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Reflecting a Triangle in the Plane

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Abstract. We prove that if the three angles of a triangle T in the plane are different from $(60°, 60°, 60°)$, $(30°, 30°, 120°)$, $(45°, 45°, 90°)$,$(30°, 60°, 90°)$, then the set of vertices of those triangles which are obtained from T by repeating 'edge-reflection' is everywhere dense in the plane.

Introduction

An *edge-reflection* of a triangle T_1 is a triangle T_2 which is symmetric to T_1 with respect to the line determined by an edge of T_1 (see Fig. 1). By a *chain of triangles* we mean a sequence of triangles

$$T_1, T_2, T_3, \ldots$$

such that $T_i (i \geq 2)$ is an edge-reflection of T_{i-1}, and $T_i \neq T_{i-2}$ for $i \geq 3$. Two triangles ABC and PQR are *equivalent* to each other if $ABC = PQR$ or there is a finite chain of triangles $T_1, \ldots, T_n$ such that $T_1 = ABC$ and $T_n = PQR$. This is clearly an equivalence relation.

Let us denote by Ω_{ABC} (or simply by Ω) the set of vertices of the triangles equivalent to a given triangle ABC. Figure 2 shows part of Ω for four types of triangles with angles

$$(60°, 60°, 60°), (30°, 30°, 120°),$$

$$(45°, 45°, 90°), (30°, 60°, 90°).$$

We are going to prove that except for the above four types of triangles, Ω is everywhere dense in the plane (Theorems 2 and 3).

The Angles of a Triangle

In this paper all angles are measured by degree (°). A triangle ABC is called *rational* if its three angles are all rational angles, otherwise, ABC is called *irrational*. It is obvious that if ABC is irrational, then at least two angles are irrational.

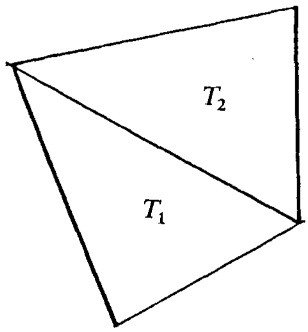

Fig. 1

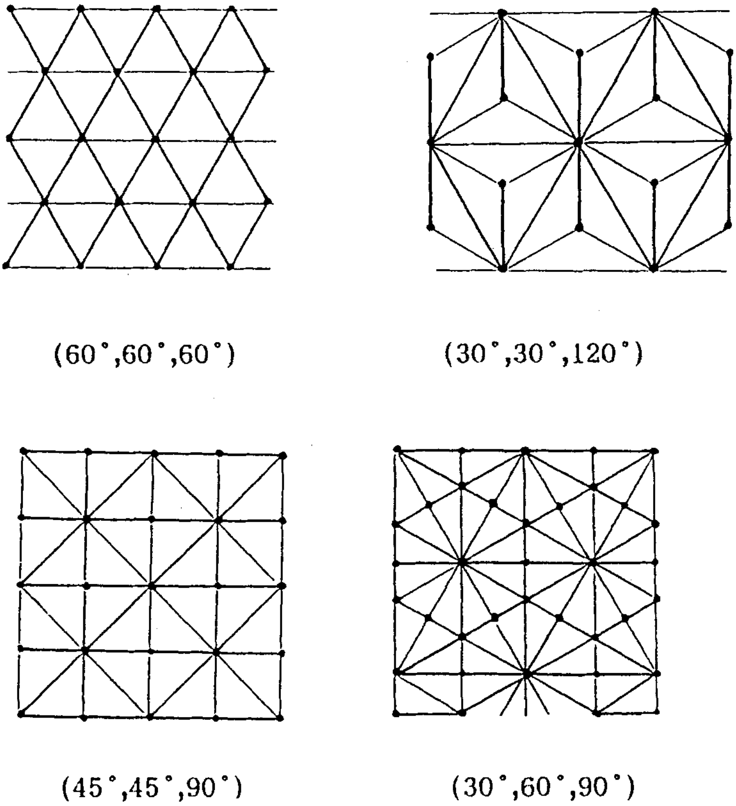

Fig. 2

Let α be a rational angle and m/n be the irreducible fraction equatl to $\alpha/180°$. If m is even, then the angle α is called *even-type*, otherwise it is called *odd-type*. Further, an odd-type angle is called *(odd/odd)-type* or *(odd/even)-type* accordingly as the denominator of the irreducible fraction is odd or even. For example, 30° is (odd/even)-type and 60° is (odd/odd)-type. If 2α is odd-type, then clearly α is (odd/even)-type.

Theorem 1. *Among the three angles α, β, γ of a rational triangle ABC*:

(1) *At least one angle is odd-type.*
(2) *The number of (odd/even)-type angles is $\neq 1$.*
(3) *The number of (odd/odd)-type angles is $\neq 2$.*

Proof. First suppose that the three angles α, β, γ are all even-type. Then since $\alpha/180° + \beta/180° + \gamma/180° = 1$, we have

$$\text{even/odd} + \text{even/odd} + \text{even/odd} = 1$$

which implies even + even + even = odd, a contradiction. Thus (1) follows.

Next, suppose that α is (odd/even)-type, but β, γ are not. Then, from $2\alpha/180° + 2\beta/180° + 2\gamma/180° = 2$, we have

$$\text{odd}/n + \text{even/odd} + \text{even/odd} = 2$$

which implies odd $= n \cdot$ even, a contradiction. Thus (2) follows.

Finally, suppose α, β are (odd/odd)-type, but γ is even-type. Then we have

$$\text{odd/odd} + \text{odd/odd} + \text{even/odd} = 1$$

which implies that even = odd + odd + odd, a contradiction. Thus (3) follows.

□

Corollary 1. *In a rational triangle ABC, one of the following three cases occurs*:

(1) *Two or three angles are (odd/even)-type.*
(2) *One angle is (odd/odd)-type, and the other two are even-type.*
(3) *Three angles are (odd/odd)-type.*

Rational Triangles

Lemma 1. *Let ABC be a rational triangle. Let $AB'C'$ be the triangle symmetric to ABC with respect to the point A, and let $AB''C''$ be the triangle symmetric to $AB'C'$ with respect to the bisector of the angle $\angle A = \alpha$ (see Fig. 3)*:

(1) *If α is (odd/even)-type, then $AB'C'$ is equivalent to ABC.*
(2) *If α is (odd/odd)-type, then $AB''C''$ is equivalent to ABC.*
(3) *If α is even-type, then $AB''C''$ is equivalent to $AB'C'$.*

Proof. Suppose α is (odd/even)-type, i.e., $\alpha/180° = (2m + 1)/(2n)$. Then $2n\alpha = (2m + 1)180° \equiv 180° \pmod{360°}$. Hence, in a chain of triangles

$$ABC, ABC_1, AB_1C_1, AB_1C_2, AB_2C_2, \ldots, AB_nC_n$$

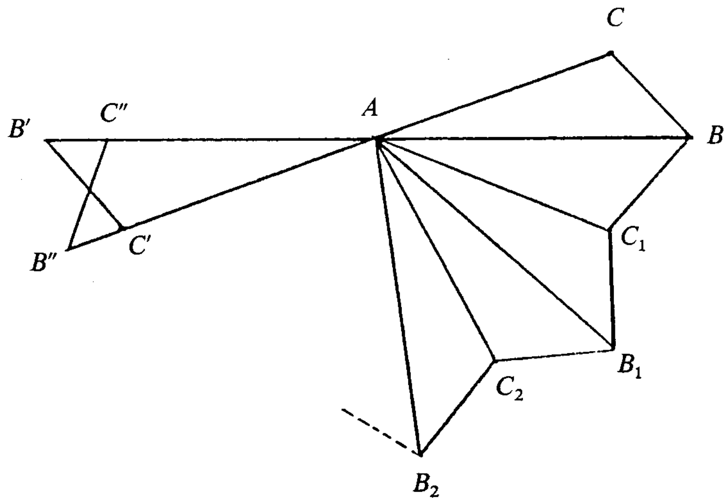

Fig. 3

with common vertex A (Fig. 3), the last triangle AB_nC_n will coincide with $AB'C'$ or $AB''C''$. However, since an even number of edge-reflections results in a congruent triangle of the same 'sense', we must have $AB_nC_n = AB'C'$. This proves (1). Similarly, we can get (2) (3). □

If $\theta = 60°$, $90°$, $120°$, or $180°$, then $\cos\theta = 1/2, 0, -1/2$, or -1. And it is well known that if $|\cos\theta|$ is a rational number other than 0, 1, 1/2, then θ is irrational (see, e.g., Hadwiger-Debrunner [1], problem 8).

Lemma 2. *Let* θ $(0° < \theta \leq 180°)$ *be a rational angle. Then* $\cos\theta$ *is a rational number if and only if* $\theta = 60°, 90°, 120°$, *or* $180°$.

Lemma 3. *For a given angle* α *and a real number* $r > 0$, *let* $A(r, \alpha)$ *denote the set of all points represented by linear combinations of plane-vectors*

$$(r \cdot \cos k\alpha, r \cdot \sin k\alpha), \qquad k = -1, 0, 1, 2,$$

with integral coefficients. If $\cos\alpha$ *is irrational, then* $A(r, \alpha)$ *is everywhere dense in the plane.*

Proof. Suppose that $\lambda := 2\cos\alpha$ is irrational. Since

$$(r \cdot \cos\alpha, r \cdot \sin\alpha) + (r \cdot \cos(-\alpha), r \cdot \sin(-\alpha)) = (\lambda r, 0)$$

$A(r, \alpha)$ contains

$$\{m(\lambda r, 0) + n(r, 0) : m, n \in Z\}$$

Hence the closure $\bar{A}(r, \alpha)$ of $A(r, \alpha)$ contains the x-axis. Similarly, $\bar{A}(r, \alpha)$ contains the line determined by the vector $(\cos\alpha, \sin\alpha)$. Further, since $A(r, \alpha)$ is closed under the addition, we have the lemma. □

In the rest of this section, let ABC be a *rational triangle with vertex A at the* *origin.* We use the same notation $A(r, \alpha)$ as in Lemma 3. The angles and edges of the triangle ABC are denoted by α, β, γ and a, b, c, as usual.

Lemma 4. *If α, β are (odd/even)-type, then Ω contains $A(2c, 2\alpha)$. Hence, if $\alpha \neq 30°$, $45°, 90°$, then Ω is dense in the plane.*

Proof. From Lemma 1(1) and Fig. 4, it follows that Ω contains $A(2c, 2\alpha)$. The latter part follows from Lemmas 2, 3 and $\alpha \neq 60°$. □

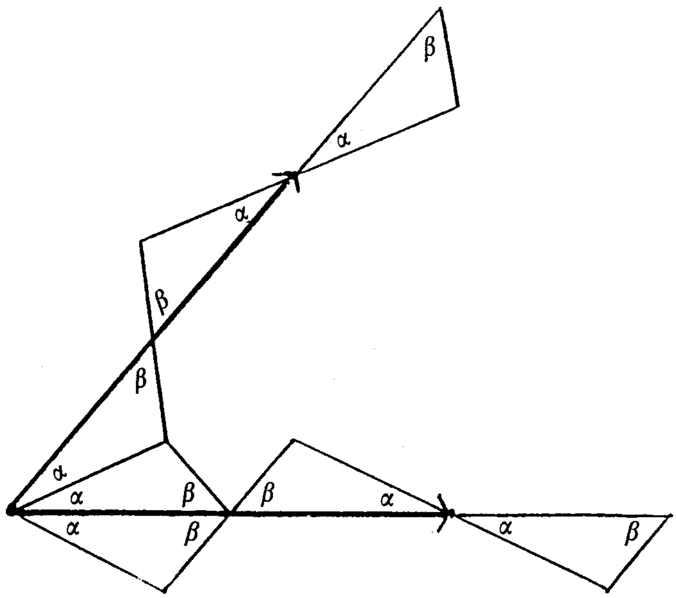

Fig. 4

Lemma 5. *If α, β, γ are all (odd/odd)-type, then Ω contains $A(a + b + c, \alpha)$. Hence, if $\alpha \neq 60°$, then Ω is dense in the plane.*

Proof. From Lemma 1(2) and Fig. 5, Ω contains $A(a + b + c, \alpha)$. The latter part follows from Lemmas 2, 3. □

Lemma 6. *If α, β are even-type and γ is (odd/odd)-type, then Ω contains $A(a + b - c, \alpha)$. Hence Ω is dense in the plane.*

Proof. From Lemma 1(2), (3) and Fig. 6, Ω contains $A(a + b - c, \alpha)$. Since one of α, β is less than 120°, the latter part follows from Lemmas 2,3. □

Theorem 2. *Let ABC be a rational triangle with angles $\alpha \leq \beta \leq \gamma$. If*

$$(\alpha, \beta, \gamma) \neq (60°, 60°, 60°), \qquad (30°, 30°, 120°)$$
$$(45°, 45°, 90°), \qquad (30°, 60°, 90°)$$

then Ω_{ABC} is everywhere dense in the plane.

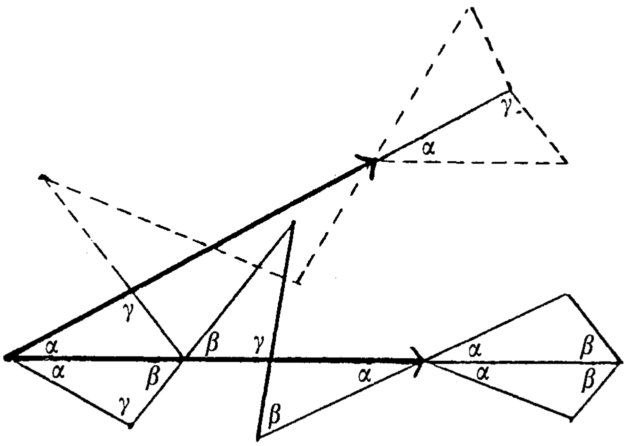

Fig. 5

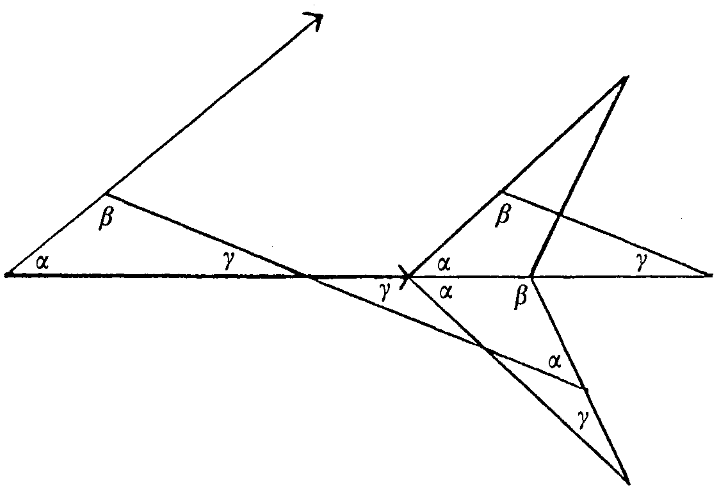

Fig. 6

Proof. Let us consider the three cases (1), (2), and (3) of Corollary 1.

Case (1). If ABC has two (odd/even)-type angles different from 30°, 45°, 90°, then Ω is dense in the plane by Lemma 4. If the two angles are 30°, 45°, then the third angle is 105°, which is (odd/even)-type. Hence, unless $(\alpha, \beta, \gamma) = (30°, 30°, 120°)$, $(45°, 45°, 90°)$, Ω is dense in this case.

Case (2). If ABC has one (odd/odd)-type angle and two even-type angles then Ω is dense in the plane by Lemma 6.

Case (3). If ABC has three (odd/odd)-type angles, and ABC is not equilateral, then Ω is dense in the plane by Lemma 5. □

Irrational Triangles

The following lemma will be obvious.

Lemma 7. *Let*

$$PQR, PQR_1, PQ_1R_1, PQ_1R_2, PQ_2R_2, \ldots$$

be an infinite chain of triangles with common vertex P. Then:

(1) *for any point $X \neq P$, there exists an n such that*

$$\min(\angle XPQ_n, \angle XPR_n) \leq 60°$$

and

(2) *if $\angle QPR$ is irrational, then the closures of the sets*

$$\{Q, Q_1, Q_2, Q_3, \ldots\} \quad \text{and} \quad \{R, R_1, R_2, R_3, \ldots\}$$

are concentric circles with center P.

Lemma 8. *For any irrational triangle PQR and a point $Y \neq P$, there is a triangle PUV equivalent to PQR with irrational angle $\angle U$ and $30° \leq \angle YPU \leq 150°$.*

Proof. Note that an irrational triangle has at least two irrational angles. Hence, if $\angle QPR$ is irrational then the lemma follows from Lemma 7(2). In the case $\angle P$ rational, the two angles $\angle Q$, $\angle R$ are both irrational, whence, applying Lemma 7(1) for a point X such that $\angle XPY = 90°$, we have the lemma. □

Theorem 3. *If ABC is an irrational triangle, then Ω_{ABC} is everywhere dense in the plane.*

Proof. Suppose there is a point Y in the plane for which

$$d := \inf\{|Y - W|: W \in \Omega\}$$

is positive. Then for any $\varepsilon > 0$, there is a triangle PQR which is equivalent to ABC and

$$|P - Y| < d + \varepsilon.$$

By Lemma 8, we may always suppose that $\angle Q$ is irrational and $30° \leq \angle YPQ \leq 150°$. Hence, if ε is sufficiently small, the circle with center Q and radius PQ cuts the circle with center Y and radius d. Therefore, by Lemma 7(2), there is a point P' of Ω with distance $< d$ from Y, a contradiction. □

Remarks

Remark 1. For a triangle $T = ABC$, let $\Phi(T)$ denote the set of triangles obtained from T by repeated reflections. Describing a triangle $T' \in \Phi(T)$ by its vertex $A' \in R^2$, the angle of side $A'B'$ and x-axis, and its orientation, we can identify $\Phi(T)$ with a set in $R^2 \times [0, 360) \times \{-1, 1\}$. Our result asserts that except for the four exceptional cases, the canonical projection of $\Phi(T)$ into R^2 is everywhere dense in R^2.

The same proof gives the following: If T is irrational, then $\Phi(T)$ is everywhere dense in $R^2 \times [0, 360) \times \{-1, 1\}$. If T is rational, then the canonical projection of $\Phi(T)$ onto $[0, 360)$ takes only finitely many values, say $\alpha_1, \alpha_2, \ldots, \alpha_n$. Further, except for the four exceptional cases, $\Phi(T) \cap (R^2 \times \{\alpha_i\} \times \{-1, 1\})$ is everywhere dense in $R^2 \times \{\alpha_i\} \times \{-1, 1\}$ for every i.

Remark 2. In [2], Laczkovich studied the problem of tiling polygons with *similar* triangles. A triangle T is said to tile the polygon P, if P can be decomposed into finitely many non-overlapping triangles similar to T. Among others, he proved that, except right triangles, only three types of triangles with angles

$$(22.5°, 45°, 112.5°), \qquad (45°, 60°, 75°) \qquad \text{or} \qquad (15°, 45°, 120°)$$

can tile the square. Further, among *rational* right triangles, only two types of triangles with angles (45°, 45°, 90°), (15°, 75°, 90°) can tile the square.

Remark 3. A sequence of (at least two) congruent regular tetrahedra in R^3 is called a *tetrahedral snake* if two consecutive tetrahedra share exactly one face, and every three consecutive tetrahedra are distinct. In 1956, Steinhaus posed the question: In a tetrahedral snake of finite length, can the last tetrahedron be a translation of the first one? This problem was solved negatively by Swierczkowski (see Wagon [4], p. 68).

It was proved in [3] that the set of those points which are obtained as the vertices of tetrahedra in tetrahedral snakes starting from a fixed regular tetrahedron is everywhere dense in the space. Analogous results hold in any dimension $n \geq 3$.

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