i, j \in \mathcal{V}$ form an edge of $\mathcal{G}$ if the caps $K_i(T)$ and $K_j(T)$ are "close to each other", in a well-defined sense. This definition is crucial, and will be explained in Sections 6 and 7. Also, it will be shown that the maximal degree of $\mathcal{G}$ is $\ll F(P)^6 \operatorname{lln}^{6(d-1)} \eta$.

Step 4. Assume that the event A holds which, as mentioned above, implies 'sandwiching'. Define the random variables ζ_i, $i \in \mathcal{V}$ and check that $\mathcal{G}$ is indeed a dependency graph. The cases of $\zeta = V(\Pi_\eta)$ and $\zeta = f_\ell(\Pi_\eta)$ have to be handled somewhat differently. Next we check that the conditions of Rinott's theorem hold. This will be done in Section 7. This proves the CLT for ζ given A.

Step 5. Remove the conditioning on A. This is simpler for $\zeta = V(\Pi_\eta)$, as it is bounded, while $\zeta = f_\ell(\Pi_\eta)$ is not. Section 8 is devoted to this task. The CLT for ζ follows from the CLT for $\zeta|A$ via the following transference lemma from [10], which has been used in an implicit form in [19] and in [23], and perhaps elsewhere as well.

Lemma 3.2 *Let ξ_η and ξ'_η be two series of random variables with means μ_η and μ'_η, variances σ^2_η and ${\sigma'}^2_\eta$, respectively. Assume that there are functions $\varepsilon_1(\eta), \varepsilon_2(\eta), \varepsilon_3(\eta), \varepsilon_4(\eta)$, all tending to zero as η tends to infinity such that*

(i) $|\mu'_\eta - \mu_\eta| \le \varepsilon_1(\eta)\sigma_\eta$.
(ii) $|{\sigma'}^2_\eta - \sigma^2_\eta| \le \varepsilon_2(\eta)\sigma^2_\eta$.
(iii) For every x, $|\mathbb{P}(\xi'_\eta \le x) - \mathbb{P}(\xi_\eta \le x)| \le \varepsilon_3(\eta)$.
(iv) For every x,

$$\left|\mathbb{P}\Big(\frac{\xi'_\eta - \mu'_\eta}{\sigma'_\eta} \le x\Big) - \Phi(x)\right| \le \varepsilon_4(\eta).$$

Then there is a positive constant c such that for every x,

$$\left|\mathbb{P}\Big(\frac{\xi_\eta - \mu_\eta}{\sigma_\eta} \le x\Big) - \Phi(x)\right| \le c\sum_{i=1}^{4} \varepsilon_i(\eta).$$

The transference lemma asserts that if ξ'_η satisfies the CLT (the fourth condition) and ξ_η is sufficiently close to ξ'_η in distribution (the first three conditions), then ξ_η also satisfies the CLT.

Remark. In [10] the transference lemma is stated with σ'_η and ${\sigma'}^2_\eta$ on the right hand side of conditions (i) and (ii). It is easy to see that the present conditions imply the ones with σ'_η: (ii) shows that ${\sigma'}^2_\eta/\sigma^2_\eta$ tends to 1 as $n \to \infty$. Thus $\sigma^2_\eta < 2{\sigma'}^2_\eta$ for large enough n. Then (ii) implies $|{\sigma'}^2_\eta - \sigma^2_\eta| \le 2\varepsilon_2(\eta){\sigma'}^2_\eta$, and similarly, (i) implies $|\mu'_\eta - \mu_\eta| \le \sqrt{2}\varepsilon_1(\eta)\sigma'_\eta$.

To apply the central limit theorem and the transference lemma we need a lower bound on $\operatorname{Var}\zeta$. In the companion paper [9] we prove a lower bound for general convex bodies in terms of the volume of the floating body: Theorem 3.1 in [9] says that the variance of $V(\Pi_\eta)$ is bounded from below by

$\eta^{-1}V(K(v \le \eta^{-1}))$, and $\operatorname{Var} f_\ell(\Pi_\eta)$ is bounded by $\eta V(K(v \le \eta^{-1}))$. Using Theorem 2.7 this gives the following.

Theorem 3.3 *Assume P is a polytope of volume one. Then*

$$F(P)\eta^{-2}\ln^{d-1}\eta \ll \operatorname{Var} V(\Pi_\eta),$$

$$F(P)\ln^{d-1}\eta \ll \operatorname{Var} f_\ell(\Pi_\eta).$$

Here we use Vinogradov's $\gg$ notation, that is, we write $f(\eta) \gg g(\eta)$ if there is a constant $c > 0$, independent of η, such that $cf(\eta) > |g(\eta)|$ for all $\eta \ge \eta_0$. The constants c and η_0 may, and usually do, depend on the dimension, but not on K.

The main achievements of this paper, besides the central limit theorems, are the precise sandwiching of Π_η, the novel definition of the dependency graph, and the proof that its maximal degree is bounded by a power of $\operatorname{lln}\eta$. The latter is based on structural properties of the wet part $P(v \le t)$ for polytopes.

4. On the boundary structure of convex polytopes. In this section we state some facts about the boundary structure of the polytope P and its floating body. All proofs of this section, except for those of Claim 4.6 and Lemma 4.7 which are given here, are postponed to Section 9.

So the polytope P is fixed, its volume is 1. We need to consider two parameters T, s which have already been defined in (3.1). But this is not important for the time being, we only assume that $2s \le T$, say.

Let $z \in P$ be a point with $v(z) \le T$ and write $[x, z]$ for the closed segment joining z and a point x. The following definition is crucial, and was used by Vu [22] as well. Set

$$S(z,T) = \{x \in P : \ [x, z] \cap P(v \ge T) = \emptyset\}.$$

This is the set of points that are *visible from z within $P(v < T)$*. We are interested in the size of $S(z, T)$.

We again use the notation $g(s) \ll f(s)$ if $|g(s)| < cf(s)$ for all $0 < s \le t_0$ with constants c and t_0 depending on the dimension, but not on the underlying convex set.

Lemma 4.1 *If $0 < v(z) \le \frac{1}{2}$, $2v(z) \le T$ then*

$$V(S(z,T)) \ll F(P)\, T \ln^{d-1}\left(\frac{T}{v(z)}\right).$$

Note that since $S(z,T) \subset P(v \le T)$, Theorem 2.7 immediately implies the inequality $V(S(z,T)) \ll F(P)\, T \ln^{d-1}(1/T)$. The improvement from $1/T$ to $T/v(z)$ is significant in the range we are interested in.

Consider the economic cap covering from Theorem 2.5 for $P(v \le s)$, $s \le s_0$, where s_0 is defined in (2.1). The caps $K_i(s)$ come from a saturated system $Z(s) = \{z_1, \dots, z_{m(s)}\} \subset P(v = s)$ which is fixed together with P and s as we agreed just before the cap covering theorem. We want to know how many $z_i \in Z(s)$ can be contained in a fixed cap C of volume T.

Lemma 4.2 *Assume C is a cap of P of volume T. Then for $0 < s \le s_0$, $2s \le T$ we have*

$$|Z(s) \cap C| \ll F(P) \ln^{d-1}\left(\frac{T}{s}\right).$$

Consider next the economic cap covering theorem for $P(v \le T)$. The saturated system $Y(T) = \{y_1, \dots, y_{m(T)}\}$ on $P(v = T)$ is again fixed, and so are the corresponding covering caps $K_j(T)$. (We use the notation $Y(T)$, $y_j(T)$ and $m(T)$ in order to avoid confusion with $Z(s), z_i(s)$ and $m(s)$.) We will need a bound on the number of those $y_j \in Y(T)$ for which $K_j^\lambda(T)$ contains a fixed $z \in P(v = s)$. Here λ is a constant that depends only on d.

Lemma 4.3 *Let $\lambda \ge 1$ be a constant depending only on d. Assume $0 \le 2s \le T \le (6\lambda)^{-d} s_0$. If $z \in P(v = s)$ then*

$$\left|\{y_j \in Y(T) :\ z \in K_j^\lambda(T)\}\right| \ll F(P) \ln^{d-1}\left(\frac{T}{s}\right).$$

The constant in $\ll$ depends on λ and thus again only on the dimension.

We will also need a bound on the number of points $z_j \in Z(s)$ that are contained in $S(z,T)$ when $z \in P(v = s)$.

Lemma 4.4 *Assume $z \in P(v = s)$ and $0 < s \le s_0,\ 2s \le T$. Then*

$$|Z(s) \cap S(z,T)| \ll F(P) \ln^{d-1}\left(\frac{T}{s}\right).$$

The following fact will be needed in the sandwiching step and concerns convex hulls of random points in $K_i'(T)$, the small sets in the cap covering theorem. Set $T^* = d6^d T$, $T \le s_0$. Choose in each $K_i'(T)$ a point x_i arbitrarily.

Claim 4.5 *Under the above conditions*

$$P(v \ge T^*) \subset [x_1, \dots, x_{m(T)}].$$

We mention in passing that the caps $K_i^\gamma(T)$ cover $P(v \le T^*)$, where $\gamma = 3d^3 6^d$:

$$P(v \le T^*) \subset \bigcup_1^{m(T)} K_i^\gamma(T). \tag{4.1}$$

This follows directly from Claim 2.6.

The system $Z(s) = \{z_1, \dots, z_{m(s)}\}$ on $P(v = s)$ is saturated, so for each $a \in P(v = s)$ there is a z_i with $M(z_i, \frac{1}{2}) \cap M(a, \frac{1}{2}) \neq \emptyset$. For each a we fix such a z_i and denote it by $z(a)$.

Claim 4.6 *If a cap C contains the point $a \in P(v = s)$, then $M(z(a), 1) \subset C^6$.*

Proof. This is very simple. As $z(a)$ satisfies $M(z(a), \frac{1}{2}) \cap M(a, \frac{1}{2}) \neq \emptyset$ by definition, Lemma 2.1 and Lemma 2.2 imply that

$$M(z(a), 1) = P \cap M(z(a), 1) \subset P \cap M(a, 5) \subset C^6.$$

□

The following lemma helps to bound the maximal degree of the dependency graph.

Lemma 4.7 *Assume $a, b \in P(v = s)$ and the segment $[a, b]$ is disjoint from $P(v \ge T)$. Then the segment $[z(a), z(b)]$ is disjoint from $P(v \ge T^*)$.*

Proof. Both $[a, b]$ and $P(v \ge T)$ are convex so they can be separated by a hyperplane since they are disjoint. This hyperplane cuts off a cap, say C, from K containing $[a, b]$ and disjoint from $P(v \ge T)$. So $V(C) \le dT$ by Lemma 2.4. Further, Claim 4.6 implies $z(a), z(b) \in C^6$. Consequently, $[z(a), z(b)] \subset C^6$, and $V(C^6) \le d6^d T = T^*$ follows from the trivial volume estimate. □

5. Sandwiching Π_η.

Recall that the Poisson polytope, Π_η, is the convex hull of $X \cap P$ where $X = X(\eta)$ is a Poisson point process of intensity η. We are going to use the well known fact that with high probability the boundary of Π_η is contained in a small strip close to the boundary of P. Results of this type have been proved in [7] and in [22]. Here we need a slightly different, perhaps more refined estimate.

We make (3.1) more precise and set

$$T = T_\eta = \alpha \frac{\operatorname{lln} \eta}{\eta}, \text{ with } \alpha = (6d)^d (4d^2 + d - 1).$$

In the following we assume that $\eta \geq \eta_0$ where η_0 is chosen such that $T \leq s_0$ with s_0 defined in (2.1). Let $Y(T)$ be the fixed saturated point set $\{y_1, \dots, y_{m(T)}\}$ on $P(v = T)$ according to Theorem 2.5. We get an economic cap covering with caps $K_j(T)$ and half Macbeath-regions $K'_j(T)$, $j = 1, \dots, m(T)$. For simpler writing set $K_j = K_j(T)$, $K'_j = K'_j(T)$ and $m_\eta = m(T)$.

Let A' be the event that each K'_j contains at least one point of X, the Poisson point process with intensity η. Since the number of points in K'_j is Poisson distributed with parameter $\eta V(K'_j)$, it follows from the fact that $(6d)^{-d}T \leq V(K'_j) \leq 2^{-d}T$, that

$$\mathbb{P}(K'_j \cap X) = \emptyset) = e^{-\eta V(K'_j)} \leq e^{-(6d)^{-d}\eta T}.$$

Let $\overline{A'}$ denote the complement of the event A'. By Theorem 2.7, $m_\eta \ll F(P) \ln^{d-1} \eta$, so by Boole's inequality

$$\mathbb{P}(\overline{A'}) \leq m_\eta e^{-(6d)^{-d}\eta T} \ll F(P)(\ln \eta)^{-(6d)^{-d}\alpha + d - 1} = F(P) \ln^{-4d^2} \eta \tag{5.1}$$

follows from the choice of α.

We mention for later reference that

$$\mathbb{P}(K'_j \cap X = \emptyset) \geq e^{-2^{-d}\eta T} = \ln^{-2^{-d}\alpha} \eta \geq \ln^{-(3d)^{d+2}} \eta. \tag{5.2}$$

Now Claim 4.5 and (5.1) show that, with high probability, Π_η contains the floating body $P(v \geq T^*)$. (Recall that $T^* = d6^d T$.)

$$\mathbb{P}(\Pi_\eta \text{ does not contain } P(v \geq T^*)) \leq \mathbb{P}(\overline{A'}) \ll F(P) \ln^{-4d^2} \eta.$$

This is the first half of sandwiching. For the second half we make the definition of s_n in (3.1) more precise and set

$$s = s_\eta = \frac{1}{\eta \ln^\beta \eta}, \text{ where } \beta = 4d^2 + d - 1.$$

We claim that with high probability $P(v \leq s)$ contains no point of X. Indeed, $\eta V(P(v \leq s)) \ll F(P)(\ln \eta)^{-\beta + d - 1}$ by Theorem 2.7, and we get

$$\mathbb{P}(X \cap P(v \leq s) \neq \emptyset) = 1 - e^{-\eta V(P(v \leq s))} \ll F(P) \ln^{-4d^2} \eta. \tag{5.3}$$

What we just proved is that Π_η is sandwiched between $P(v \geq s)$ and $P(v \geq T^*)$ with high probability:

$$1 - \mathbb{P}\Big(P(v \geq T^*) \subset \Pi_\eta \subset P(v \geq s)\Big) \ll F(P) \ln^{-4d^2} \eta.$$

The proof of the CLT for $V(\Pi_\eta)$ could go via conditioning on A'. For $f_\ell(\Pi_\eta)$ we need a stronger condition, to be called A, which will work for $V(\Pi_\eta)$ as well. Set

$$\gamma = 3d^3 6^d.$$

For $j = 1, \ldots, m_\eta$ let $S_j = S_j(T)$ be pairwise internally disjoint closed sets with $\bigcup S_j = P$, $K'_j \subset S_j$, and $S_j \cap P(v \le T^*) \subset K_j^\gamma$. (Recall from Claim 2.6 that the sets K_j^γ cover $P(v \le T^*)$.) Set $S'_j = S'_j(T) = S_j \cap P(v \le T^*)$.

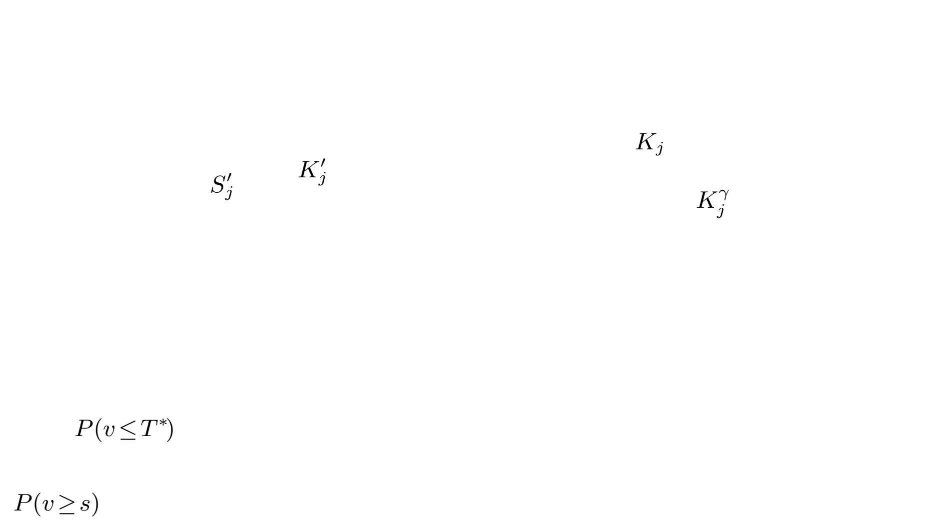

Figure 1: *Definition of* S'_j

Before defining A, observe that the expected number of points of X lying in S'_j is $\eta V(S'_j)$. Trivial volume estimates show that

$$(5.4) \qquad (6d)^{-d}\alpha \operatorname{lln}\eta \le \eta V(K'_j) \le \eta V(S'_j) \le \eta V(K_j^\gamma) \le (6\gamma)^d \alpha \operatorname{lln}\eta.$$

Define A to be the event that each K'_j contains at least one point, $P(v \le s)$ contains no point, and each S'_j contains at most $3(6\gamma)^d\alpha \operatorname{lln}\eta$ points of X, $(j = 1, \ldots, m_\eta)$. The following two claims are essential for our proof. We collect the properties of Π_η given the event A, and estimate the probability of A.

Claim 5.1 *Given A we have $P(v \ge T^*) \subset \Pi_\eta \subset P(v \ge s)$ and $|P(v \le T^*) \cap X| \ll F(P) \ln^{d-1}\eta \operatorname{lln}\eta$.*

Proof. This follows immediately from the definition of A and from the estimate on the volume of $P(v \le T^*)$. □

Claim 5.2 $\ln^{-(3d)^{d+2}} \eta \ll \mathbb{P}(\overline{A}) \ll F(P) \ln^{-4d^2} \eta$.

Proof. The lower bound follows from $\mathbb{P}(\overline{A}) \ge \mathbb{P}(K_1'(T) \cap X = \emptyset)$ and from (5.2). For the upper bound recall formulae (5.1) and (5.3): In (5.1) we showed that $K_j' \cap X = \emptyset$ for some j has probability $\ll F(P) \ln^{-4d^2} \eta$. Inequality (5.3) shows that $X \cap P(v \le s) \ne \emptyset$ with probability $\ll F(P) \ln^{-4d^2} \eta$.

So we only have to estimate the probability that, for some j, the set S_j' contains more than $3(6\gamma)^d \alpha \ln\ln \eta \ge \eta V(S_j')$ points. Let N denote a Poisson random variable with parameter p. Then (see, e.g., [19])

$$\mathbb{P}(N \ge 3p) \le \frac{3}{3-e} e^{-p}.$$

This inequality implies, by setting $p = \eta V(S_j')$, that the probability that S_j' contains more than $3(6\gamma)^d \alpha \ln\ln \eta \ge 3p$ points from X is bounded from above by

$$\frac{3}{3-e} e^{-p} \le \frac{3}{3-e} \exp\left(-(6d)^{-d} \alpha \ln\ln \eta\right) \ll \ln^{-(4d^2+d-1)} \eta.$$

Combining this with the bound $m_\eta \ll F(P) \ln^{d-1} \eta$ from Theorem 2.7 finishes the proof. □

Set

$$U = U_\eta = \frac{\ln \eta}{\eta} \text{ and } U^* = d6^d U.$$

Since we are assuming $V(P) = 1$, Theorem 2.7 tells us that

$$b_1 \frac{\ln^d \eta}{\eta} \le V(P(v \le U^*)) \le b_2 F(P) \frac{\ln^d \eta}{\eta}$$

with positive constants b_1, b_2 depending only on d. Let B be the event that $P(v \ge U^*) \subset \Pi_\eta$ and that $P(v \le U^*)$ contains at most $3b_2 F(P) \ln^d \eta$ points from X. The following estimate will be useful in Section 8. Its proof is similar to the ones above, actually even simpler (no need to worry about α) and is therefore left to the reader.

Lemma 5.3 $\mathbb{P}(\overline{B}) \ll F(P) \eta^{-3d}$.

□

6. The dependency graph. It is high time to define the dependency graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The values of s, T, T^* have been given in the previous section. The sets $K_i = K_i(T)$ and $K_i' = K_i'(T)$ come from the cap covering theorem. The vertex set, $\mathcal{V}$, of the dependency graph is just $\{1, \dots, m_\eta\}$.

Define the set L_i as the union of all S_k' such that there are points

$$a \in S_i' \cap P(v \geq s),\ b \in S_k' \cap P(v \geq s) \\ \text{with } [a,b] \text{ disjoint from } P(v \geq T^*). \tag{6.1}$$

Note that $S_i' \subset L_i$ for all i. Also, $S_k' \subset L_i$ holds if and only if $S_i' \subset L_k$. Now distinct vertices $i, j \in \mathcal{V}$ form an edge in $\mathcal{G}$ if L_i and L_j contain at least one set S_k' in common,

$$ij \in \mathcal{E} \quad \Leftrightarrow \quad \exists k \in \{1, \dots, m_\eta\} :\ S_k' \subset L_i \cap L_j \tag{6.2}$$

That this defines a dependency graph for the suitably chosen random variables is proved later, in Lemma 7.1. The main result of this section is an upper bound on the maximal degree D in $\mathcal{G}$.

Some preparation is needed. We need a bound on the number of sets $S_k' \subset L_i$.

Lemma 6.1 $|\{k :\ S_k' \subset L_i\}| \ll F(P)^3(\operatorname{lln}\eta)^{3(d-1)}$.

Proof. We show first that if $S_k' \subset L_i$, then there are also points

$$a' \in K_i^\gamma \cap P(v = s),\, b' \in K_k^\gamma \cap P(v = s) \\ \text{with } [a', b'] \text{ disjoint from } P(v \geq T^*). \tag{6.3}$$

To simplify notation we write $C' = C \cap P(v \geq s)$ when C is a cap of P. Clearly, C' is a cap of $P(v \geq s)$. We are going to use the fact that if two caps of a convex body have a point in common, then they have a point in common from the boundary of the convex set as well.

Since the segment $[a,b]$ is disjoint from $P(v \geq T^*)$, there is a cap C, also disjoint from $P(v \geq T^*)$, such that $[a,b] \subset C'$. Now

$$a \in C' \cap K_i^\gamma{}',\ b \in C' \cap K_k^\gamma{}'.$$

Every one of the two sets above is a nonempty intersection of two caps of $P(v \geq s)$. So each has a point, a' and b' respectively, on the boundary of $P(v \geq s)$ which is $P(v = s)$. As the segment $[a', b'] \subset C$, it is disjoint from $P(v \geq T^*)$ which proves (6.3).

Recall that a saturated system $Z(s) = \{z_1, \dots, z_{m(s)}\}$ has been chosen in $P(v = s)$. Also, for each $x \in P(v = s)$ we fixed a point $z(x) \in Z(s)$ so that $M(x, \frac{1}{2}) \cap M(z(x), \frac{1}{2}) \neq \emptyset$. We have points $a', b' \in P(v = s)$ satisfying (6.3). Claim 4.6 shows the existence of points $z(a'), z(b') \in Z(s)$ such that $z(a') \in K_i^{6\gamma}$, $z(b') \in K_j^{6\gamma}$, and by Lemma 4.7 the segment $[z(a'), z(b')]$ is disjoint from $P(v \geq T^o)$ where $T^o = d6^d T^*$.

We bound the number of sets S'_k in L_i in three steps. In view of Lemma 4.2, with $C = K_i^{6\gamma}$, we have

$$|Z(s) \cap K_i^{6\gamma}| \ll F(P) \ln^{d-1} \left(\frac{V(K_i^{6\gamma})}{s} \right) \ll F(P) \operatorname{lln}^{d-1} \eta$$

where the upper bound for $V(K_i^{6\gamma})$ comes from (5.4). This is an upper bound on how many $z(a') \in K_i^{6\gamma}$ can there be, given that the segment $[z(a'), z(b')]$ starts at $K_i^{6\gamma}$.

In the second step we estimate, for a fixed $z(a')$, the number of $z(b') \in Z(s)$ such that $[z(a'), z(b')]$ is disjoint from $P(v \geq T^o)$. All such $z(b')$ lie in $S(z(a'), T^o)$. So by Lemma 4.4, the number of such $z(b')$ is

$$\ll F(P) \ln^{d-1} \left(\frac{T^o}{s} \right) \ll F(P) \operatorname{lln}^{d-1} \eta.$$

In the third step we estimate the number of $K_j^{6\gamma}$ that contain a fixed $z(b') \in Z(s)$. Lemma 4.3 implies, with $\lambda = 6\gamma$, that this number is

$$\ll F(P) \ln^{d-1} \left(\frac{T}{s} \right) \ll F(P) \operatorname{lln}^{d-1} \eta.$$

This argument shows that for a set S'_i there are at most $\ll F(P)^3 \operatorname{lln}^{3(d-1)} \eta$ sets S'_k which can be connected by some segment $[a', b']$. Since every set $S'_k \subset L_i$ is connected to S'_i by some segment, the number of sets S'_k in L_i is $\ll F(P)^3 \operatorname{lln}^{3(d-1)} \eta$. □

Here comes the upper bound on the maximal degree D.

Theorem 6.2 $D \ll F(P)^6 (\operatorname{lln} \eta)^{6(d-1)}$.

Proof. By (6.2) we have $ij \in \mathcal{E}$ if $L_i \cap L_j$ contains some set S'_k. Clearly, if $S'_k \subset L_j$ then by the definition (6.1) also $S'_j \subset L_k$. Thus $ij \in \mathcal{E}$ if there is some k such that $S'_k \subset L_i$, $S'_j \subset L_k$ which gives

$$D \leq \max_i \sum_{k:\, S'_k \subset L_i} |\{j :\ S'_j \subset L_k\}|.$$

Combined with Lemma 6.1 this gives the bound on the degree of $\mathcal{G}$. □

Thus the graph $\mathcal{G}$ has been defined, its maximal degree has been bounded. In the next section we define the random variables ζ_i and show that $\mathcal{G}$ is a dependency graph.

7. The central limit theorem under condition A. **Proof** of the CLT for $V(\Pi_\eta)|A$. We introduce m_η random variables ζ_j in the following way. For simpler notation we keep writing K_j' for $K_j'(T)$, K_j for $K_j(T)$, S_j for $S_j(T)$, and S_j' for $S_j'(T)$. Define ζ_j as the missed volume in the set S_j,

$$\zeta_j = V(S_j) - V(S_j \cap \Pi_\eta).$$

and ζ as the missed volume in the polytope P,

$$\zeta = \sum_{j=1}^{m_\eta} \zeta_j = V(P) - V(\Pi_\eta) = 1 - V(\Pi_\eta).$$

In order to prove the CLT for $V(\Pi_\eta)|A$ we simply check the conditions of Rinott's theorem. We start with the weak independence condition.

Lemma 7.1 *Given disjoint subsets $\mathcal{W}_1, \mathcal{W}_2$ of $\mathcal{V}$ with no edge between them, the random variables $\{\zeta_i : i \in \mathcal{W}_1\}$ are independent of the random variables $\{\zeta_j : j \in \mathcal{W}_2\}$ under the conditional distribution of X given that A holds.*

Proof. Under condition A the boundary of Π_η lies in $P(s < v \leq T^*)$ and thus $\zeta_j = V(S_j') - V(S_j' \cap \Pi_\eta)$. The intersection $S_i' \cap \Pi_\eta$ is determined by the facets ($(d-1)$-dimensional faces) of Π_η intersecting S_i'. These facets are determined by their vertices. Thus all vertices that may determine a facet that intersects S_i' are contained in L_i. In other words, $S_i' \cap \Pi_\eta$ is the same as the intersection of S_i' with the convex hull of $X \cap L_i$.

Set now $L^k = \bigcup_{i \in \mathcal{W}_k} L_i$ for $k = 1, 2$. By definition L^1 and L^2 are unions of sets S_k' and have disjoint interiors. Given A, the ζ_i, $i \in \mathcal{W}_1$, are determined by $L^1 \cap X$ and the ζ_i, $i \in \mathcal{W}_2$, are determined by $L^2 \cap X$. Since $L^1 \cap X$ is independent of $L^2 \cap X$, conditional on A and otherwise, the claim follows. $\square$

We have to check two more conditions of Rinott's theorem.

Claim 7.2 *Under condition A, $M = \max \|\zeta_j\|_\infty \ll (\,\mathrm{lln}\,\eta)/\eta$.*

Proof. This is very simple: $\zeta_j \leq V(S_j') \ll T \ll (\,\mathrm{lln}\,\eta)/\eta$. $\square$

Claim 7.3 *For $\ln \eta \gg F(P)^{1/d^2}$ we have $\mathrm{Var}\,(V(\Pi_\eta)|A) \gg F(P)\eta^{-2} \ln^{d-1} \eta$.*

This claim is an easy corollary of Theorem 3.3 and (8.6) from the next section.

Bounds on $|\mathcal{V}|, D, \zeta_j$ and $\mathrm{Var}\,\zeta = \mathrm{Var}\,(V(\Pi_\eta)|A)$ have been established. Rinott's theorem can be applied. For $\ln \eta \gg F(P)^{1/d^2}$, the dominating error term is

$$\frac{|\mathcal{V}|D^2M^3}{\mathrm{Var}\,(V(\Pi_\eta)|A)^{\frac{3}{2}}} \ll F(P)^{11.5}\frac{(\,\mathrm{lln}\,\eta)^{12d-9}}{(\ln\eta)^{\frac{d-1}{2}}},$$

as a simple computation shows. If $\ln\eta$ equals $F(P)^{1/d^2}$ the right hand side is already $\gg 1$ which proves that this error term is valid for all η. □

Proof of the CLT for $f_\ell(\Pi_\eta)|A$. The dependency graph remains the same. The random variables ζ_i are to be defined, just like in [19], the following way. Let F be an ℓ-dimensional face of Π_η having $f_0(S_i, F)$ vertices in S_i, and set

$$\zeta_i = \frac{1}{\ell+1}\sum_{\text{all } F} f_0(S_i, F).$$

Since with probability one no point from X lies in two S_j, and each face F is a simplex with probability one, the sum of the ζ_i is equal to $f_\ell(\Pi_\eta)$ almost surely. The analogue of Lemma 7.1 for the new variables ζ_i is proved the same way.

We need to bound $\max\|\zeta_i\|_\infty$ from above and, also, $\mathrm{Var}\,\zeta = \mathrm{Var}\, f_\ell(\Pi_\eta)$ from below.

Claim 7.4 *For* $\ln\eta \gg F(P)^{1/d}$ *we have* $\mathrm{Var}\,(f_\ell(\Pi_\eta)|A) \gg F(P)\ln^{d-1}\eta$.

Again, this follows from Theorem 3.3 and (8.8) in the next section.

Claim 7.5 $M = \max\|\zeta_i\|_\infty \ll F(P)^{3d}(\,\mathrm{lln}\,\eta)^{3d^2}$.

Proof. (Similar to the one in Reitzner [19].) Condition A ensures that all vertices of Π_η lie in $P(s < v \le T^*)$. As we have seen in the proof of Lemma 7.1, each face F intersecting S_i' has all of its vertices in L_i: if $x \in S_i'$ and $y \in S_j'$ are vertices of F, then $y \in L_i$. Under condition A, S_j' contains $\ll \mathrm{lln}\,\eta$ points from X. Thus the number of vertices contributing to ζ_i is $\ll F(P)^3(\,\mathrm{lln}\,\eta)^{(3d-2)}$ by Lemma 6.1.

The number of ℓ-faces (actually, all subsets of size $\ell+1$) on this many vertices is $\ll (F(P)^3(\,\mathrm{lln}\,\eta)^{(3d-2)})^{\ell+1}$. Each such ℓ-face contributes at most one to the value of ζ_i. Consequently,

$$\zeta_i \ll (F(P)^3(\,\mathrm{lln}\,\eta)^{(3d-2)})^{\ell+1} \ll F(P)^{3d}(\,\mathrm{lln}\,\eta)^{d(3d-2)}$$

since $\ell+1 \le d$. □

All condition of Rinott's theorem have been established. The dominating error term is again the third one and we get the CLT for $f_\ell(\Pi_\eta)|A$ with error term

$$\frac{|\mathcal{V}|D^2M^3}{\mathrm{Var}\,(f_\ell(\Pi_\eta)|A)^{\frac{3}{2}}} \ll F(P)^{15d}\frac{(\,\mathrm{lln}\,\eta)^{15d^2}}{(\ln\eta)^{\frac{d-1}{2}}},$$

as a simple and generous computation shows. □

8. Removing the conditioning. We are going to use the transference Lemma 3.2:

Lemma 8.1 *The random variables* $\xi_\eta = V(\Pi_\eta)$ *and* $\xi'_\eta = V(\Pi_\eta)|A$ *satisfy the conditions of Lemma 3.2 with*

$$\sum \varepsilon_i(\eta) \ll F(P)^{11.5}\ln^{-\frac{d-1}{2}+o(1)}\eta.$$

Lemma 8.2 *The random variables* $\xi_\eta = f_\ell(\Pi_\eta)$ *and* $\xi'_\eta = f_\ell(\Pi_\eta)|A$ *satisfy the conditions of Lemma 3.2 with*

$$\sum \varepsilon_i(\eta) \ll F(P)^{15d}\ln^{-\frac{d-1}{2}+o(1)}\eta.$$

In both cases, the fourth condition of the transference lemma has been proved in the previous section with $\varepsilon_4 \ll F(P)^{11.5}\ln^{-(d-1)/2+o(1)}\eta$ for the case of volume and with $\varepsilon_4 \ll F(P)^{15d}\ln^{-(d-1)/2+o(1)}\eta$ for the number of faces. So our main theorem for Π_η follows once the first three conditions of the transference lemma have been checked for the volume and for the number of faces. We will make use of a simple claim:

Claim 8.3 *If* ζ *is a non-negative random variable and* A *is an event, then*

$$|\mathbb{E}(\zeta)-\mathbb{E}(\zeta|A)| \le \Big(\mathbb{E}(\zeta|A)+\mathbb{E}(\zeta|\overline{A})\Big)\,\mathbb{P}(\overline{A}).$$

Proof. It is clear that $\mathbb{E}(\zeta) = \mathbb{E}(\zeta|A)\mathbb{P}(A) + \mathbb{E}(\zeta|\overline{A})\mathbb{P}(\overline{A})$. Replacing $\mathbb{P}(A)$ by $1-\mathbb{P}(\overline{A})$ here gives

$$\mathbb{E}(\zeta)-\mathbb{E}(\zeta|A) = \Big(-\mathbb{E}(\zeta|A)+\mathbb{E}(\zeta|\overline{A})\Big)\,\mathbb{P}(\overline{A}),$$

and the claim follows. □

Proof of Lemma 8.1. We need some preparations. We use Claim 8.3 with $\zeta = 1-V(\Pi_\eta)$. We estimate first $\mathbb{E}(\zeta^k|\overline{A})$ for $k=1,2$, the first two moments of $\zeta|\overline{A}$. (We will have to do a lot of similar estimations later.) Note that $0 \le \zeta^k \le 1$.

This is where we use the last paragraph of Section 5. Recall that B denotes the event that $P(v \geq U^*) \subset \Pi_\eta$ and $P(v \leq U^*)$ contains at most $3b_2 F(P) \ln^d \eta$ points from X. Here $U = (\ln \eta)/\eta$ and $U^* = d6^d U$. Lemma 5.3 says that $\mathbb{P}(\overline{B}) \ll F(P)\eta^{-3d}$. Let $I(B)$ denote the indicator function of the event B. Observe that $\zeta^k I(B) \leq V(P(v \leq U^*))^k$. Moreover, $V(P(v \leq U^*)) \ll F(P)(\ln \eta)^d/\eta$ by Theorem 2.7. So we have

$$\begin{aligned} \mathbb{E}(\zeta^k|\overline{A}) &= \mathbb{E}(\zeta^k(1-I(B))|\overline{A}) + \mathbb{E}(\zeta^k I(B)|\overline{A}) \\ &\leq \mathbb{E}((1-I(B))|\overline{A}) + V(P(v \leq U^*))^k \\ &\ll \mathbb{P}(\overline{B}|\overline{A}) + \left(F(P)\frac{\ln^d \eta}{\eta}\right)^k \ll \left(F(P)\frac{\ln^d \eta}{\eta}\right)^k . \end{aligned} \tag{8.1}$$

Here we used the estimate

$$\mathbb{P}(\overline{B}|\overline{A}) \leq \frac{\mathbb{P}(\overline{B})}{\mathbb{P}(\overline{A})} \ll \frac{F(P)\eta^{-3d}}{(\ln \eta)^{-(3d)^{d+2}}} \ll F(P)\eta^{-3d+1} \tag{8.2}$$

where the lower bound for $\mathbb{P}(\overline{A})$ comes from Claim 5.2.

As for $\mathbb{E}(\zeta^k|A)$, Claim 5.1 tells us that

$$\mathbb{E}(\zeta^k|A) \leq V(P(v \leq T^*))^k \ll \left(F(P)\frac{\ln^d \eta}{\eta}\right)^k .$$

Thus we get, using Claim 8.3

$$|\mathbb{E}(\zeta^k|A) - \mathbb{E}(\zeta^k)| \ll \left(F(P)\frac{\ln^d \eta}{\eta}\right)^k \mathbb{P}(\overline{A}). \tag{8.3}$$

We check condition (ii) first. Since $\mathrm{Var}\,(V(\Pi_\eta)) = \mathrm{Var}\,(1-\zeta) = \mathrm{Var}\,(\zeta) = \mathbb{E}(\zeta^2) - (\mathbb{E}(\zeta))^2$ and similarly for $\mathrm{Var}\,(V(\Pi_\eta)|A)$, the target is to estimate

$$\begin{aligned} &|(\mathbb{E}(\zeta^2|A) - (\mathbb{E}(\zeta|A))^2) - (\mathbb{E}(\zeta^2) - (\mathbb{E}(\zeta))^2)| \\ &\qquad \leq |\mathbb{E}(\zeta^2|A) - \mathbb{E}(\zeta^2)| + |(\mathbb{E}(\zeta|A))^2 - (\mathbb{E}(\zeta))^2|. \end{aligned} \tag{8.4}$$

The first term in the last line is bounded in (8.3) with $k = 2$. For the second we have

$$\begin{aligned} |(\mathbb{E}(\zeta|A))^2 \;-\; (\mathbb{E}(\zeta))^2| &= |\mathbb{E}(\zeta|A) + \mathbb{E}(\zeta)| \cdot |\mathbb{E}(\zeta|A) - \mathbb{E}(\zeta)| \\ &\ll F(P)\frac{\ln^d \eta}{\eta} |\mathbb{E}(\zeta|A) - \mathbb{E}(\zeta)| \ll \left(F(P)\frac{\ln^d \eta}{\eta}\right)^2 \mathbb{P}(\overline{A}), \end{aligned} \tag{8.5}$$

where (8.3) and (8.1) have been applied with $k = 1$. We need now the lower bound $\mathrm{Var}\,(V(\Pi_\eta)) \gg F(P)\eta^{-2}\ln^{d-1}\eta$ from Theorem 3.3. Combining this lower bound, formulae (8.3), (8.4), (8.5), and Claim 5.2 yield

$$\begin{aligned} |\mathrm{Var}\,(V(\Pi_\eta)|A) \quad &- \quad \mathrm{Var}\,(V(\Pi_\eta))| \ll \left(F(P)\frac{\ln^d \eta}{\eta}\right)^2 \mathbb{P}(\overline{A}) \\ &\ll \quad F(P)^2 \frac{\ln^{d+1}\eta}{\ln^{4d^2}\eta}\mathrm{Var}\,(V(\Pi_\eta)). \end{aligned} \tag{8.6}$$

This shows that condition (ii) of Lemma 8.1 is satisfied with $\varepsilon_2(\eta) \ll F(P)^2 \ln^{-4d^2+d+1}\eta$. This and Theorem 3.3 also proves immediately Claim 7.3, that is, $\mathrm{Var}\,(V(\Pi_\eta)|A) \gg F(P)\eta^{-2}\ln^{d-1}\eta$ when $\ln^{4d^2-d-1}\eta \gg F(P)^2$.

Finally (8.3) with $k = 1$ gives

$$\begin{aligned} |\mathbb{E}(V(\Pi_\eta)|A) \quad &- \quad \mathbb{E}(V(\Pi_\eta)| = |\mathbb{E}(\zeta|A) - \mathbb{E}(\zeta)| \ll F(P)\frac{\ln^d\eta}{\eta}\mathbb{P}(\overline{A}) \\ &\ll \quad F(P)^2\frac{\ln^d\eta}{\eta\ln^{4d^2}\eta} \ll F(P)^{\frac{3}{2}}\frac{\ln^{\frac{d+1}{2}}\eta}{\ln^{4d^2}\eta}\sqrt{\mathrm{Var}\,(V(\Pi_\eta))}. \end{aligned}$$

Thus condition (i) is also satisfied with $\varepsilon_1(\eta) \ll F(P)^{\frac{3}{2}}\ln^{-4d^2+(d+1)/2}\eta$.

Condition (iii) is the simplest to check: Set $\zeta = I(V(\Pi_\eta)) \le x)$ and apply Claim 8.3. Then

$$\begin{aligned} |\mathbb{E}(\zeta|A) - \mathbb{E}(\zeta)| &= |\mathbb{P}(V(\Pi_\eta)) \le x|A) - \mathbb{P}(V(\Pi_\eta)) \le x)| \\ &\le 2\mathbb{P}(\overline{A}) \ll F(P)\ln^{-4d^2}\eta. \end{aligned}$$

and thus (iii) holds with $\varepsilon_3(\eta) \ll F(P)\ln^{-4d^2}\eta$. □

Proof of Lemma 8.2. This proof is similar to the previous one and we only point out the main differences. Set $\zeta = f_\ell(\Pi_\eta)$. We want to estimate, for $k = 1, 2$,

$$\mathbb{E}(\zeta^k|\overline{A}) = \mathbb{E}(\zeta^k(1 - I(B))|\overline{A}) + \mathbb{E}(\zeta^k I(B)|\overline{A}) \tag{8.7}$$

Note that, given B, Π_η can have at most $F(P)\ln^d\eta$ vertices, implying that $\zeta^k I(B) \ll (F(P)\ln^d\eta)^{k(\ell+1)}$ which is an upper bound for the second term in (8.7). The first term needs extra care since the random variable ζ is not bounded. Let N be a random variable which is Poisson distributed with mean η, and write E_m for the event $N = m$. Of course, $\zeta \le m^{\ell+1} \le m^d$ under condition E_m. Thus

$$\begin{aligned}
\mathbb{E}(\zeta^k(1 - I(B))|\overline{A}) &= \sum_{m=0}^{\infty} \mathbb{E}(\zeta^k(1-I(B))|\overline{A}\,E_m)\mathbb{P}(E_m) \\
&\le \sum_{0\le m<3\eta} \mathbb{E}(\zeta^k(1-I(B))|\overline{A}\,E_m)\mathbb{P}(E_m) \\
&\quad + \sum_{3\eta\le m} \mathbb{E}(\zeta^k(1-I(B))|\overline{A}\,E_m)\mathbb{P}(E_m) \\
&\le \sum_{0\le m<3\eta} (3\eta)^{kd}\mathbb{E}((1-I(B))|\overline{A}\,E_m)\mathbb{P}(E_m) \\
&\quad + \sum_{3\eta\le m} m^{kd}\mathbb{E}((1-I(B))|\overline{A}\,E_m)\mathbb{P}(E_m) \\
&\ll (3\eta)^{kd} \sum_{0\le m<3\eta} \mathbb{P}(\overline{B}|\overline{A}\,E_m)\mathbb{P}(E_m) + \sum_{3\eta\le m} m^{kd}\mathbb{P}(E_m) \\
&\ll (3\eta)^{kd}\mathbb{P}(\overline{B}|\overline{A}\,) + \sum_{3\eta\le m} m^{kd}\mathbb{P}(E_m) \ll F(P)\eta^{-d+1}
\end{aligned}$$

where we used (8.2), and the routine estimation of $\sum_{3\eta}^{\infty} m^{kd}\mathbb{P}(E_m)$ is omitted. Using this and Claim 5.1 for $\mathbb{E}(\zeta^k|A)$ yields for $k = 1, 2$

$$\mathbb{E}(\zeta^k|A), \mathbb{E}(\zeta^k|\overline{A}) \ll (F(P)\ln^d \eta)^{k(\ell+1)} \le F(P)^{kd}\ln^{kd^2}\eta.$$

We need again Claim 8.3 and the lower bound from Theorem 3.3 to show that

$$|\mathrm{Var}\,(\zeta|A) - \mathrm{Var}\,(\zeta)| \ll F(P)^{2d}\ln^{-2d^2-d+1}\eta\,\mathrm{Var}\,(\zeta). \tag{8.8}$$

So condition (ii) is satisfied. It also follows that $\mathrm{Var}\,(\zeta|A) \gg F(P)\ln^{d-1}\eta$ for $\ln\eta \gg F(P)^{1/d}$, which is Claim 7.4 from the previous section.

Checking condition (i) goes the same way and condition (iii) is straightforward. □

9. Proofs of the auxiliary lemmas.

In this section we assume that P is a fixed polytope in $\mathbb{R}^d$ whose volume is one. We prove first the following claim, where $\beta = 2ed^3 + 1$ (a β which is different from the one in Section 5).

Claim 9.1 *For all T and for all $z \in P$ satisfying $0 < v(z) < \frac{1}{2}$, $v(z) \le T$, we have*

$$S(z, T) \subset C^{\beta T/v(z)}(z).$$

Proof: Set $s = v(z)$. Let $C(z) = \{x \in \mathbb{R}^d \,|\, u\cdot x \ge h_P(u) - h_z\}$ be the minimal cap of $z \in P$, and denote by H_h the hyperplane $\{x \in \mathbb{R}^d \,|\, u\cdot x =$

$h_P(u) - h\}$. Then H_0 touches the boundary of P in a centre of the cap, H_{h_z} is the bounding hyperplane of $C(z)$, and thus $z \in H_{h_z}$. Write $Q_z = P \cap H_{h_z}$. The following simple geometric arguments show that for every $h \in [0, h_z]$, we have

$$V_{d-1}(P \cap H_h) \le 2dV_{d-1}(Q_z) \tag{9.1}$$

if $s \le \frac{1}{2}$, where $V_{d-1}(\cdot)$ stands for $(d-1)$-dimensional volume. Indeed, the Brunn-Minkowski inequality shows that for some $h_{\max}$ the volume of the sections $P \cap H_h$ is first increasing for $h \in [0, h_{\max}]$ and then decreasing for $h \in [h_{\max}, w]$. Here w denotes the width of P in direction u. Thus if $h_z \in [0, h_{\max}]$, equation (9.1) is immediate (with $2d$ replaced by 1). And if $h_z > h_{\max}$ we have to show that $V_{d-1}(P \cap H_{h_{\max}}) \le 2dV_{d-1}(Q_z)$. Clearly

$$\frac{1}{d} w V_{d-1}(P \cap H_{h_{\max}}) \le 1.$$

Since we assume $h_z > h_{\max}$ here, $V_{d-1}(P \cap H_h)$ is decreasing for $h \in [h_z, w]$ and we also have

$$(w - h_z)V_{d-1}(Q_z) \ge 1 - s \ge \frac{1}{2}.$$

Combining this gives

$$V_{d-1}(P \cap H_{h_{\max}}) \le \frac{d}{w} \le \frac{d}{w - h_z} \le 2dV_{d-1}(Q_z)$$

which is (9.1). It follows that

$$s \le 2d\, h_z V_{d-1}(Q_z).$$

Clearly the set $S(z, T)$ is the union of caps $C \subset P(v \le T)$ such that $z \in C$. Let C be such a cap. Then $V(C) \le dT$ by Lemma 2.4. If C contains a point of H_h, then

$$V(C) \ge \frac{1}{d}\,(h - h_z)V_{d-1}(C \cap Q_z).$$

As is well-known (see for instance [13]), $z \in C$ is the centre of gravity of Q_z. Then a result of Grünbaum [15] says that $V_{d-1}(C \cap Q_z) \ge \frac{1}{e} V_{d-1}(Q_z)$. Thus

$$\frac{1}{2ed^2}\,\frac{s(h - h_z)}{h_z} \le \frac{1}{ed}\,(h - h_z)V_{d-1}(Q_z) \le V(C) \le dT.$$

Hence the distance between an arbitrary point of $S(z, T)$ and H_0 is at most $h = \frac{T}{s} 2ed^3 h_z + h_z \le (2ed^3 + 1)\frac{T}{s} h_z$, which shows that, indeed, $S(z, T) \subset C^{\beta T/s}(z)$. □

Proof of Lemma 4.1. Set again $v(z) = s$, then the condition is $0 < s \leq \frac{1}{2}$, $2s \leq T$. Choose β as in Claim 9.1. Let $C(z)$ be the minimal cap of z and set $C^* = C^{\beta T/s}(z)$, $V^* = V(C^*)$, and note that C^* is a polytope. By trivial volume estimates $V^* \leq (\beta T/s)^d V(C(z)) = (\beta T)^d / s^{d-1}$. Assume first that $C^* = P$ and thus $1/T \leq \beta^d (T/s)^{d-1}$. Then since $S(z,T) \subset P(v \leq T)$ we have

$$V(S(z,T)) \ll F(P) T \ln^{d-1}\left(\frac{1}{T}\right)$$

for $T \leq s_0$ by Theorem 2.7, which gives

$$V(S(z,T)) \ll F(P) T \ln^{d-1}\left(\frac{T}{s}\right).$$

for any T with $2s \leq T$. For $C^* \neq P$ trivial volume estimates show that $V(C^*) \geq (\beta T/ds) V(C(z)) = (\beta/d) T$. Claim 9.1 shows that

$$S(z,T) \subset P(v_P \leq T) \cap C^* \subset C^*(v_{C^*} \leq T),$$

where we wrote v_{C^*} to emphasize that the underlying convex set now is C^*. By Theorem 2.7 there is a constant s_0 such that for $T \leq s_0 V^*$

$$\begin{aligned} V(S(z,T)) \leq V(C^*(v_{C^*} \leq T)) &\ll F(C^*) T \ln^{d-1}\left(\frac{V^*}{T}\right) \\ &\leq F(P) T \ln^{d-1}\left(\frac{V^*}{T}\right). \end{aligned}$$

Here we used the fact that $F(C^*) \leq F(P)$ which can be proved quite easily (we omit the proof). In the remaining case $s_0 V^* \leq T \leq (d/\beta) V^*$ we have

$$V(C^*(v_{C^*} \leq T)) \leq V^* \leq s_0^{-1} T \ll F(P) T \ln^{d-1}\left(\frac{V^*}{T}\right).$$

The lemma follows since $V^*/T \leq \beta^d (T/s)^{d-1}$. □

Proof of Lemma 4.2. Assume $z_i \in Z(s) \cap C$. Then by Lemma 2.2 $M(z_i, 1) \subset C^2$. Thus for $s \leq s_0$ the set $K_i'(s) = M(z_i, \frac{1}{2}) \cap C(z_i)$ lies in $P(v \leq s) \cap C^2$. The sets $K_i'(s)$, $i = 1, \ldots, m(s)$, are pairwise disjoint, so the usual volume argument applies:

$$|Z(s) \cap C| \ll \frac{V(P(v \leq s) \cap C^2)}{s}$$

as $V(K_i'(s)) \gg s$. Further, $P(v \leq s) \cap C^2 \subset C^2(v_{C^2} \leq s)$, whose volume can be estimated the same way as in the previous proof. Theorem 2.7 gives

$$V(C^2(v_{C^2} \leq s)) \ll F(C^2) s \ln^{d-1}\left(\frac{V(C^2)}{s}\right) \ll F(P) s \ln^{d-1}\left(\frac{T}{s}\right)$$

for $s \leq s_0 V(C^2)$, since $F(C^2) \ll F(P)$. And for $s_0 V(C^2) \leq s \leq s_0$ the lemma follows from the fact that $V(C^2(v_{C^2} \leq s)) \leq V(C^2)$ and $s \leq 2T$. □

Proof of Lemma 4.3. Since $V(K_j^\lambda(T)) \leq \lambda^d V(K_j(T)) \leq (6\lambda)^d T$, each $y_j \in Y(T)$ with $z \in K_j^\lambda(T)$ is contained in $S(z,(6\lambda)^d T)$. It is also clear that $M(y_j, \frac{1}{2}) \cap C(y_j)$ lies in $S(z,(6\lambda)^d T)$, once $y_j \in Y(T)$. Thus the usual volume argument applies, with the upper bound on $V(S(z,(6\lambda)^d T))$ coming from Lemma 4.1. □

Proof of Lemma 4.4. Let $C(z)$ be the minimal cap of z. Claim 9.1 shows that $S(z,T)$ is contained in the cap $C := C^{\beta T/s}(z)$ with volume $V(C) \leq (\beta T)^d / s^{d-1}$. Then Lemma 4.2 applies and gives

$$|Z(s) \cap S(z,T)| \leq |Z(s) \cap C| \ll F(P) \ln^{d-1} \frac{V(C)}{s} \ll F(P) \ln^{d-1} \left(\frac{T}{s}\right)$$

for $s \leq s_0, 2s \leq V(C)$, since $V(C)/s \ll (T/s)^d$. The inequality $2s \leq V(C)$ follows from the trivial volume estimate if $C \neq P$ and from $s \leq s_0$ if $C = P$. □

Proof of Claim 4.5. Clearly it suffices to show that each cap C whose bounding hyperplane touches $P(v \geq T^*)$ contains at least one point x_i. If this is not the case then there is a cap C whose bounding hyperplane touches $P(v \geq T^*)$ with no $x_i \in C$ and thus no $M(y_i, \frac{1}{2}) \subset C$ either.

We claim now that $C^{\frac{1}{3}}$ is disjoint from all Macbeath regions $M(y_i, \frac{1}{2})$. Assume, for simpler notation, that $u \cdot x = h$ with $h > 0$ is the equation of the bounding hyperplane of C, and $u \cdot x = 0$ is the supporting hyperplane of P and C. If $u \cdot y_i = g$, then $M(y_i, 1)$ lies between hyperplanes $u \cdot x = 2g$ and $u \cdot x = 0$. Thus $M(y_i, \frac{1}{2})$ lies between hyperplanes $u \cdot x = \frac{3}{2}g$ and $u \cdot x = \frac{1}{2}g$. Here $\frac{3}{2}g > h$ holds since otherwise $M(y_i, \frac{1}{2}) \subset C$. Then $g > \frac{2}{3}h$ implying that $u \cdot x = \frac{1}{3}h$ is a separating hyperplane between $M(y_i, \frac{1}{2})$ and $C^{\frac{1}{3}}$. This proves the claim.

By trivial volume estimates $V(C^{\frac{1}{6}})$ is at least dT. Let x_0 be the point in $C^{\frac{1}{6}}$ where $v(x)$ takes its maximal value on $C^{\frac{1}{6}}$. By Lemma 2.4 $V(C^{\frac{1}{6}}) \leq dv(x_0)$, and so $v(x_0) \geq T$. This shows the existence of a point $z \in P(v = T) \cap C^{\frac{1}{6}}$. But then $M(z, \frac{1}{2}) \subset C^{\frac{1}{3}}$ is disjoint from all $M(y_i, \frac{1}{2})$ which is impossible since $Y(T) = \{y_1, \dots, y_{m(T)}\}$ is a saturated system. □

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