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On the Number of Convex Lattice Polygons

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We prove that there are at most $\exp\{cA^{1/3}\}$ different lattice polygons of area A. This improves a result of V. I. Arnol'd.

1. Introduction

Two convex lattice polygons are said to be equivalent if there is a lattice preserving affine transformation mapping one of them to the other. This is an equivalence relation. Equivalent polygons have the same area. Let us write $H(A)$ for the number of equivalence classes of convex lattice polygons having area A. Arnol'd [3] proved that

$$c_1 A^{1/3} < \log H(A) < c_2 A^{1/3} \log A \tag{1.1}$$

if A is large enough. Here, and in what follows, $c_1, c_2, \ldots$ denote absolute constants (in the following we will make no effort to make the constants best possible). We will also use Vinogradov's $\ll$ notation. Thus $f(x) \ll g(x)$ means that there are constants c_3 and c_4 such that $f(x) \leq c_3 g(x) + c_4$ for all values of x. With this notation (1.1) says

$$A^{1/3} \ll \log H(A) \ll A^{1/3} \log A.$$

The aim of this paper is to improve the upper bound.

Theorem 1. $\log H(A) \ll A^{1/3}$.

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The constant implied by $\ll$ is not too large: $\log H(A) < 11A^{1/3}$ if A is large enough. This can be established by carrying out the computations explicitly.

Theorem 1 will follow from a result concerning two-dimensional partitions (*cf.* [1]). Given two positive integers a and b, write $N(a,b)$ for the number of sets $V \subset \mathbb{Z}_+^2$ such that $\sum_{v \in V} v \leq (a,b)$. Here $\mathbb{Z}_+^2$ denotes the set of two-dimensional vectors with positive integer components.

Theorem 2. $\log N(a,b) \ll \sqrt[3]{ab}$.

This estimate is exact (apart from the implied constants) when $a \leq b \leq a^2$, and symmetrically, when $b \leq a \leq b^2$. We will obtain a better estimate for the range $a^2 < b$.

Let us denote the number of equivalence classes of d-dimensional convex lattice polytopes of volume A by $H_d(A)$. It follows from the results of [2], [3] (*cf.* [4] and [6]) that

$$A^{(d-1)/(d+1)} \ll \log H_d(A) \ll A^{(d-1)/(d+1)} \log A.$$

We think that the upper bound here can be improved to $\log H_d(A) \ll A^{(d-1)/(d+1)}$. There appear to be several points at which the approach of this paper does not extend to the d-dimensional case. This will soon be apparent to the reader.

2. Further results

Write $\mathcal{P}$ for the set of all convex lattice polygons. Define $U(h,k)$ as the rectangle $\{(x,y) \in \mathbb{R}^2 : 0 \leq x \leq h, 0 \leq y \leq k\}$, where h, k are positive integers. We will need a special element from each equivalence class in $\mathcal{P}$. The following lemma identifies one.

Lemma 3. *For every $P \in \mathcal{P}$ there is a $P_1 \in \mathcal{P}$ equivalent to P such that*

$$P_1 \subset U(h,k)$$

with $hk < 4$ Area P.

A similar fact is proved in [3]: namely, that every $P \in \mathcal{P}$ has an equivalent in the square $U(A,A)$, where $A = 36$ Area P.

Let us use vert P to denote the set of vertices of the polygon P. Arnol'd proves the upper bound in (1.1) by showing that for any $P \in \mathcal{P}$

$$|\text{vert } P| \ll (\text{ Area } P)^{1/3}. \tag{2.1}$$

Several proofs exist for this: Andrews [2] was probably the first; others are by Arnol'd [3], and Schmidt [7]. Here we give a simple proof based on the following:

Lemma 4. *Any convex polygon with n vertices and unit area has three vertices that span a triangle of area $\ll n^{-3}$.*

3. Proof of Theorem 1 using Theorem 2

We begin by proving Lemma 3.

Proof of Lemma 3. Given $u \in \mathbb{Z}^2$, $u \neq 0$, we write $L_u(x)$ for the line parallel to u and passing through x. The line $L_u(z)$ is a lattice line if $z \in \mathbb{Z}^2$. Assume $u \in \mathbb{Z}^2$ is primitive (*i.e.*, its components are relative prime) and let $v \in \mathbb{Z}^2$ be another vector that, together with u, forms a basis of $\mathbb{Z}^2$. Then all lattice lines $L_u(z)$, $z \in \mathbb{Z}^2$, are of the form $L_u(\ell v)$ with ℓ an integer.

Now choose $u \in \mathbb{Z}^2$ in such a way that the number of lattice lines $L_u(\ell v)$ that intersect P is minimal. These lines are $L_u(k_0 v)$, $L_u((k_0+1)v)$, ..., $L_u(k_1 v)$. Set $k = k_1 - k_0$. Clearly $k_0 < k_1$, since otherwise P is contained in a lattice line. Moreover, $L_u(k_0 v)$ and $L_u(k_1 v)$ contain vertices, p_0 and p_1, of P. Now let $L_u(iv)$ be a lattice line parallel with u that has the longest intersection with P. Denote the two endpoints of $L_u(iv) \cap P$ by p_2 and p_3. It is not difficult to see (we leave the details to the reader) that there are parallel supporting lines, $L_z(p_2)$ and $L_z(p_3)$, to P at the points p_2 and p_3. Clearly, $p_3 - p_2 = \alpha u$ for some $\alpha \neq 0$, and we may assume $\alpha > 0$ (exchanging the names of p_2 and p_3 if necessary). As P contains the quadrangle with vertices p_0, p_1, p_2, p_3,

$$\text{Area } P \geq \frac{1}{2} k\alpha.$$

Let us write Q_1 for the parellelogram determined by the four lines $L_u(k_0 v)$, $L_u(k_1 v)$, $L_z(p_2)$, and $L_z(p_3)$. Then $P \subset Q_1$. Write $z = \beta u + \gamma v$, where we assume $\gamma > 0$ (otherwise replace z by $-z$). Define $w = v + \delta u$, where δ denotes the integer nearest to β/γ. It is evident that u, w form a basis of $\mathbb{Z}^2$. Let $L_w(h_0)$, ..., $L_w(h_1)$ be the lattice lines intersecting P and set $h = h_1 - h_0$. Then the choice of u means that $h \geq k$.

Let us write Q for the parallelogram determined by the lines $L_u(k_0 v)$, $L_u(k_1 v)$, $L_w(h_0 u)$, and $L_w(h_1 u)$. As u, w form a basis, Q is a lattice parallelogram. Let $L_w(j_0 u)$, ..., $L_w(j_1 u)$ be the lattice lines that intersect Q_1. Since $P \subset Q_1$, we must have $j_0 \leq h_0$ and $j_1 \geq h_1$. The projection of Q_1 along w on the line $L_u(p_0)$ has length $(j_1 - j_0)\|w\|$ at least. It consists of two pieces: the projections of the two non-parallel sides of Q_1. One of them is simply $p_3 - p_2 = \alpha u$ (in vector form), so its projection has length $\alpha\|u\|$. The other is

$$k\gamma^{-1}z = k(\beta\gamma^{-1}u + v) = k\{(\beta\gamma^{-1} - \delta)u + w\},$$

whose projection has length $k|\beta\gamma^{-1} - \delta|\|u\| \leq k\|u\|/2$. This implies that $(j_1 - j_0)\|u\| \leq \alpha\|u\| + k\|u\|/2$, so

$$k \leq h \leq j_1 - j_0 \leq \alpha + k/2.$$

Then $\frac{1}{2}k \leq \alpha$, so the length of the u-side of Q is $h\|u\| \leq 2\alpha\|u\|$, implying

$$\text{Area } Q = kh \leq 2k\alpha \leq 4 \text{ Area } P.$$

We are almost done. Choose u, w as the basis $(0,1)$, $(1,0)$ of $\mathbb{Z}^2$ and translate the suitable vertex of Q to the origin. With this lattice preserving transformation, P is mapped to an equivalent P_1 and Q is mapped to $U(h,k)$. □

We now turn to the proof of Theorem 1.

Proof of Theorem 1. From each equivalence class, fix P, which is contained in $U(h,k)$ according to Lemma 3. We know from the proof that P has common points with all four sides of $U(h,k)$.

Let the vertices of P be $p_0, p_1, \dots, p_n$ (where $p_0 = p_n$) in anticlockwise order. We choose p_0 so that it is the rightmost point of P on the line $y = 0$. Let p_j be the first vertex with x-component equal to h. Then the sum of the vectors $(p_1-p_0)+(p_2-p_1)+\cdots+(p_j-p_{j-1}) \le (h,k)$, where this inequality is understood componentwise. Set $v_i = p_i - p_{i-1}$, for $i = 1, \dots, j$. The set of vectors $V = \{v_1, \dots, v_j\}$ uniquely determines the shape of P in the "South-East" corner of $U(h,k)$, and different shapes determine different set of vectors. (Actually, two sets of positive vectors may determine the same shape.) Obviously V consists of distinct positive integer vectors and satisfies $\sum_{v\in V} v \le (h,k)$. The number of such sets V is at most $N(h,k)$. The same estimate holds for the North-East, North-West, and South-West corners of $U(h,k)$ as well. Finally, there is at most one edge of P on each side of the rectangle $U(h,k)$, and the number of ways of choosing them is at most h^4k^4. So the number of convex lattice polygons in $U(h,k)$ that touch each side of it is at most $h^4k^4(N(h,k))^4$. Then the number of equivalence classes with area A is

$$H(T) \le \sum_{hk\le 4A} h^4k^4(N(h,k))^4.$$

By Theorem 2, every term here is at most $(4A)^4 \exp 4cA^{1/3}$. The number of terms is obviously $A \log A$. This proves the Theorem. □

4. Proof of Theorem 2

Proof. By symmetry, we assume that $a \le b$. We have to consider two different cases: when $a \le b \le 2a^2$ and when $2a^2 \le b$. The behaviour of $N(a,b)$ is different in each case.

Case 1: $a \le b \le 2a^2$. We assume that a divides b, since otherwise we replace b by the smallest multiple of a that is larger than b. Define $\ell(z) = xa^{-1} + yb^{-1}$, where $z = (x,y) \in \mathbb{R}^2$. For $t = 1, \dots, a$, set

$$S_t = \{z \in \mathbb{Z}_+^2 : ta^{-1} < \ell(z) \le (t+1)a^{-1}\}.$$

It is easy to see that the number of points in S_t is $M_t = tb/a$. This is where we use the fact that a divides b. It is also clear that $V \subset \mathbb{Z}_+^2 \subset \bigcup_{t=1}^{\infty} S_t$.

Now we count the number of sets $V = \{v_1, \dots, v_n\} \subset \mathbb{Z}_+^2$ satisfying $\sum_{i=1}^n v_i \le (a,b)$. Assume V has m_t vectors in S_t. Since, for $z \in S_t$, $\ell(z)$ is between ta^{-1} and $(t+1)a^{-1}$, we get

$$\sum_{t=1}^{a} m_t t a^{-1} \le \sum_{i=1}^{n} \ell(v_i) = \ell(\sum_{i=1}^{n} v_i) \le \ell(a,b) = 2.$$

So we have

$$\sum_{t=1}^{a} m_t t \le 2a. \tag{4.1}$$

The number of ways to choose $m_1, \ldots, m_a$ from $S_1, \ldots, S_a$ is $\prod_{t=1}^{a} \binom{M_t}{m_t}$. Consequently

$$N(a,b) \leq \sum \prod_{t=1}^{a} \binom{M_t}{m_t}, \tag{4.2}$$

where the summation is taken over all integers $m_t \geq 0$ that satisfy (4.1).

Claim 5. *Under conditions* (4.1)

$$\log \prod_{t=1}^{a} \binom{M_t}{m_t} \ll \sqrt[3]{ab}.$$

The proof is rather routine, so we postpone it until the final section. It is this proof, however, that reveals why $N(a,b)$ behaves differently in the two cases.

It follows that every term in (4.2) is at most $\exp\{c\sqrt[3]{ab}\}$. The number of terms is the number of possible choices of nonnegative integers $m_1, \ldots, m_a$ satisfying (4.1). This is the same as the number of partitions of all the numbers less than or equal to $2a$. It is well known (see [5] for instance) that this number is $\exp c\sqrt{2a}$. So we get

$$\log N(a,b) \ll \sqrt{a} + \sqrt[3]{ab} \ll \sqrt[3]{ab}.$$

Case 2: $2a^2 \leq b$. We are going to estimate the number of sets $V \subset \mathbb{Z}_+^2$ such that $\sum_{v \in V} v \leq (a,b)$. Let $V = \{v_1, \ldots, v_n\}$, where the vectors $v_i = (x_i, y_i)$ are indexed so that $0 < y_1 \leq \ldots \leq y_n$. Clearly, given $y_1, \ldots, y_n$, the integers $x_1, \ldots, x_n \in \{1, \ldots, a\}$ can be chosen in at most

$$\binom{a+n-1}{n} < \binom{2a}{a} < 4^a$$

different ways, since $n \leq \sum_{i=1}^{n} x_i \leq a$.

Let $P(b,a)$ denote the number of partitions of b into at most a positive summands. Obviously, the sequence $0 < y_1 \leq \ldots \leq y_n$ can be chosen in at most $P(b,a)$ different ways. To estimate $P(b,a)$, we are going to use the following asymptotic formula due to Szekeres [8]. Define

$$d = b - a(a+1)/4, \quad \text{and} \quad \alpha = (a+1/2)^2/d.$$

The function $r(\alpha)$ is the inverse of

$$\begin{aligned} \alpha(r) &= r^2 \left(\int_0^r (s/2) \coth(s/2) ds \right)^{-1}, \quad \text{i.e.,} \\ r(\alpha) &= \alpha + \frac{1}{36}\alpha^3 + \frac{41}{32400}\alpha^5 + \cdots, \end{aligned}$$

which is valid for $|\alpha| < 4$. Then, Szekeres's result says

$$\begin{aligned} \log P(b,a) &= a\Big(2\frac{r(\alpha)}{\alpha} - \log[2\sinh(r(\alpha)/2)]\Big) - \log d + \frac{r(\alpha)}{\alpha} \\ &\quad -\frac{1}{2}\log\left(\frac{\sinh(r(\alpha)/2)}{r(\alpha)/2}\right) + \frac{1}{2}\log r'(\alpha) - \log(2\pi) + O(a^{-1}) \end{aligned}$$

uniformly for $\alpha < 2.598\ldots$. Here,

$$2\frac{r(\alpha)}{\alpha} - \log[2\sinh(r(\alpha)/2)] \leq 2.5 - \log\alpha$$

when $\alpha \leq 1$, say, and the terms after $-\log d$ are bounded. So for a large enough,

$$\log P(b,a) \leq a(3 - \log\alpha) - \log d.$$

Now, $1 \geq \alpha = (a+1/2)^2/d$ is the same as $b \geq (a+1/2)^2 + a(a+1)/4$, which follows from the $b \geq 2a^2$ condition. Moreover, $\alpha \geq a^2/b$. So for a large enough, we get

$$\begin{aligned}\log P(b,a) &\leq 3a + a\log(b/a^2) - \log d \\ &\leq \sqrt[3]{ab}\left[3\sqrt[3]{a^2/b} + \sqrt[3]{a^2/b}\log(b/a^2)\right] < 3\sqrt[3]{ab},\end{aligned}$$

since on substituting $s = \sqrt[3]{a^2/b}$, the expression in $[\cdots]$ is equal to $3s(1-\log s)$, which is less than 3 when $0 < s < 1/\sqrt[3]{2}$.

So we get

$$\log N(a,b) \leq \log_4 a + 3\sqrt[3]{ab} \ll \sqrt[3]{ab}.$$ □

5. Proof of Lemma 4 and (2.1)

Proof of Lemma 4. Let Q be the convex polytope with n vertices and unit area. We assume that the Löwner–John ellipsoid of Q is a circle. This can be achieved by an area-preserving linear transformation. It is easy to see, then, that Q is contained in a circle of radius 1. As the perimeter of Q is at most 2π, 90 percent of its edges have length at most $20\pi/n \ll n^{-1}$. Since the sum of the outer angles of Q is 2π, 90 percent of them are $\ll n^{-1}$. Then there are two consecutive "short" edges with the outer angle between them $\ll n^{-1}$, so the triangle spanned by these edges has area $\ll n^{-3}$. □

Remark. A sharper form of this Lemma follows from a result of Rényi and Sulanke [6], which says that among all convex polygons with n vertices and of unit area, the geometric mean of the areas of the n triangles spanned by consecutive triplets of vertices is maximal for the (affine) regular n-gon. The proof above does not give such an exact estimate, although it shows the existence of "many" triangles of area $\ll n^{-3}$.

Proof of (2.1). Let P be a convex lattice polytope with $|\text{vert } P| = n$ vertices. Lemma 4, applied to P, says that some three (consecutive) vertices of P span a triangle $\triangle$ with "relative" area $\ll n^{-3}$, *i.e.*,

$$\frac{\text{Area }(\triangle)}{\text{Area }(P)} \ll n^{-3}.$$

On the other hand, any lattice triangle has area at least 1/2. This shows

$$n = |\text{vert } P| \ll (\text{ Area } P)^{1/3}.$$ □

6. Proof of Claim 5

Proof. Observe that

$$\binom{M}{m} \le \frac{M^M}{m^m(M-m)^{M-m}},$$

where $0^0 = 1$. Replace the integer variable m_t by the real variable $s_t \ge 0$. Now we want to estimate the maximum of

$$\prod_{t=1}^{a} \frac{M_t^{M_t}}{s_t^{s_t}(M_t - s_t)^{M_t - s_t}} \tag{6.1}$$

under the conditions

$$\sum_{t=1}^{a} t s_t \le 2a \text{ , and } \; s_t \ge 0. \tag{6.2}$$

Write

$$f(s) = -\sum_{t=1}^{a} s_t \log s_t + (M_t - s_t)\log(M_t - s_t),$$

where s stands for the vector $(s_1, \ldots, s_a)$. $f(s)$ is just the logarithm of the product in (6.1) minus a constant. We want to solve the following conditional extremum problem: maximize f subject to (6.2). Denote (one of) its solutions by s. We check first that none of the s_t is zero. Assume that $s_i = 0$ and choose an $s_j \neq 0$. Define s' by $s'_i = s_i + \varepsilon j$, $s'_j = s_j - \varepsilon i$, and $s'_t = s_t$ otherwise. This s' is feasible. Set $F(\varepsilon) = f(s')$. By the mean value theorem (even though F is not differentiable at 0),

$$\frac{F(\varepsilon) - F(0)}{\varepsilon} = F'(\theta\varepsilon) = j \log \frac{M_i - j\varepsilon\theta}{j\varepsilon\theta} - i \log \frac{M_j - s_j + i\varepsilon\theta}{s_j - i\varepsilon\theta},$$

where $0 < \theta < 1$. Since the last expression tends to infinity when ε goes to zero, we get a contradiction.

Now we know that $s > 0$. If s is in the interior of the feasible region, the gradient of f at s is 0. Thus $s_t = M_t/2$ for all t, which contradicts condition (6.2) if $a \le b$ and a is large enough.

Then s satisfies

$$\sum_{t=1}^{a} t s_t = 2a, \tag{6.3}$$

and there is $\lambda > 0$ such that

$$\log \frac{M_t - s_t}{s_t} = \lambda t \text{ , i.e., } \; s_t = \frac{M_t}{1 + e^{\lambda t}},$$

for all $t = 1, \ldots, a$. The number λ will be determined, or rather estimated, from (6.3), which says

$$\sum_{t=1}^{a} \frac{t^2}{1 + e^{\lambda t}} = 2a^2/b. \tag{6.4}$$

The left-hand side is monotone increasing in λ. At $\lambda = 0$ it is larger than the right-hand side, while it is 0 at infinity. So there is a unique solution λ_0 to (6.4). We now show that

$$\lambda_0 > \lambda_1 := \sqrt[3]{\frac{b}{1.6a^2}}. \tag{6.5}$$

Notice that $\lambda_1 \leq \sqrt[3]{1.25} = 1.0772\ldots$. The function $t^2/(1+e^{\lambda t})$ takes its maximal value on $[0,\infty)$ when $(\lambda t - 2)e^{\lambda t} = 2$, *i.e.*, $\lambda t = 2.217\ldots$. Set $t_1 = 2.217\ldots/\lambda_1$. We show that the left-hand side of (6.4) at λ_1 is larger than the right-hand side. This will prove (6.5). The function $t^2/(1+e^{\lambda_1 t})$ increases in $[0, t_1]$ and decreases afterwards. Thus

$$\sum_{t=1}^{a} t^2/(1+e^{\lambda_1 t}) > \int_0^{a+1} \frac{t^2 dt}{1+e^{\lambda_1 t}} - \frac{t_1^2}{1+e^{\lambda_1 t_1}} = \lambda_1^{-3}\left(\int_0^{\lambda_1(a+1)} \frac{t^2 dt}{1+e^t} - \lambda_1 \frac{(\lambda_1 t_1)^2}{1+e^{\lambda_1 t_1}}\right)$$
$$> \frac{1.6a^2}{b}\left(\int_0^{\lambda_1 a} \frac{t^2 dt}{1+e^t} - 1.0772\ldots 0.483\ldots\right) > 2a^2/b$$

if a is large enough and $a \leq b \leq 2a^2$, since the last integral tends to $1.80305\ldots$ as a goes to infinity. This proves (6.5).

Next we estimate (6.1) with m_t replaced by $s_t = M_t/(1+e^{\lambda_0 t})$.

$$\log \prod_{t=1}^{a} \frac{M_t^{M_t}}{s_t^{s_t}(M_t - s_t)^{M_t - s_t}} = \sum_{t=1}^{a} M_t \left(\frac{\log(1+e^{\lambda_0 t})}{1+e^{\lambda_0 t}} + \frac{\log(1+e^{-\lambda_0 t})}{1+e^{-\lambda_0 t}}\right)$$
$$\leq \sum_{t=1}^{a} M_t \left(\lambda_0 t e^{-\lambda_0 t} + e^{-\lambda_0 t}\right) = ba^{-1}\sum_{t=1}^{a} t(1+\lambda_0 t)e^{-\lambda_0 t}.$$

The function in the last sum increases in $t \in [0, (1+\sqrt{5})/(2\lambda_0)]$ and decreases afterwards. We continue the last formula using (6.5)

$$\leq ba^{-1}\lambda_0^{-2}\left(\int_0^{\lambda_0(a+1)} x(x+1)e^{-x}dx + \lambda_0 \max_{x\in[0,\infty)} x(x+1)e^{-x}\right) \ll ba^{-1}\lambda_1^{-2} \ll \sqrt[3]{ab},$$

if a is large enough. □

References

[1] Andrews, G. E. (1976) *The Theory of Partitions*, Addison-Wesley.

[2] Andrews, G. E. (1965) A lower bound for the volumes of strictly convex bodies with many boundary points. *Trans. Amer. Math. Soc.* **106** 270–279.

[3] Arnol'd, V. I. (1980) Statistics of integral lattice polygons (in Russian). *Funk. Anal. Pril.* **14** 1–3.

[4] Konyagin, S. B. and Sevastyanov, K. A. (1984) Estimation of the number of vertices of a convex integral polyhedron in terms of its volume. *Funk. Anal. Pril.* **18** 13–15.

[5] Rademacher, G. (1973) *Topics in Analytic Number Theory*, Springer.

[6] Rényi, A. and Sulanke, R. (1933) Über die konvexe Hülle von n zufällig gewählten Punkten. *Z. Wahrscheinlichkeitstheorie verw. Gebiete* **x 3** 75–84.

[7] Schmidt, W. (1985) Integer points on curves and surfaces. *Monatshefte Math.* **99** 45–82.

[8] Szekeres, G. (1951) On the theory of partitions. *Quarterly J. Math. Oxford, Second Series* **2** 85–108.