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A GENERALIZATION OF CARATHÉODORY'S THEOREM

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The following theorem is proved. If the sets $V_1, \ldots, V_{n+1} \subset \mathbb{R}^n$ and $a \in \bigcap_{i=1}^{n+1} \operatorname{conv} V_i$, then there exist elements $v_i \in V_i$ $(i = 1, \ldots, n+1)$ such that $a \in \operatorname{conv}\{v_1, \ldots, v_{n+1}\}$. This is a generalization of Carathéodory's theorem. By applying this and similar results some open questions are answered.

1. Introduction

The well-known Carathéodory's theorem says that, given a set $V \subset \mathbb{R}^n$ and a point $a \in \operatorname{conv} V$ (the convex hull of V), there exists a subset $A \subset V$ such that $|A| \leq n+1$ and $a \in \operatorname{conv} A$. This simple theorem has many applications and generalizations (see, for instance, [4, 9, 10]).

The aim of this paper is to give a new generalization of Carathéodory's theorem and to present some consequences of this generalization. The paper is organized as follows. The second section contains the main theorems. The third section is about a generalization of Helly's theorem. The fourth section deals with systems of linear inequalities and simple polytopes. The next section answers a question of Boros and Füredi [2]. In the last section we apply our results to convex functions.

2. The main theorems

Theorem 2.1. *Suppose* $V_1, \ldots, V_{n+1} \subset \mathbb{R}^n$ *and* $a \in \operatorname{conv} V_i$ *for* $i = 1, \ldots, n+1$. *Then there exist vectors* $v_i \in V_i$ $(i = 1, \ldots, n+1)$ *such that* $a \in \operatorname{conv}\{v_1, \ldots, v_{n+1}\}$.

This theorem is sharp in the sense that the number of V_i's cannot be decreased. This is shown by the example $V_i = \{e_i, -e_i\}$ $(i = 1, \ldots, n)$ and $a = 0$, where e_i is the ith basis vector of $\mathbb{R}^n$.

Theorem 2.1 does indeed generalize Carathéodory's theorem: put simply $V_1 = V_2 = \cdots = V_{n+1} = V$, then for $A = \{v_1, \ldots, v_{n+1}\}$ we have $|A| \leq n+1$ and $a \in \operatorname{conv} A$. However, while proving the theorem we shall make use of Carathéodory's theorem.

There is a cone-version of Carathéodory's theorem stating that, given a set

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$V \subset \mathbb{R}^n$ and a point $a \in \operatorname{pos} V$ (the convex cone hull of V), there exists a subset A of V such that $|A| \leq n$ and $a \in \operatorname{pos} A$. This cone-version has the following generalization.

Theorem 2.2. *Suppose $V_1, \ldots, V_n \subset \mathbb{R}^n$ and $a \in \operatorname{pos} V_i$ for $i = 1, \ldots, n$. Then there exist elements $v_i \in V_i$ for each i such that $a \in \operatorname{pos}\{v_1, \ldots, v_n\}$.*

Theorem 2.3. *Suppose $V_1, \ldots, V_n \subset \mathbb{R}^n$ and $a \in \operatorname{conv} V_i$ for $i = 1, \ldots, n$. Let v_0 be an arbitrary element of $\mathbb{R}^n$. Then there exists a choice $v_i \in V_i$ $(i = 1, \ldots, n)$ such that $a \in \operatorname{conv}\{v_0, v_1, \ldots, v_n\}$.*

We mention that Theorem 2.1 can be deduced directly from Theorem 2.2. However, when using Theorem 2.3 we can guarantee that for any prescribed element $v \in \bigcup_{i=1}^{n+1} V_i$ these exists a choice $\{v_1, \ldots, v_{n+1}\}$. containing v and such that $a \in \operatorname{conv}\{v_1, \ldots, v_{n+1}\}$.

Corollary 2.4. *Suppose $V_1, \ldots, V_m \subset \mathbb{R}^n$ and $a \in \operatorname{conv} V_i$ for $i = 1, \ldots, m$. Given nonnegative integers $k_1, \ldots, k_m$ with $\sum_{i=1}^m k_i = n + 1$, there exist subsets $A_i \subseteq V_i$, $|A_i| \leq k_i$ such that $a \in \operatorname{conv} \bigcup_{i=1}^m A_i$.*

This corollary contains both Theorem 2.1 (when $m = n + 1$ and $k_1 = \cdots = k_{n+1} = 1$) and Carathéodory's theorem (when $m = 1$ and $k_1 = n + 1$).

Proof of Theorem 2.2. We suppose (by Carathéodory's theorem) that each V_i is finite. Let us define for any choice $v_1 \in V_1, \ldots, v_n \in V_n$

$$d(v_1, \ldots, v_n) = \rho(a, \operatorname{pos}\{v_1, \ldots, v_n\}),$$

or, less formally, d is the distance between the point a and the convex cone $C = \operatorname{pos}\{v_1, \ldots, v_n\}$. We have to prove that $d = 0$ for some choice.

Suppose, to the contrary, that the choice $v_1, \ldots, v_n$ gives the minimal value of d and $d > 0$. Then there exists a (uniquely determined) $z \in C$ with $d = \|a - z\|$. In fact, z is the projection of the point a to the cone C. Clearly, putting $b = a - z$ the hyperplane $\{x \in \mathbb{R}^n : \langle b, x\rangle = 0\}$ separates a and C, i.e.,

$$\langle b, a\rangle > 0, \qquad \langle b, z\rangle = 0$$

and

$$\langle b, v_i\rangle \leq 0 \quad \text{for } i = 1, \ldots, n.$$

The point z can be written as $z = \sum_{i=1}^n \gamma_i v_i$ with $\gamma_i \geq 0$. Moreover, this representation can be chosen so that $\gamma_j = 0$ for some $j = 1, \ldots, n$. This is true either because C is an n-dimensional cone, consequently the minimum of $\|a - x\|$ over $x \in C$ is attained on the boundary of C, or because C lies on a hyperplane and then every point of C can be expressed with some $\gamma_j = 0$ (by Carathéodory's theorem). Without loss of generality we suppose that $\gamma_1 = 0$. The condition $a \in \operatorname{pos} V_1$ implies that there exists an element $v \in V_1$ such that $\langle b, v\rangle > 0$.

Now we show that $d(v, v_2, \ldots, v_n) < d$ contradicting to the minimality of d. Indeed, for $0 \leqslant t \leqslant 1$

$$z + t(v - z) \in \operatorname{pos}\{v, v_2, \ldots, v_n\},$$

and

$$\begin{aligned} d^2(v, v_2, \ldots, v_n) &\leqslant \|a - [z + t(v - z)]\|^2 \\ &= d^2 - 2t\langle b, v - z\rangle + t^2 \|v - z\|^2, \end{aligned}$$

and this is less than d^2 if $t > 0$ is sufficiently small because $\langle b, v - z\rangle = \langle b, v\rangle > 0$.

Proof of Theorem 2.3. Suppose, and we may do so without loss of generality, that $a = 0$. Further, using Carathéodory's theorem we assume that $|V_i| \leqslant n + 1$ for each $i = 1, \ldots, n$.

First, we prove the theorem for the case when $D \in \operatorname{int} \operatorname{con} V_i$ $(i = 1, \ldots, n)$. In this case, clearly, for some small $\varepsilon > 0$ we have

$$-\varepsilon v_0 \in \operatorname{int} \operatorname{conv} V_i,$$

and consequently

$$-\varepsilon v_0 \in \operatorname{pos} V_i \quad (i = 1, \ldots, n).$$

Now, by Theorem 2.2, for some choice $v_i \in V_i$ and $\alpha_i \geqslant 0$

$$-\varepsilon v_0 = \sum_{i=1}^{n} \alpha_i v_i.$$

Dividing here by $\varepsilon + \sum_{i=1}^{n} \alpha_i$ we get

$$0 = \sum_{i=0}^{n} \alpha_i' v_i,$$

where $\alpha_1' \geqslant 0$, and

$$1 = \sum_{i=0}^{n} \alpha_i'.$$

This is clearly the same as $0 \in \operatorname{conv}\{v_0, \ldots, v_n\}$.

In order to prove Theorem 2.3 from this special case one should approximate each set V_i with a sequence of $(n+1)$-membered sets $V_i(j)$ such that $0 \in \operatorname{int} \operatorname{conv} V_i(j)$, and then present a usual continuity argument. We omit the details.

Finally we mention that a good many generalization of Carathéodory's theorem admit further generalizations of the same 'multiplied' version as Theorems 2.1 and 2.2. For instance, S. Dancs observed [3] that Steinitz's theorem [4] can be generalized in this way. An application of Theorem 2.1 can be found in [1]. A further application of Theorem 2.1 is due to A. Frank and L. Lovász [7]:

Theorem. *Let $C_1, \ldots, C_n$ be directed cycles in the directed graph $D(V, A)$ where $|V| = n$. Then there exist arrows $a_1, \ldots, a_n \in A$, the ith belonging to C_i $(i = 1, \ldots, n)$ such that the set of arrows $\{a_1, \ldots, a_n\}$ contains a directed cycle.*

3. Two Helly-type theorems

It is well-known that Carathéodory's theorem and Helly's theorem imply each other. So it is not surprising that Theorem 2.2 yields a Helly-type theorem. This theorem was first observed by L. Lovász [7].

Theorem 3.1. *Let $\mathscr{C}_1, \ldots, \mathscr{C}_{n+1}$ be nonempty families of compact convex sets from $\mathbb{R}^n$ and suppose that for any choice $C_1 \in \mathscr{C}_1, \ldots, C_{n+1} \in \mathscr{C}_{n+1}$ the intersection $\bigcap_{i=1}^{n+1} C_i$ is not empty. Then for some $i = 1, \ldots, n+1$ all the sets in the family $\mathscr{C}_i$ have a point in common.*

Helly's theorem follows from this theorem putting $\mathscr{C}_1 = \cdots = \mathscr{C}_{n+1}$. We postpone the proof to Section 6, although the theorem could be proved right now using the following well-known fact (see [10]).

The compact convex sets C_i $(i = 1, \ldots, p)$ have no point in common if and only if there exist closed half-spaces $D_1, \ldots, D_p$ such that

$$C_i \subset D_i \quad (i = 1, \ldots, p) \quad \text{and} \quad \bigcap_{i=1}^{p} D_i = \emptyset.$$

Using this fact one can even show that Theorem 3.1 implies Theorem 2.2.

We mention that using Theorem 3.1 (or directly 2.2) we can get further 'multiplied' versions of several generalization of Helly's theorem. (For these generalizations see [4] for instance.)

We mention further that Theorem 3.1 can be regarded as a further (though small as it may be) step towards characterizing the possible types of intersection of families of convex sets from $\mathbb{R}^n$. To be more precise let $\mathscr{F} = \{C_1, \ldots, C_p\}$ be a family of convex (compact) sets in $\mathbb{R}^n$. The incidence function of $\mathscr{F}$, F is defined on the subsets of $\{1, \ldots, p\}$ as

$$F(I) = \begin{cases} 1 & \text{for } I \subseteq \{1, \ldots, p\} \text{ if } \bigcap_{i \in I} C_i \neq \emptyset \\ 0 & \text{otherwise.} \end{cases}$$

The question posed in [4] or [6] is to determine the necessary and sufficient conditions on $F: 2^{\{1, \ldots, p\}} \to \{0, 1\}$ so that F be the incidence function of some family C. This question is answered for $n = 1$ only. By Helly's theorem F is determined by its values $F(I)$ for $I\subseteq\{1,\ldots,p\}$, $|I|\leqslant n+1$. Theorem 3.1 says that the incidence function satisfies the following implication. If $I_1,\ldots,I_{n+1}\subseteq\{1,\ldots,p\}$ and $F(\{i_1,\ldots,i_{n+1}\})=1$ for each choice $i_1\in I_1,\ldots,i_{n+1}\in I_{n+1}$, then $F(I_j)=1$ for some $j=1,\ldots,n+1$.

Now we prove a theorem which is related to a result by Berge and Ghouila-Houri (see [4]). This result says that if $C_1,\ldots,C_m\subset\mathbb{R}^n$ are convex compact sets such that $\bigcup_{i=1}^m C_i$ is convex and any $(m-1)$ of these sets have a point in common, then $\bigcap_{i=1}^m C_i$ is nonempty.

Theorem 3.2. *Let $\mathscr{C}_1,\ldots,\mathscr{C}_m$ $(m>1)$ be nonempty families of nonempty sets of $\mathbb{R}^n$. Assume that the sets in $\mathscr{C}_1$ and $\mathscr{C}_2$ are compact. If for every choice $C_i\in\mathscr{C}_i$ $(i=1,\ldots,m)$ the union $\bigcup_{i=1}^m C_i$ is convex, then for some $i=1,\ldots,m$ the intersection $\bigcap\mathscr{C}_i$ is nonempty.*

Proof. First we reduce m to 2 by a backward induction. Suppose that for some choice $C_1\in\mathscr{C}_1,\ldots,C_{m-1}\in\mathscr{C}_{m-1}$ the union $\bigcup_{i=1}^{m-1}C_i$ is not convex. Then every set of $\mathscr{C}_m$ has to contain the 'hole' in $\bigcup_{i=1}^{m-1}C_i$, consequently $\bigcap C_m$ is nonempty. So either we are done or the conditions of the theorem hold for $\mathscr{C}_1,\ldots,\mathscr{C}_{m-1}$. This shows that we have to prove the theorem for $m=2$ only. Put, for brevity $\mathscr{C}_1=\mathscr{A}$ and $\mathscr{C}_2=\mathscr{B}$. The above argument also gives that the sets in $\mathscr{A}$ and $\mathscr{B}$ are convex.

Assume now that $\bigcap\mathscr{A}=\emptyset$. Then $|\mathscr{A}|\geqslant 2$ and there are elements $A_0,A_1,\ldots,A_j$ of $\mathscr{A}$ such that

$$A=A_1\cap\cdots\cap A_j\neq\emptyset\quad\text{but}\quad A_0\cap A=\emptyset.$$

Further, for any $B\in\mathscr{B}$ the set $A_0\cup B$ is convex, and $A\cup B$ is convex, too, because $A\cup B=\bigcap_{i=1}^j(A_i\cup B)$.

Now A_0 and A are disjoint, hence there are open halfspaces H_0 and H (with boundaries L_0 and L) such that $A_0\subset H_0$, $A\subset H$ and $S=\mathbb{R}^n\setminus(H\cup H_0)$ is a strip of positive width.

We are going to show that for any $B\in\mathscr{B}$

$$D=S\cap\operatorname{conv}(A_0\cup A)\subset B.$$

As D is not empty, this will prove that $\bigcap\mathscr{B}\neq\emptyset$.

First we observe that $H_0\cap B$ contains a point b_0 since otherwise A_0 and B would be separated strictly and $A_0\cup B$ could not be convex.

Now consider any point $x\in D$. Clearly $x=ta_0+(1-t)a$ where $a_0\in A_0$ and $a\in A$ and $0<t<1$. The line segment $[b_0,a]$ belongs to $B\cup A$, consequently it meets L in a point $b_1\in B$. Now the line segment $[a_0,b_1]$ belongs to $A_0\cup B$, so it meets L_0 in a point $b_2\in B$. Further the line segment $[b_2,a]$ belongs to $B\cup A$ so it meets L in a point $b_3\in B$ and so on. It is not difficult to check that the segments $[b_0,b_1]$, $[b_2,b_1]$, $[b_2,b_3],\ldots$ tend to $S\cap[a_0,a]$. This shows that B contains a point in every neighbourhood of x, i.e., B is dense in D. But then the compactness of B implies $D\subset B$.

We mention that the same method yields a bit more:

$$\operatorname{conv}(\operatorname{conv}(A_0 \cup A)\setminus(A_0 \cup A)) \subset B.$$

There is a graph-theoretic analogue of the above theorem. A folklore result in graph theory says that given some subtrees $T_1, \ldots T_m$ of a tree T and if any two of them intersect, then $\bigcap_{i=1}^m T_i$ is nonempty. As $T_i \cap T_j \neq \emptyset$ is equivalent to the connectedness of $T_i \cup T_j$, we have the following generalization.

Theorem 3.3. *Let T be a tree and let us given $m\,(\geq 2)$ nonempty families, $\mathscr{G}_1, \ldots, \mathscr{G}_m$, of nonempty subgraphs of T. Suppose that for each choice $G_1 \in \mathscr{G}_1, \ldots, G_m \in \mathscr{G}_m$ the union $\bigcup_{i=1}^m G_i$ is connected (i.e., a tree). Then for some $j = 1, \ldots, m$ the intersection $\bigcap \mathscr{G}_j$ is nonempty.*

The proof is similar to the above one and is omitted.

4. About the face lattice of simple polytopes

The following reformulation of Theorem 2.2 is due to S. Dancs [3].
Consider the linear system

$$Ax = a, \qquad x \geqslant 0 \tag{1}$$

where A is an n by m matrix (with rank n, suppose), $a \in \mathbb{R}^n$ and $x \in \mathbb{R}^m$. For a given solution x to (1) put

$$I(x) = \{j \in \{1, \ldots, m\}: x_j > 0\},$$

i.e., $I(x)$ is the set of indices where the inequality is strict in (1). Let us given $x^1, \ldots, x^n$ feasible solutions to (1). Then, by Theorem 2.2 there is a feasible solution x^0 and there are indices $i_1 \in I(x^1), \ldots, i_n \in I(x^n)$ such that

$$I(x^0) \subseteq \{i_1, i_2, \ldots, i_n\}.$$

The next application of Theorem 2.2 was inspired by the above observation. It is about the face structure of simple polytopes. A d-dimensional polytope is called simple provided every vertex belongs to exactly d facets.

Theorem 4.1. *Let P be a d-dimensional simple polytope having f facets and suppose that we are given $n = f - d$ vertices, $c^1, \ldots, c^n$ of P. Then there exists a vertex, c, and facets, $L_1, \ldots, L_n$ of P such that c and c^i do not lie on L_i for $(i = 1, \ldots, n)$.*

Proof. We can suppose (see [5]) that P is given by the following linear system:

$$Ax = a, \qquad x \geqslant 0 \tag{2}$$

where the vector $a \in \mathbb{R}^n$ and the n by f matrix A are chosen appropriately. The facets of P are given as

$$F_j = \{x \in \mathbb{R}^f : x_j = 0\} \cap P.$$

It is clear that a solution x to (2) is a vertex of P if and only if $I(x) = n$.

Now let $a_i \in \mathbb{R}^n$ be the ith column of the matrix A. Put

$$V_j = \{a_i : i \in I(c^j)\} \quad \text{for } j = 1, \ldots, n.$$

Clearly, $a \in \operatorname{pos} V_j$ for each j. Then by Theorem 2.2 there exists vectors $a_{i_k} \in V_k$ $(k = 1, \ldots, n)$ such that

$$a \in \operatorname{pos}\{a_{i_1}, \ldots, a_{i_n}\}.$$

This can be written as $a = \sum_{k=1}^n x_{i_k} a_{i_k}$ with $x_{i_k} \geq 0$. Put now $c = \sum_{k=1}^n x_{i_k} e_{i_k}$ where e_i is the ith basic vector of $\mathbb{R}^f$. Obviously $c \in P$ and, using the fact that P is simple, it is easy to check that c is a vertex of P. This also implies that each $x_{i_k} > 0$. Putting now $L_k = F_{i_k}$ we are done: $c \notin L_k$ and $c_k \notin L_k$ for $k = 1, \ldots, n$.

The polytope $P = \operatorname{conv}\{e_1, e_2, -e_1, -e_2, e_3\} \subset \mathbb{R}^3$ shows that the theorem does not remain true for non-simple polytopes. From the other hand, using polarity one can state a similar theorem about simplicial polytopes.

5. On the number of covers in the convex hull

Carathéodory's theorem says that for $V \subset \mathbb{R}^n$, $|V| \geq n+1$ the set $\operatorname{conv} V$ is covered by n-dimensional simplices (or n-simplices for short) of V, i.e., by simplices of the type $\operatorname{conv} A$, $A \subseteq V$ and $|A| = n+1$. Now we are interested in the following question. How many times do the n-simplices of V cover the points of $\operatorname{conv} V$? More precisely, let $f(V, x)$ denote the number of n-simplices of V covering the point X and set

$$f(V) = \max_{x \in \mathbb{R}^n} f(V, x).$$

Clearly, $f(V)$ cannot be larger than $\binom{|V|}{n+1}$, the number of n-simplices of V. From the other hand we are going to prove that

$$f(V) \geq c(n) \binom{|V|}{n+1}.$$

This question was raised in [2]. Boros and Füredi showed in [2] that for $V \subset \mathbb{R}^2$, $|V| = N$

$$\frac{2}{9}\binom{N}{3} + O(N^2) \leq f(V),$$

and for $V \subset \mathbb{R}^2$ not containing three points on a line

$$f(V) \leq \frac{1}{4}\binom{N}{3} + O(N^2),$$

and the constants $\frac{2}{9}$ and $\frac{1}{4}$ are the best possible. (In fact, they gave the exact upper bound for $f(V)$.)

Theorem 5.1. *For each* $V \subset \mathbb{R}^n$, $|V| = N$

$$\frac{1}{(n+1)^{n+1}}\binom{N}{n+1} + O(N^n) \leq f(V).$$

Proof. We shall use Tverberg's theorem which says that any set $V \subset \mathbb{R}^n$, $|V| \geq (r-1)(n+1)+1$ can be partitioned into disjoint sets $S_1, \ldots, S_r$ such that $\bigcap_{i=1}^r \operatorname{conv} S_i \neq \emptyset$.

Put now

$$r = \left[\frac{N-1}{n+1}\right] + 1$$

and consider the above partition of V and a point $x_0 \in \bigcap_{i=1}^r \operatorname{conv} S_i$. By Theorem 2.1 (or Corollary 2.4) for any $S_{i_1}, \ldots, S_{i_{n+1}}$ $(1 \leq i_1 < i_2 < \cdots < i_{n+1} \leq r)$ there exists an n-simplex of V, $\operatorname{conv}\{v^1, \ldots, v^{n+1}\}$ containing x_0 where $v^j \in S_{i_j}$ $(j = 1, \ldots, n+1)$. This simplex is clearly different for different index-sets $\{i_1, \ldots, i_{n+1}\}$. This gives

$$\binom{r}{n+1} = \frac{1}{(n+1)^{n+1}}\binom{N}{n+1} + O(N^n)$$

different n-simplices of V containing x_0. We mention that using the remark after Theorem 2.3 one can improve this to

$$f(V, x_0) \geq \frac{1}{(n+1)^n}\binom{N}{n+1} + O(N^n),$$

but I think this is far from being the best possible constant.

Another question of interest is as follows. Let $f_0(V, x)$ denote the number of n-simplices of V containing x in their interior. The question is to determine bounds for

$$f_0(V) = \max_{x \in \mathbb{R}^n} f_0(V, x).$$

In this case, of course, one has to assume that the points of V are in general position, i.e., no $n+1$ of them lie on a hyperplane. If this is so, then any point $x \in \mathbb{R}^n$ lies on the boundary of at most $\binom{|V|}{n}$ n-simplices. Thus Theorem 5.1 implies

$$f_0(V) \geq \frac{1}{(n+1)^{n+1}}\binom{N}{n+1} + O(N^n)$$

if V is in general position. From the other hand we have

Theorem 5.2. *For any $V \subset \mathbb{R}^n$ with $|V| = N$*

$$f_0(V) \leq \begin{cases} \dfrac{2N}{N+n+1} \dbinom{\frac{1}{2}(N+n+1)}{n+1} & \text{if } N-n \text{ is odd,} \\ \dfrac{2(N-n)}{N+n+2} \dbinom{\frac{1}{2}(N+n+2)}{n+1} & \text{if } N-n \text{ is even} \end{cases}$$

and this bound is sharp.

Proof. Let $V_0 \subset \mathbb{R}^n$, $|V_0| = N$ and $x_0 \in \mathbb{R}^n$ be the extremal system, that is,

$$f_0(V_0, x_0) = \max_x f_0(V_0, x) = \max_{|V|=N} \max_x f_0(V, x).$$

It is not difficult to see that here V_0 and x_0 can be chosen so that the set $V_0 \cup \{x_0\}$ is in general position.

Let $V_0 = \{v_1, \ldots, v_N\}$ and considers the polytope P determined by the linear system

$$A\xi = (x_0; 1), \qquad \xi \geq 0,$$

where A is the $(n+1)$ by N matrix with ith column $(V_i; 1)$ and $\xi \in \mathbb{R}^N$. (Here we use the notation $(x; t)$ for the $(n+1)$-dimensional vector whose jth component equals that of $x \in \mathbb{R}^n$ and the last component equals t.) The general position of $V_0 \cup \{x_0\}$ implies that $x_0 \in \operatorname{conv}\{v_{i_1}, \ldots, v_{i_{n+1}}\}$ if and only if for some (uniquely determined) $\xi \in P$, $I(\xi) = \{i_1, \ldots, i_{n+1}\}$. So there is a one-to-one correspondence between the vertices of P and the n-simplices of V containing x_0 in their interior. Now P is $(N-n-1)$ dimensional and has at most N facets.

The question is how many vertices P can have.

McMullen's Upper Bound Theorem [8] gives the exact upper bound of the number of facets of a d-dimensional polytope which has a given number of vertices. (And not only for the number of facets but of the k-dimensional faces as well.) If this theorem is applied to the polar of P, then we get the formula in the theorem.

6. Applications to convex functions

In this last section we are going to prove two theorems concerning families of convex functions $\mathbb{R}^n \to \mathbb{R}$. We use the terminology of the theory of convex functions (see [10]). Thus a function $f : \mathbb{R}^n \to \mathbb{R}$ may take the values $+\infty$ and $-\infty$. For a family $\mathcal{F}$ of functions $\mathbb{R}^n \to \mathbb{R}$ $\bigvee_{f \in \mathcal{F}}$ denotes the largest *convex* function that is not larger than any one of the functions $f \in \mathcal{F}$, and $\bigwedge_{f \in \mathcal{F}}$ denotes the smallest *convex* function that is not smaller than any one of the functions $f \in \mathcal{F}$.

Theorem 6.1. *Let $\mathcal{F}_1, \dots, \mathcal{F}_{n+1}$ be families of functions $\mathbb{R}^n \to \mathbb{R}$ and suppose that for each $i = 1, \dots, n+1$*

$$\Big(\bigwedge_{f\in\mathcal{F}_i} f\Big)(a) < t.$$

Then there are functions $f_i \in \mathcal{F}_i$ $(i = 1, \dots, n+1)$ such that

$$\Big(\bigwedge_{i=1}^{n+1} f_i\Big)(a) < t.$$

Theorem 6.2. *Let $\mathcal{F}_1, \dots, \mathcal{F}_{n+1}$ be finite families of convex functions $\mathbb{R}^n \to \mathbb{R}$ that are finite on the whole space. Suppose that for each $i = 1, \dots, n+1$ and $x \in \mathbb{R}^n$*

$$\Big(\bigwedge_{f\in\mathcal{F}_i} f\Big)(x) \geq 0.$$

Then there exist functions $f_i \in \mathcal{F}_i$ $(i = 1, \dots, n+1)$ such that

$$\Big(\bigwedge_{i=1}^{n+1} f_i\Big)(x) \geq 0 \quad \textit{for each } x \in \mathbb{R}^n$$

Proof of Theorem 6.1. Take $t' \in \mathbb{R}$ so that

$$\Big(\bigvee_{f\in\mathbb{R}_i} f\Big)(a) \leq t' < t \quad \text{for each } i.$$

This implies (see [10]) that there exist functions $f^1, f^2, \dots, f^{n+1} \in \mathcal{F}_i$ and points $(a_j; t_j) \in \mathbb{R}^n \times \mathbb{R}$ with $t_j \geq f^j(a_j)$ $(j = 1, \dots, n+1)$ such that for some $\alpha_i \geq 0, \sum \alpha_i = 1$

$$\sum_{j=1}^{n+1} \alpha_j a_j = a \quad \text{and} \quad \sum_{j=1}^{n+1} \alpha_j t_j = t'.$$

This shows that for $V_i = \{(a_1; t_1), \dots, (a_{n+1}; t_{n+1})\}$

$$(a; t') \in \operatorname{conv} V_i.$$

Now this is true for $i = 1, \dots, n+1$, so by Theorem 2.3 applied to $V_1, \dots, V_{n+1} \subset \mathbb{R}^{n+1}$ and the point $(a; t) \in \mathbb{R}^n \times \mathbb{R}$ we have

$$(a; t') \in \operatorname{conv}\{(a, t), (a_{j_1}; t_{j_1}), \dots, (a_{j_{n+1}}; t_{j_{n+1}})\}$$

where $(a_{j_i}, t_{j_i}) \in V_i$. But then, for the functions $f_{j_1} \in \mathcal{F}_1, \dots, f_{j_{n+1}} \in \mathcal{F}_{n+1}$ we have

$$\Big(\bigwedge_{i=1}^{n+1} f_{j_i}\Big)(a) \leq t' < t.$$

This proof also shows that if the families $\mathcal{F}_I$ are finite, then $(\bigvee_{f\in\mathcal{F}_i} f)(a) \leq t$ $(i = 1, \dots, n+1)$ implies the existence of functions $f_i \in \mathcal{F}_i$ $(i = 1, \dots, n+1)$ with $(\bigvee_{i=1}^{n+1} f_i)(a) \leq t$. This fact is needed in the next proof.

Proof of Theorem 6.2. It is well-known [10] that for a convex function $h:\mathbb{R}^n \to \mathbb{R}$ that is finite over $\mathbb{R}^n$

$$h(x) \geq 0 \quad \text{for each } x \in \mathbb{R}^n$$

if and only if

$$h^*(0) \leq 0,$$

where h^* is the convex conjugate of h. (The convex conjugate of the function $h:\mathbb{R}^n \to \mathbb{R}$ is a (convex) function $h^*:\mathbb{R}^n \to \mathbb{R}$ defined by

$$h^*(y) = \sup_{x \in \mathbb{R}^n} \{\langle x, y\rangle - h(x)\}).$$

Now our assumption is equivalent to

$$\left(\bigwedge_{f \in \mathcal{F}_i} f\right)^*(0) \leq 0, \quad i = 1, \dots, n+1.$$

But $(\bigwedge_{f \in \mathcal{F}_i} f)^* = \bigvee_{f \in \mathcal{F}_i} f^*$ (see again [10]). By the remark at the end of the previous proof we have that there are functions $f_i \in \mathcal{F}_i$ such that $(\bigvee_{i=1}^{n+1} f_i^*)(0) \leq 0$. But this is, again, equivalent to $(\bigwedge_{i=1}^{n+1} f_i)(x) \geq 0$ for each $x \in \mathbb{R}^n$.

Finally we prove the generalization of Helly's theorem from the third section.

Proof of Theorem 3.1. By Helly's theorem we can suppose that each family $\mathcal{C}_i$ contains at most $n+1$ sets. Put now for $i = 1, \dots, n+1$

$$\mathcal{F}_i = \{f_c : C \in \mathcal{C}_i\}$$

where $f_c : \mathbb{R}^n \to \mathbb{R}$ is defined by

$$f_c(x) = \rho(x, C).$$

Suppose that $\bigcap \mathcal{C}_i = \emptyset$ for each i, then

$$\left(\bigwedge_{c \in \mathcal{C}_i} f_c\right)(x) \geq \varepsilon \quad \text{for each } x \in \mathbb{R}^n,\ i = 1, \dots, n+1$$

with some suitable $\varepsilon > 0$. Now Theorem 6.2 implies the existence of a choice $C_1 \in \mathcal{C}_i, \dots, C_{n+1} \in \mathcal{C}_{n+1}$ with

$$\left(\bigwedge_{i=1}^{n+1} f_{c_i}\right)(x) \geq \varepsilon \quad \text{for each } x \in \mathbb{R}^n$$

and this, in turn, shows that $\bigcap C_i = \emptyset$, contradicting to the assumptions.

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