A case when the union of polytopes is convex

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Abstract

We present a necessary and sufficient condition for the union of a finite number of convex polytopes in \( \mathbb{R}^d \) to be convex. This generalises two theorems on convexity of the union of convex polytopes due to Bemporad et al.

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1. Introduction

A convex polytope or simply polytope is the convex hull of a finite set of points in Euclidean space \( \mathbb{R}^d \). Bemporad et al. [2] studied various necessary and sufficient conditions for the union of several polytopes in \( \mathbb{R}^d \) to be convex. In particular, it was...
shown that for two polytopes $P_1$ and $P_2$ in $\mathbb{R}^d$, their union $P_1 \cup P_2$ is convex if and only if the line segment $[v_1, v_2]$ is contained in $P_1 \cup P_2$ for each vertex $v_1$ of $P_1$ and each vertex $v_2$ of $P_2$. The main objective of the present paper is to give a natural extension of this theorem to the general case of several polytopes.

Assume that $P_i$ is a polytope in $\mathbb{R}^d$ whose vertex set is $X_i$, for $i \in [n]$. Here $[n]$ is a shorthand for the set $\{1, 2, \ldots, n\}$. Define $P = \bigcup_{i=1}^n P_i$, $X = \bigcup_{i=1}^n X_i$, and $Q = \text{conv } X$. Clearly, $P \subset Q$ always, and $P$ is convex if and only if it coincides with $Q$. Obviously, if $P$ is convex, then $\text{conv } S \subset P$ for every $S \subset X$. Our main theorem is a converse to this simple statement.

**Theorem 1.** Assume $d \geq 1$, $n \geq 2$. Then $P = Q$ if and only if $\text{conv } S \subset P$ for each $S \subset X$ with $|S| \leq d + 1$ and $|S \cap X_i| \leq 1$ for each $i \in [n]$.

It should be noted that Bemporad et al. gave another weaker version [2–Theorem 5] of the theorem in which the last condition “$|S \cap X_i| \leq 1$ for each $i \in [n]$” is replaced by the weaker condition “$|S| \leq n$”. It appears that our stronger theorem is substantially harder to prove.

Given sets $A_1, \ldots, A_n$, a transversal is a set $\{a_1, \ldots, a_n\}$ with $a_i \in A_i$ for all $i$. We are going to use the following theorem known as the Colourful Carathéodory theorem.

**Theorem 2** (The Colourful Carathéodory Theorem [1]). Given points $a, v \in \mathbb{R}^d$ and sets $A_i \subset \mathbb{R}^d$, $i \in [d]$, with $a \in \cap_i \text{conv } A_i$, there exists a transversal $\{a_1, \ldots, a_d\}$ of the $A_i$ such that

$$a \in \text{conv } \{a_1, \ldots, a_d, v\}.$$ 

2. **Preparations**

Carathéodory’s theorem (see [3]) says that the convex hull of $S \subset \mathbb{R}^d$ is the union of simplices $\text{conv } T$ with $T \subset S$ and $|T| \leq d + 1$. We will call such a simplex colourful if its vertices constitute a transversal of a subsystem of the $X_i$ ($i \in [n]$). In this terminology what we want to prove is the following: $P$ is convex if it contains every colourful simplex.

The statement is invariant under nondegenerate affine transformations, so we may apply any such transformation even during the proof.

We will need a following simple lemma:

**Lemma 3.** Assume $T \subset \mathbb{R}^d$ is a polytope with nonempty interior and $E \subset T$ is an ellipsoid. Assume $b_1, \ldots, b_s$ are the common points of $E$ and $\partial T$ and the outer unit normal to $E$ at $b_i$ is $u_i$. If $0 \notin \text{int conv } \{u_1, \ldots, u_s\}$, then there is another ellipsoid $E' \subset T$, arbitrarily close to $E$, with $\text{Vol } E' > \text{Vol } E$. 

Proof. The statement is invariant under nondegenerate affine transformations so we may assume that $E$ is just $B_r$, the ball of radius $r$ centered at the origin. If $B_r \cap \partial T = \emptyset$, then any $B_{\rho}$ with $\rho$ slightly larger than $r$ will do for $E'$.

Now assume that the set $B_r \cap \partial T = \{b_1, \ldots, b_s\}$ is nonvoid. It is clear that $u_i = b_i/r$, and $T$ has a facet with outer normal $u_i$ at distance $r$ from the origin (for all $i$), and all other facets are farther away.

Suppose $0 \notin \text{int conv} \{u_1, \ldots, u_s\}$ and set $C = \text{conv} \{b_1, \ldots, b_s\}$. Then $0 \notin \text{int} C$ as well, and there is a unit vector $u$ such that the hyperplane $u \cdot x = 0$ separates $C$ and $0$, that is, $u \cdot x \leq 0$ for every $x \in C$.

If the separation is strict, that is, $u \cdot x < 0$ for all $x \in C$, then, for sufficiently small $\varepsilon > 0$, the point $\varepsilon u$ is farther than $r$ from each facet of $T$. In this case the ball $\varepsilon u + B_r$ is disjoint from the boundary of $T$ if $\varepsilon > 0$ is small enough. Then $\varepsilon u + B_{\rho}$ will do for $E'$ for all $\rho$ slightly larger than $r$.

If the separation is not strict, then $0$ is in the relative interior of the convex hull of a subset of $\{b_1, \ldots, b_s\}$. Say $0 \in \text{relint} \{b_1, \ldots, b_j\}$. Then, for all small enough $\varepsilon > 0$, the ball $\varepsilon u + B_r$ touches all facets of $T$ that contain $b_i$, $i \in [j]$ and is disjoint from all other facets. The set $\text{conv} \{\varepsilon u + B_r \cup B_r\}$ is contained in $T$, and contains an ellipsoid $E'$, arbitrarily close to $B_r$ and of larger volume than $B_r$. □

The following fact can be proved easily by induction on $n$.

Lemma 4. If $Q$ and $P_i$, $i \in [n]$ are polytopes in $\mathbb{R}^d$, then the closure of $Q \setminus \bigcup_1^n P_i$ can be written as a finite union of simplices.

3. Proof of the main theorem

The statement is true for $d = 1$ and any $n \geq 2$. We use induction on $d$ so assume $d \geq 2$ and the statement is true in dimension $d - 1$.

Assume the contrary: suppose polytopes $P_1, \ldots, P_n$ in $\mathbb{R}^d$ form a counterexample to the theorem with minimal $n$. The induction hypothesis implies that $Q$ is full dimensional. Further, let $F$ be a facet of $Q$. Then, in the hyperplane containing $F$, the induction hypothesis can be used to show that $F \subset P$. This implies that $\partial Q \subset P$.

The set $G = Q \setminus P$ is open. By Lemma 4, its closure $\text{cl} G$ can be written as a finite union of full dimensional simplices $F_i$. Let $V$ denote the set of vertices of the $F_i$. Choose a unit vector $u \in \mathbb{R}^d$ so that

$$\min\{u \cdot x : x \in \text{cl} G\} = \min\{u \cdot x : x \in V\}$$

is reached on a unique vertex $a \in V$. Assume that $a$ coincides with the origin (otherwise apply a suitable affine transformation). Write $H(t) = \{x \in \mathbb{R}^d : u \cdot x = t\}$. Clearly, $H(t) \cap Q = H(t) \cap P$ for $t \leq 0$. Let $t_1 = \min\{u \cdot x : x \in V, x \neq 0\}$. Then $t_1 > 0$ and for $t \in (0, t_1)$
where the union is taken over all simplices $F_i$ with $0 \in F_i$. No polytope $P_i$ contains the origin in its interior. But $0 \notin P_i$ for some $i \in [n]$ since otherwise $0 \notin P$.

We clean the picture further. Fix $t \in (0, t_1)$ very small (to be specified soon) and set, again for $0 \in F_i$ in the union,

$$Z = H(t) \cap \text{cl} \ G = H(t) \cap \bigcup F_i,$$

$Z$ is the union of $(d-1)$-dimensional simplices in $H(t)$ which is a copy of $R^{d-1}$. Choose an ellipsoid $E \subset Z$ of maximum $(d-1)$-dimensional volume. Such an ellipsoid clearly exists and has finitely many points $z \in P$ on its boundary. The segment $[0, z] \subset P$ since otherwise the interior of some simplex $F_i$ with $0 \in F_i$ contains a point from the segment, but then the whole segment is contained in int $F_i$. Then $[0, \gamma z] \subset P_i$ for some $i \in [n]$ with a suitable $\gamma > 0$. So if $t$ is chosen small enough, then $[0, z] \subset P_i$. Here $z$ is determined by $P_i$ uniquely, we set $b_i = z$ for concreteness.

Assume, for simpler writing, that the set of indices $i$ with $b_i$ on the boundary of $E$ is just $[k]$. Then $1 \leq k \leq n$. Write $h_i$ for the halfspace (in $H(t)$) which contains $E$ and whose boundary hyperplane contains $b_i$. Then $T = \bigcap_{i \in [k]} h_i$ is a polytope,

$E \subset T$ and $E$ is at a positive distance from all $P_i, i \notin [k]$.

Let $u_i$ be the outer unit normal to $E$ at $b_i \ (i \in [k])$. We claim that

$$0 \in \text{int conv} \{u_1, \ldots, u_k\}.$$

Indeed, if this were not case, then Lemma 3 implies the existence of another ellipsoid $E' \subset T$ arbitrarily close to $E$ with $\text{Vol} E' > \text{Vol} E$. Such an ellipsoid is contained in $Z$ and has larger volume than $E$, contradicting the choice of $E$.

The claim shows that $d \leq k$ (otherwise $\text{int conv} \{u_1, \ldots, u_k\}$ is empty). Thus $d \leq k \leq n$. Note that we are finished with the case $n < d$.

Now we apply a nondegenerate linear transformation (to all polytopes $P_i, P$ and $Q$) that keeps the hyperplane $H(0)$ fixed and moves the ellipsoid $E$ to a ball $B$ in $H(t)$ whose center, $b$, is orthogonal to $H(0)$. We keep the same notation, so the images of $P, Q, P$, and the points $b_i$ will go under the same name. This should cause no confusion as we won’t return to their preimages.

We write $C = \text{pos} \ B$, this is a closed circular cone whose axis is the halfline $L(b) = \{ \lambda b : \lambda \geq 0 \}$. The cone $C$ is separated from each $P_i \ (i \in [k])$ and the (unique) separating hyperplane is tangent to $C$ along the halfline $L(b_i)$. Moreover, $C \cap P_i$ is contained in $L(b_i)$. It is also clear that $b \in \text{int} C$. We define $C^* = C \cap \{ x : u \cdot x \leq t \}$ and note that $\text{int} C^* \cap P = \emptyset$.

Claim 5. For distinct $i, j \in [k]$, the rays $L(b_i)$ and $L(b_j)$ are distinct.

Proof. Assume, on the contrary that $L(b_1) = L(b_2)$, say. Set $P_0 = \text{conv} (X_1 \cup X_2)$. Since $P_1$ and $P_2$ are separated by the same hyperplane from $C$, $P_0$ is separated from $C$ by that hyperplane. It is not hard to check now that $P_0, P_3, \ldots, P_n$ is another
counterexample to the theorem with \( n - 1 \) polytopes, contrary to the minimality of \( n \).

We claim now that \( k < n \). Indeed, the halfline \( L(b) \) intersects \( \partial Q \) at the point \( b^* \), say. As \( C \) and \( P_i \) are separated for all \( i \in [k] \), \( b^* \notin P_i \). Now

\[
    b^* \in \partial Q \subset P = \bigcup_{i=1}^n P_i
\]

so \( b^* \) is contained in one of the polytopes \( P_i \), \( i > k \). Assume, for concreteness, that \( b^* \in P_n \). (Note that we are finished with the case \( n \leq d \) now.)

We can now apply Theorem 2: \( 0 \in \text{conv} X_i \) for each \( i \in [k] \). The last vector \( v \in \mathbb{R}^d \) can be anything; it will often but not always come from \( X_n \). As \( k \) may be larger than \( d \) we will consider partial transversals of the system \( X_i \). A partial transversal, or \( I \)-transversal, of this system is \( \{ x_i : i \in I \} \), here \( I \) can be any subset of \( [k] \).

Theorem 2 says now the following:

**Lemma 6.** For every \( I \subset [k] \) with \( |I| = d \) and every \( v \in \mathbb{R}^d \) there exists an \( I \)-transversal \( \{ x_i : i \in I \} \) such that

\[
    0 \in \text{conv} (\{ x_i : i \in I \} \cup \{ v \})
\]

We have to distinguish some cases.

**Case 1.** There is a colourful simplex whose interior contains the origin.

By the conditions of Theorem 1 this colourful simplex is contained in \( P \), so a full neighbourhood of 0 lies in \( P \), contradicting \( \text{int} C^* \cap P = \emptyset \).

Thus 0 does not lie in the interior of any colourful simplex. Then by Lemma 6, for every \( x_n \in X_n \) and for every \( I \subset [k], |I| = d \), 0 is on the boundary of the convex hull of an \( I \)-transversal and \( x_n \). Thus 0 is contained in the relative interior of the convex hull of a partial transversal plus possibly \( x_n \).

**Case 2.** When 0 is not contained in the convex hull of any \( I \)-transversal, with \( |I| \leq d \), of the \( X_i, i \in [k] \). Then \( k = d \) since otherwise Lemma 6 can be applied to \( X_1, \ldots, X_d \) with \( v \) equal to an arbitrary \( x_{d+1} \in X_{d+1} \) and the convex hull of the transversal and \( x_{d+1} \) either contains 0 in its interior (which is impossible since Case 1 is excluded by now), or 0 is contained in the convex hull of some partial transversal from \( [k] \) which is impossible in Case 2.

So \( k = d \). Choose a point \( x_n \in X_n \). There are finitely many transversals \( \{ x_1, \ldots, x_d \} \) such that the set \( F = \text{conv} \{ x_1, \ldots, x_d, x_n \} \) contains the origin. Write \( \mathcal{F} \) for the collection of such sets.

**Claim 7.** The union of these sets contains a small neighbourhood of the origin.

**Proof.** Write \( W \) for the set of unit vectors \( w \) such that, for all small \( \varepsilon \in (0, \varepsilon_0] \), \( \varepsilon w \) is not contained in the affine hull of any \( d \) points from \( X \). (All unit vectors, except those in a finite union of hyperplanes, satisfy this requirement if \( \varepsilon_0 \) is chosen small.
enough.) Now apply Lemma 6 to the system $X_i, i \in [d]$ and the point $v = x_n - \varepsilon w$ where $w \in W$. The result is a transversal $x_i \in X_i (i \in [d])$ such that

$$0 \in \text{conv}([x_i : i \in [d]] \cup \{x_n - \varepsilon w\}).$$

In other words, there are $\gamma_i \geq 0$ (for $i \in [d] \cup \{n\}$) that sum to one with

$$0 = \gamma_1 x_1 + \cdots + \gamma_d x_d + \gamma_n (x_n - \varepsilon w).$$

We claim that all $\gamma_i > 0$ here. First, $\gamma_n = 0$ would show that 0 is in the convex hull of $x_1, \ldots, x_d$, contrary to the assumptions of Case 2. So $\gamma_n > 0$ (for every small positive $\varepsilon$), and

$$\gamma_n \varepsilon w = \gamma_1 x_1 + \cdots + \gamma_d x_d + \gamma_n x_n.$$

If some $\gamma_i = 0$ here, then $\gamma_n \varepsilon w$ is in the affine hull of $d$ or fewer vectors from $X$, contrary to the choice of $w$. Thus all $\gamma_i > 0$.

The last equation shows that the simplex $F = \text{conv} \{x_1, \ldots, x_d, x_n\}$ contains the segment $[0, \delta(w)w]$ for some small $\delta(w) > 0$. Let $r(F)$ be the distance from the origin to the union of the facets of $F$ not containing the origin. Thus if $F$ contains $[0, r(F)w]$, then it contains $[0, \delta(w)w]$ as well.

Set $r = \min \{r(F) : F \in \mathcal{F}\}$. Then for each $w \in W$ the segment $[0, rw]$ is contained in some $F \in \mathcal{F}$. This holds then for all unit vectors $w$ as the union of $F \in \mathcal{F}$ is a closed set. \hfill \square

The Claim shows that the union of the colourful simplices contains a small neighbourhood of the origin. This contradicts, again, the assumption that $\text{int} C^*$ is disjoint from $P$.

Case 3. When 0 is in the relative interior of the convex hull of some $I$-transversal with $I \subset [k], |I| \leq d$.

This is very simple if $|I| = d$. Assume the $I$-transversal is $\{x_1, \ldots, x_d\}$. Then the affine hull of $x_1, \ldots, x_d$ is a hyperplane, and $b$ and $b^*$ are on the same side of it. As $b^* \in \text{conv} X_n$, there is an $x_n \in X_n$ on the same side, and then the colourful simplex \(\text{conv} \{x_1, \ldots, x_d, x_n\}\) contains $\varepsilon b$ for some small $\varepsilon > 0$. Hence the segment $[0, \varepsilon b]$ is in $P$ and $\text{int} C^* \cap P$ is nonempty, a contradiction again.

Assume $|I| < d$ for all $I$ in Case 3 and consider the family, $\mathcal{F}$, of transversals of the form

$$T = \{x_{i_1}, \ldots, x_{i_d}, x_n\} \quad \text{with} \ 0 \in \text{conv} T,$$

where $1 \leq i_1 < \cdots < i_d \leq k$. Our target is to show that for some $T \in \mathcal{F}$, $\text{conv} T$ intersects the interior of $C$.

This would finish the proof as follows: Let $x$ be a common point of $\text{conv} T$ and $\text{int} C$. Both $C$ and $\text{conv} T$ contain the origin, so the segment $(0, x)$ is contained in $\text{int} C \cap \text{conv} T$. But $\text{conv} T$ is a colourful simplex, so it is contained in $P$, yet $\text{int} C^*$ should be disjoint from $P$.

For $T \in \mathcal{F}$ let $v(T)$ be the point in $\text{conv} T$ nearest to $b$, so $v(T)$ is on the boundary of $\text{conv} T$. If $b$ is short enough, then the whole segment $[0, v(T)]$ lies on the boundary.
of $\text{conv } T$. This can be reached for all $T \in \mathcal{F}$, if we fix $t \in (0, t_1)$ small enough. Our target is to show that $v(T) \in \text{int } C$ for some $T \in \mathcal{F}$.

Set $w(T) = b - v(T)$, then $v(T)$ and $w(T)$ are orthogonal, and $v(T)$ is the orthogonal projection of $b$ onto the $(d - 1)$-dimensional subspace whose normal is $w(T)$. Moreover, if $\text{conv } T \cap \text{int } C = \emptyset$, then the vector $w(T)$ separates $C$ and $\text{conv } T$, that is, $x \cdot w(T) \leq 0$ for all $x \in \text{conv } T$ and $x \cdot w(T) \geq 0$ for all $x \in C$. Further, if $x \cdot w(T) = 0$ for some $x \in C$, then $x$ lies on a unique extreme ray of the circular cone $C$.

Note further, that $[0, v(T)]$ lies on the boundary of $\text{conv } T$. By Carathéodory’s theorem (cf. [3]) implies the existence of $\{v(T)\}$ with $\{u \in \text{conv } T \mid u \cdot w(T) = \|w(T)\|\}$.

Thus $w(T)$ is the (unique) normal to the cone $C$. Clearly, $T/y_i$ is not longer than $w(T)$ since $v(T) \in \text{conv } (T/y_i)$.

Choose now $S \in \mathcal{F}$ so that

$$\|w(S)\| = \min\{\|w(T)\| : T \in \mathcal{F}\}.$$ 

Observe that $v(S) \neq 0$ or, equivalently, $w(S) \neq b$. Indeed, there is a partial transversal, $\{x_1, \ldots, x_d\}$ say, containing 0 in its convex hull. Further, there is $x_n \in X_n$ with $u \cdot x_n > 0$ as $u \cdot b^* > 0$ and $b^* \in \text{conv } X_n$ where $u$ is the unit vector from the very beginning of this proof. So $T = \{x_1, \ldots, x_d, x_n\} \in \mathcal{F}$ and $v(T)$ is shorter than $b$ as the distance between $b$ and $[0, x_n]$ is shorter than $b$.

**Claim 8.** $v(S) \in \text{int } C$.

**Proof.** Assume the contrary, then $\text{conv } S$ and $C$ are separated by the hyperplane $H = \{x : x \cdot w(S) = 0\}$.

Assume first that $v(S) \notin C$. Let $x_i \in S$ be a vector with $[0, v(S)] \subset \text{conv } (S \setminus \{x_i\})$. Then $C \cap \text{conv } S = \emptyset$ and $b_1 \in C$, $b_1 \cdot w(S) > 0$. Further, $b_1 \in \text{conv } X_i$, so there is a $y_i \in X_i$ with $y_i \cdot w(S) > 0$. But then $\|w(S/y_i)\| < \|w(S)\|$ showing that $v(S) \in C$, or rather, $v(S) \in \partial C$. (For $i = n$ one should take $b$ instead of $b_1$.)

Now with $v(S) \in \partial C$, the assumption $b_1 \cdot w(S) > 0$ leads to the same contradiction. Thus $b_1 \cdot w(S) = 0$ so $w(S)$ is the (unique) normal to the cone $C$ at the point $b_1$. This implies that $\beta b_1 = v(S)$ for some positive $\beta$. Note that this $i$ is unique, otherwise $\gamma b_j = v(S)$ (for some $\gamma > 0$) and then rays $L(b_i)$ and $L(b_j)$ coincide which is impossible by Claim 5. Assume for concreteness that $i = 1$.

Set now $V = S \setminus \{x_1\}$, then $V \subset H$ and $|V| = d$ and the segment $[0, v(S)] \subset \text{conv } V$. Next, $[0, b_1] \subset H \cap P_1$ since $0, b_1 \in P_1$. The set $P_1$ is separated from $C$ by $H$. The set $Y = X_1 \cap H \neq \emptyset$ since $b_1 \in \text{conv } Y$, let $y_1 \in Y$ arbitrary. Note that both $V$ and $Y$ are contained in $H$, which is a copy of $R^{d-1}$.

For every vector $a \in (0, v(S)) \cap (0, b_1)$,

$$a \in \text{conv } V \subset \text{conv } (V \cup \{y_1\}).$$

The last set lies in $H$ and contains $d + 1$ elements. A well-known version of Carathéodory’s theorem (cf. [3]) implies the existence of $v \in V$ such that $a \in \text{conv } (V \cup$
\{y_1\} \setminus \{v\}). The vector \(v\) may depend on \(a\) but it is the same for infinitely many \(a\) while \(a\) tends to 0, say \(v = x_d \in X_d\). Then

\[0, a] \subset \text{conv}(S \cup \{y_1\} \setminus \{x_1, x_d\})

for some \(a\) on the segment \((0, v(S))\). This new partial transversal contains \(v(S)\) in its convex hull, and contains no point of \(X_d\). As \(b_d \cdot w(S) > 0, y_d \cdot w(S) > 0\) for a suitable \(y_d \in X_d\). This gives a new transversal \(S^* = S \cup \{y_1, y_d\} \setminus \{x_1, x_d\}\) with \(w(S^*)\) shorter than \(w(S)\). This final contradiction shows that \(v(S) \in \text{int} C\). \(\square\)

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