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A Positive Fraction Erdős–Szekeres Theorem*

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Abstract. We prove a fractional version of the Erdős–Szekeres theorem: for any k there is a constant $c_k > 0$ such that any sufficiently large finite set $X \subset \mathbb{R}^2$ contains k subsets $Y_1, \ldots, Y_k$, each of size $\geq c_k|X|$, such that every set $\{y_1, \ldots, y_k\}$ with $y_i \in Y_i$ is in convex position. The main tool is a lemma stating that any finite set $X \subset \mathbb{R}^d$ contains "large" subsets $Y_1, \ldots, Y_k$ such that all sets $\{y_1, \ldots, y_k\}$ with $y_i \in Y_i$ have the same geometric (order) type. We also prove several related results (e.g., the positive fraction Radon theorem, the positive fraction Tverberg theorem).

1. Introduction

The Erdős–Szekeres theorem [ES1] says that among sufficiently many points in general position in the plane one can find k that are in convex position. It is a classical result in combinatorial geometry with a number of generalizations and extensions (see, e.g., [S2] and [EP]). This paper increases this number by one: we prove a fractional version of the Erdős–Szekeres theorem.

A finite set in $\mathbb{R}^d$ is *in general position* if it contains no $d+1$ points lying in a hyperplane. A finite set $Y \subset \mathbb{R}^d$ is *in convex position* if every $y \in Y$ is a vertex of conv Y. Given k sets $Y_1, \ldots, Y_k$, a set $\{y_1, \ldots, y_k\}$ is called a *transversal* of the Y_i, if

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$y_1 \in Y_1, \ldots, y_k \in Y_k$. We write $[n] = \{1, \ldots, n\}$. The fractional version of the Erdős–Szekeres theorem follows:

Theorem 1. *For every integer $k \geq 4$ there is a constant $c_k > 0$ with the following property. Every sufficiently large finite set $X \subset \mathbb{R}^2$ in general position contains k subsets $Y_1, \ldots, Y_k$ with $|Y_i| \geq c_k|X|$ $(i \in [k])$ such that every transversal of the Y_i is in convex position.*

The proof is based on what we like to call the same type lemma. With further applications in mind we present it in colored version and in arbitrary dimension. Two m-tuples $(x_1, \ldots, x_m)$ and $(y_1, \ldots, y_m)$ $(x_i, y_i \in \mathbb{R}^d)$ are said to have *the same* (*order*) *type* if the orientations of the simplices $x_{i_1} \cdots x_{i_{d+1}}$ and $y_{i_1} \cdots y_{i_{d+1}}$ are the same for every $1 \leq i_1 < \cdots < i_{d+1} \leq m$. This is the same as saying that the signs of $\det\left[\binom{x_{i_1}}{1} \cdots \binom{x_{i_{d+1}}}{1}\right]$ and $\det\left[\binom{y_{i_1}}{1} \cdots \binom{y_{i_{d+1}}}{1}\right]$ are equal. Properties of order types have been intensively studied, mainly in relation to computational geometry; a survey on these investigations can be found in [GP1] or in [GP2].

Theorem 2 (Same Type Lemma). *For every two natural numbers d and m there is a constant $c(d, m) > 0$ with the following property. Given finite sets $X_1, \ldots, X_m \subset \mathbb{R}^d$ such that $X_1 \cup X_2 \cup \cdots \cup X_m$ is in general position, there are subsets $Y_i \subset X_i$ with $|Y_i| \geq c(d, m)|X_i|$ such that all transversals of the Y_i have the same type.*

We mention without elaborating that the sets $X, X_1, \ldots, X_m$ in the above theorems could be replaced by probability measures. Then the subsets Y_i would be of measure at least c_k or $c(d, m)$, respectively.

Recently, Theorem 1 was proved for $k = 4$ by Nielsen (personal communication). Solymosi (unpublished) found the following weaker version of Theorem 1: given n points in general position in the plane, one can always choose a sequence of length $c_k n$ from among them such that any k consecutive members of this sequence are in convex position.

The proofs of the above two theorems, followed by a discussion on direct consequences, are given in the next two sections. Related results (e.g., the positive fraction Radon theorem, the positive fraction Tverberg theorem) are described in Section 4.

2. Proof of Theorem 2

It is enough to work with the case $m = d+1$, the theorem would then follow by applying the case $m = d + 1$ to every $(d + 1)$-tuple $X_{i_1}, \ldots, X_{i_{d+1}}$ $(1 \leq i_1 < \cdots < i_{d+1} \leq m)$. So assume $m = d + 1$.

Partition $[d + 1]$ into all possible unordered pairs of (nonempty) subsets: $(I_1, J_1), \ldots, (I_{2^d-1}, J_{2^d-1})$. For any $i \in [d + 1]$, we will find a chain of subsets $X_i = X_i^0 \supset X_i^1 \supset \cdots \supset X_i^{2^d-1} = Y_i$ such that, for all $\alpha \in [2^d - 1]$,

$$|X_i^\alpha| \geq \frac{1}{d+1}|X_i^{\alpha-1}|. \tag{1}$$

We proceed in $2^d - 1$ steps. In step α we find the subsets X_i^α in the following way. Let z_i be the center of $X_i^{\alpha-1}$ in the sense of [DGK], i.e., every open half-space containing z_i contains at least $[1/(d+1)]|X_i^{\alpha-1}|$ points of $X_i^{\alpha-1}$. We may assume that the set $\{z_1, \ldots, z_{d+1}\}$ is in general position, since otherwise we may achieve it by a small perturbation of the sets $X_i^{\alpha-1}$. Consider the hyperplane H_α parallel with $\text{aff}\{z_i\colon i \in I_\alpha\}$ and with $\text{aff}\{z_i\colon i \in J_\alpha\}$ and positioned half-way between them. Write H_α^I and H_α^J for the two half-spaces bounded by H_α so that $H_\alpha^I \supset \text{aff}\{z_i\colon i \in I_\alpha\}$ and $H_\alpha^J \supset \text{aff}\{z_i\colon i \in J_\alpha\}$. Take H_α^I closed and H_α^J open, say. Define

$$X_i^\alpha = \begin{cases} H_\alpha^I \cap X_i^{\alpha-1} & \text{for} \quad i \in I_\alpha, \\ H_\alpha^J \cap X_i^{\alpha-1} & \text{for} \quad i \in J_\alpha. \end{cases}$$

Inequality (1) follows now from the property of the centers z_i. So at the end we have $Y_i = X_i^{2^d-1} \subset X_i$ with

$$|Y_i| \geq (d+1)^{-(2^d-1)}|X_i|. \tag{2}$$

We claim now that every simplex with vertices $y_1 \in Y_1, \ldots, y_{d+1} \in Y_{d+1}$ has the same orientation. Suppose the contrary and let $y_1' y_2' \cdots y_{d+1}'$ be another simplex with a different orientation. Then, for a suitable $t \in (0, 1)$, the points $u_i = t y_i + (1-t) y_i'$ $(i \in [d+1])$ all lie on a hyperplane H. By Radon's theorem [R], applied in H to the points $u_1, \ldots, u_{d+1}$, there is a partition (I, J) of $[d+1]$ with

$$\text{conv}\{u_i\colon i \in I\} \cap \text{conv}\{u_i\colon i \in J\} \neq \emptyset. \tag{3}$$

Now $(I, J) = (I_\alpha, J_\alpha)$ for some α. We have $\text{conv}\{u_i\colon i \in I\} \subset \text{conv} \bigcup \{Y_i\colon i \in I\} \subset \text{conv} \bigcup \{X_i^\alpha\colon i \in I\} \subset H_\alpha^I$ and similarly $\text{conv}\{u_j\colon j \in J\} \subset H_\alpha^J$, a contradiction with (3).

The argument in the last paragraph was used for a different purpose by Goodman *et al.* [GPW]. □

Remark 1. Denote by $c(d, m)$ the infimum of the constants for which Theorem 2 is true. The above proof gives

$$c(d, m) \geq (d+1)^{-(2^d-1)\binom{m-1}{d}}. \tag{4}$$

A slight improvement on (1) and consequently on (2) and (4) comes from using the ham–sandwich theorem instead of the center point theorem.

Remark 2. In the plane, (4) can be improved to

$$c(2, m) \geq \frac{1}{m} 2^{-\binom{m-1}{2}}. \tag{5}$$

To see this observe first that the sets $X_1, \ldots, X_m$ may be reordered so that there are vertical (say) lines $l_0, l_1, \ldots, l_m$ (in this order from left to right) such that X_i has at least $(1/m)|X_i|$ elements between l_{i-1} and l_i. Write X_i' for the set of points of X_i between

l_{i-1} and l_i. Now, for any triple $1 \le p < q < r \le m$, only X'_q has to be separated from X'_p and X'_r (l_p separates X'_p from the other two, and l_q separates X'_r from the other two). This can be reached by a line l that halves X'_p and X'_r simultaneously. l cuts X'_q into two parts. Keep the larger part and half of X'_p and of X'_r on the other side of l.

Remark 3. There is a cone version to the same type lemma. This states, under the same conditions, the existence of $Y_i \subset X_i$, $|Y_i| \ge c'(d,m)|X_i|$ such that

$$\det(y_{i_1}, \dots, y_{i_d})$$

has the same sign for all choices $y_{i_1} \in Y_{i_1}, \dots, y_{i_d} \in Y_{i_d}$. The proof is essentially the same, starting with the case $m = d$. However, as a first step, halve $Y_1, \dots, Y_d$ by a hyperplane and keep those halves that are on the other side to the origin. Then use two partitions of $[d]$ and separating hyperplanes that pass through the origin.

Remark 4. It is clear from the proof that the statement of Theorem 2 is also valid for transversals of the conv Y_i. The same is true in the case of Theorem 1.

Remark 5. With some effort, Theorem 2 can also be proved when $X_1 \cup X_2 \cup \dots \cup X_m$ is not in general position.

Remark 6. It follows from Theorem 2 that for any k and any finite point set X in general position in R^d there exist k positive fraction subsets $X_1, \dots, X_k$ so that the convex hull of every choice is combinatorially the cyclic polytope on k vertices.

3. Proof of Theorem 1

Let $m = m(k)$ be the Erdős–Szekeres number for k. Choose vertical lines $l_0, l_1, \dots, l_m$ (listed from left to right) so that at least $\lfloor (1/m)|X| \rfloor$ points of X lie between l_{i-1} and l_i ($i \in [m]$); denote by X_i the set of these points. Apply the same type lemma to obtain subsets $Y_i \subseteq X_i$ such that all transversals of the Y_i are of the same type and, of course, $|Y_i| \ge c(2,m)|X_i|$ ($i \in [m]$).

For every $i \in [m]$, fix $y_i \in Y_i$. The Erdős–Szekeres theorem implies that some $y_{i_1}, \dots, y_{i_k}$ are in convex position. Then, by the same type lemma, every transversal of the Y_{i_j} is in convex position. □

Remark. Again, write c_k for the infimum of the constants for which Theorem 1 is true. The above proof gives

$$c_k \ge \frac{1}{m(k)} 2^{-\binom{m(k)-1}{2}}$$

which is doubly exponential in k: it is known that $2^k + 1 \le m(k) \le \binom{2k-4}{k-2} + 1$ (see [ES1]

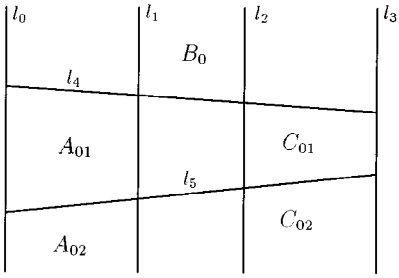

Fig. 1. The regions $A_{01}, A_{02}, C_{01}, C_{02}$.

and [ES2]). For $k = 4$ and 5 we can do better. We give the proof of $c_4 \geq \frac{1}{22}$ and invite the reader to prove or improve $c_5 \geq \frac{1}{352}$.

Proof of $c_4 \geq \frac{1}{22}$. Assume $|X|$ is divisible by 22 and set $|X| = 22n$. Choose vertical lines l_0, l_1, l_2, l_3 (listed from left to right) so that writing A, B, C for the set of points between l_0 and l_1, l_1 and l_2, and l_2 and l_3, respectively, we have $|A| = 10n$, $|B| = 2n$, $|C| = 10n$. The halving line, l_4, of A and C bisects B. Assume at least half of B is above l_4, and denote this subset of B by B_0. Let A_0, C_0 be the half of A, C below l_4, respectively. Take the line l_5 that bisects A_0 into two subsets A_{01}, A_{02}, $|A_{01}| = n$, $|A_{02}| = 4n$, and C_0 into two subsets C_{01}, C_{02}, $|C_{01}| = 3n$, $|C_{02}| = 2n$, as in Fig. 1. Now push the line l_3 toward l_2 and stop when it passed either n points of C_{01} or n points of C_{02} (whichever comes first). Further, halve the set A_{02} by a vertical line. Denote the obtained regions as in Fig. 2. We know that $|A_{01}| = n$, $|A_1| = |A_2| = 2n$, $|B_0| \geq n$, $|C_1| \geq 2n$, $|C_3| \geq n$, and $\max\{|C_2|, |C_4|\} = n$. We now distinguish two possible cases.

Case 1: $|C_2| = n$. The sets A_{01}, B_0, C_2, and C_3 are "convexly independent" sets of size $\geq n$ in this case.

Case 2: $|C_4| = n$. Take the halving line of A_1 and C_1. It bisects A_1, A_2, and C_1 into upper and lower parts to be denoted by A_1^u, A_2^u, C_1^u, and A_1^l, A_2^l, C_1^l. Now either $|A_2^u| \geq n$, in which case A_1^l, A_2^u, C_1^l, C_4 are "convexly independent" of size $\geq n$, or $|A_2^l| > n$, in which case A_1^u, A_2^l, C_1^u, B_0 are "convexly independent" of size $\geq n$. □

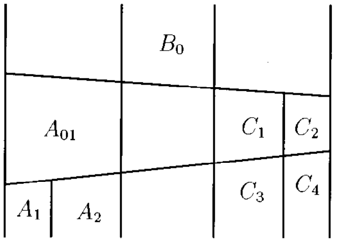

Fig. 2. The regions A_i, C_i.

4. Further Consequences

4.1. *Positive Fraction Radon Theorem*

A simple consequence of the same type lemma is a positive fraction Radon theorem saying that the sets $Y_1, \ldots, Y_m$ obtained have the following property as well. Any $(d+2)$-set $D \subset [m]$ has a two-partition $D = I \cup J$ such that the Radon partition of every set $\{y_i \in Y_i \colon i \in D\}$ is $\{y_i \colon i \in I\} \cup \{y_i \colon i \in J\}$.

The proof is straightforward. The Radon partition is induced by the signs of the coefficients in the affine dependence

$$\sum_{i \in D} \alpha_i y_i = 0, \qquad \sum_{i \in D} \alpha_i = 0.$$

The sign of α_i is just the sign of $\det[\binom{y_j}{1} \colon j \in D \backslash \{i\}]$ which depends only on $D \backslash \{i\}$ (and not on the choice).

4.2. *Positive Fraction Tverberg Theorem*

With a little effort, one can get a positive fraction Tverberg theorem as well. For simplicity, we state it when $m = (d+1)(r-1)+1$. A partition $Z = Z_1 \cup \cdots \cup Z_r$ of a finite set $Z \subset \mathbb{R}^d$ is called a *Tverberg partition* if

$$\bigcap_{i=1}^{r} \operatorname{conv} Z_i \neq \emptyset.$$

Theorem 3. *Assume $d, r \geq 2$, and let $m = (d+1)(r-1)+1$ and $X_1, \ldots, X_m \subset \mathbb{R}^d$. Then there are positive fraction subsets $Y_i \subset X_i$ $(i \in [m])$ and r-partitions $I_1^\alpha \cup \cdots \cup I_r^\alpha$, $\alpha \in [a]$, of $[m]$ (with $a \geq 1$) such that all Tverberg r-partitions of any set of the form $\{y_i \colon i \in [m]\}$ where $y_i \in Y_i$ are $\bigcup_{j=1}^r \{y_i \colon i \in I_j^\alpha\}$, $\alpha \in [a]$.*

Proof. Let $v_1, \ldots, v_r \in \mathbb{R}^{r-1}$ be r vectors such that their only linear dependence is

$$v_1 + \cdots + v_r = 0. \tag{6}$$

For $x \in \mathbb{R}^d$, write $\overline{x} = \binom{x}{1} \in \mathbb{R}^{d+1}$. The tensor product $v_j \otimes \overline{x}$ is an $r-1$ by $(d+1)$ matrix and is regarded as an element of $\mathbb{R}^{m-1}$. Further, let $x_1, x_2, \ldots, x_m \in \mathbb{R}^d$ and $g \colon [m] \to [r]$.

We make use of the following observation [BO] and [S1]: Tverberg partitions of $\{x_1, \ldots, x_m\}$ are in one-to-one correspondence with linear dependences of the form

$$\sum_{i=1}^{m} \alpha_i v_{g(i)} \otimes \overline{x_i} = 0, \qquad \alpha_i \geq 0. \tag{7}$$

To see this assume (7) holds. Then the sets $I_j = \{i \colon g(i) = j\}$ partition $[m]$. We claim that $\bigcap_{j \in [r]} \operatorname{conv}\{x_i \colon i \in I_j\} \neq \emptyset$, i.e., the sets $\{x_i \colon i \in I_j\}$ form a Tverberg partition.

Equation (7) can be written as

$$0 = \sum_{j=1}^{r} v_j \otimes \sum_{i \in I_j} \alpha_i \overline{x_i}.$$

Multiplying from the left by vectors $u^\top \in \mathbb{R}^{r-1}$ orthogonal to $r-2$ of the vectors $v_1, \ldots, v_r$ shows, using (6), the existence of $x \in \mathbb{R}^{d+1}$ with

$$x = \sum_{i \in I_1} \alpha_i \overline{x_i} = \cdots = \sum_{i \in I_r} \alpha_i \overline{x_i}.$$

Checking the last components gives $x_{d+1} = \sum_{i \in I_1} \alpha_i = \cdots = \sum_{i \in I_r} \alpha_i$ so that, indeed,

$$\bigcap_{j=1}^{r} \operatorname{conv}\{x_i \colon i \in I_j\} \neq \emptyset.$$

The argument can be reversed showing that a Tverberg partition gives rise to a linear dependency of the form (7).

Returning to the proof of Theorem 3, consider the rm sets $\{v_j \otimes \overline{x_i} \colon x_i \in X_i\}$, to be denoted by $v_j \otimes X_i$. Choose $k \in [m]$ and a map $g \colon [m] \setminus \{k\} \to [r]$ and apply the proof of the same type lemma (cone version) to the sets $v_{g(i)} \otimes X_i$ $(i \in [m] \setminus \{k\})$ with the following extra requirement. When $v_{g(i)} \otimes X_i^{\alpha-1}$ is to be replaced by the subset $v_{g(i)} \otimes X_i^{\alpha}$, replace $v_j \otimes X_i^{\alpha-1}$ by $v_j \otimes X_i^{\alpha}$ for every $j \in [r]$. Do this for every $k \in [m]$ and every $g \colon [m] \setminus \{k\} \to [r]$. The outcome is positive fraction subsets $Y_i \subset X_i$ $(i \in [m])$ such that for every $k \in [m]$ and every $g \colon [m] \setminus \{k\} \to [r]$ the sign of

$$\det[v_{g(i)} \otimes \overline{y_i} \colon i \in [m] \setminus \{k\}]$$

(where $y_i \in Y_i$) depends only on k and g (and not on the choice of y_i). To finish the proof observe that solutions to (7) are determined by the above determinants. □

4.3. *Tverberg-Type Result on Multicolored Simplices*

Pach [P] used a modification of the same type lemma to prove the following. Given sets $X_1, \ldots, X_{d+1} \subset \mathbb{R}^d$ there are subsets $Y_i \subseteq X_i$ with $|Y_i| \geq C(d)|X_i|$ $(i \in [d+1])$ and a point $p \in \mathbb{R}^d$ such that for every choice $y_i \in Y_i$ $(i \in [d+1])$ the point p lies in $\operatorname{conv}\{y_1, \ldots, y_{d+1}\}$. This was proved in the plane by [BFL] with $C(2) = \frac{1}{12}$ but was not known for $d > 2$.

Here is a sketch of a modified version of Pach's neat argument. (It differs from Pach's proof by applying a different point selection theorem and by applying the same type lemma instead of a weaker separation argument.) Consider the complete $(d+1)$-partite hypergraph $\mathcal{H} = (V, E)$ with vertex set $V = X_1 \cup \cdots \cup X_{d+1}$. The "point selection" theorem of [ABFK] implies the existence of a point $z \in R^d$ and an edge set $E' \subset E$, $|E'| \geq p|E|$, where $p = p(d) > 0$, such that $z \in \operatorname{conv} e$ for each $e \in E'$. By a weak form of the hypergraph version of Szemerédi's regularity lemma (see [KS] or [P] for this particular case), for every $\eta > 0$ there are subsets $Z_i \subset X_i$ with $|Z_i| \geq b(p, \eta)|X_i|$ for all $i \in [d+1]$ (where $b(p, \eta) > 0$ is a constant) such that for every choice of subsets $Y_i \subset Z_i$ with $|Y_i| \geq \eta |Z_i|$, there is an edge $\{y_1, \ldots, y_{d+1}\} \in E'$ with $y_i \in Y_i$. Choose $\eta = c(d, d+2)$ from Theorem 2, and apply Theorem 2 to the sets $Z_0, Z_1, \ldots, Z_{d+1}$ where Z_0 consists of "many" copies of the point z. We get $Y_i \subset Z_i$, $|Y_i| \geq \eta |Z_i|$ $(i = 0, 1, \ldots, d+1)$, such that all transversals of the Y_i have the same type. There is an edge $\{y_1^*, \ldots, y_{d+1}^*\} \in E'$ with $y_i^* \in Y_i$. We have $z \in \text{conv}\{y_1^*, \ldots, y_{d+1}^*\}$, and consequently $z \in \text{conv}\{y_1, \ldots, y_{d+1}\}$ for each choice $y_i \in Y_i$.

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Note added in proof: J. Solymosi found a new and nice proof of Theorem 1 that gives a better constant for c_k as well. His constant is roughly 2^{-16k^2}.