In this note we prove a conjecture of Bisztriczky and Schaer [1] about convex sets in the real projective plane $\mathbb{P}^2$. It will be simpler to formulate the result for convex cones in $\mathbb{R}^3$ and then show that it implies the conjecture. A cone $C \subset \mathbb{R}^3$ is called point-
ed if it contains no line, i.e., when $x \in C$ and $-x \in C$ imply $x=0$. Here is the result:

**Theorem 1.** Assume $n \geq 3$ and $C_1, \ldots, C_n \subset \mathbb{R}^3$ are closed, pointed, convex cones with common apex the origin $O$. Assume that for $i \neq j$ ($i,j=1,2,\ldots,n$) there is an $e(i,j) \in \{ -1, +1 \}$ such that for all $k=1,\ldots,n$, $k \neq i,j$ and for both $e=1,-1$

$$(i,j;k,e) \quad (eC_k) \cap (C_1+e(i,j)C_j) = \{ O \}.$$

Then there is a plane $P$ through $O$ such that for all $i=1,\ldots,n$, $P \cap C_i = \{ O \}$.

We will now translate this theorem from $\mathbb{R}^3$ to $\mathbb{P}^2$. For a convex pointed cone $C \subset \mathbb{R}^3$ set $S(C) = S^2 \cap C$ where $S^2$ is the unit sphere of $\mathbb{R}^3$. $\mathbb{P}^2$ is obtained from $S^2$ by identifying antipodal points. With this identification the points of $S(C)$ and $-S(C)$ give rise to a set $P(C) \subset \mathbb{P}^2$. Clearly, $P(C) = P(-C)$.

A set $A \subset \mathbb{P}^2$ is called convex if there exists a line $L$ in $\mathbb{P}^2$ disjoint from $A$ and $A$ is convex in the affine plane $\mathbb{P}^2 \setminus L$ (cf. [2] or [1]). A convex set $A$ in $\mathbb{P}^2$ gives rise to two connected subsets $S^+(A)$ and $S^-(A) = -S^+(A)$ of $S^2$, whose cone hulls are $C^+(A)$ and $C^-(A)$, respectively. Evidently, $C^+(A) = -C^-(A)$. In this way one can see that $A \subset \mathbb{P}^2$ is convex if and only if $A = P(C)$ for some pointed convex cone $C \subset \mathbb{R}^3$.

Now let $A_1, A_2 \subset \mathbb{P}^2$ be convex. We want to define the convex hull of their union. Then $A_j = P(C_j)$ for some pointed convex cone $C_j \subset \mathbb{R}^3$ and also $A_j = P(-C_j)$ ($j=1,2$). So the union of $A_1$ and $A_2$ will have, in general, two convex hulls: $H_1(A_1,A_2) = P(\text{conv}(C_1,C_2))$ and $H_2(A_1,A_2) = P(\text{conv}(C_1,-C_2))$. Of course, $H_1$ and $H_2$ will be convex only if $C_1-C_2 = \text{conv}(C_1,-C_2)$ and $C_1+C_2 = \text{conv}(C_1,C_2)$ are pointed cones.

We can now formulate Theorem 1 in $\mathbb{P}^2$.

**Theorem 2.** Let $A_1, \ldots, A_n$ be closed convex sets in $\mathbb{P}^2$ ($n \geq 3$). Assume that for $i \neq j$ ($i,j=1,\ldots,n$) either $A_k \cap H_1(A_i,A_j) = \emptyset$ for all $k \neq i,j$ or $A_k \cap H_2(A_i,A_j) = \emptyset$ for all $k \neq i,j$. Then there is a line $L \subset \mathbb{P}^2$ disjoint from each $A_i$.

In [1], the collection of the sets $A_1, \ldots, A_n$ is called affinely embeddable when the conclusion of Theorem 2 holds.

In the proof of Theorem 1 we will use standard techniques from the theory of convex cones in finite dimensional spaces (cf. [3], [4] or [5]).
When proving Theorem 1 we will obtain its dual form which seems to be worth mentioning:

**THEOREM 3.** Assume $D_1, \ldots, D_n \subseteq \mathbb{R}^3$ $(n \geq 3)$ are closed, pointed, convex cones with common apex the origin. Suppose that for $i \neq j$ $(i, j = 1, \ldots, n)$ there is an \( e(i, j) \in \{-1, +1\} \) such that for all $k = 1, \ldots, n$, $k \neq i, j$ and for both $e = 1$ and $-1$ $(eD_i) \cap D_j \cap (e(i, j)D_j) \neq \{O\}$. Then there are signs $e_1, \ldots, e_n$ $(e_i = +1$ or $-1)$ and a vector $p \in \mathbb{R}^n \setminus \{O\}$ such that $p \in e_iD_i$ for all $i = 1, \ldots, n$.

**PROOF OF THEOREM 1.** Assume the theorem is false and take a counterexample $C_1, \ldots, C_n \subseteq \mathbb{R}^3$ of closed, convex, pointed cones satisfying condition $(i, j; k, e)$ such that for all planes $P$ through the origin there is an $i \in \{1, \ldots, n\}$ with $P \cap C_i \neq \{O\}$.

We will modify this counterexample. We **claim** first that for $i \neq j$ both $C_i + C_j$ and $C_i - C_j$ are pointed and closed convex cones. We prove this for $C_i + C_j$, the proof for $C_i - C_j$ is identical. By condition $(i, k; j, -1)$

\[
(C_i + C_j) \cap C_l \subseteq \{C_l + e(i, k)C_k\} = \{O\},
\]

so $C_l$ and $(-C_j)$ can be separated (strictly, because they are closed), i.e., there exists $v \in \mathbb{R}^3$ such that $v \cdot y < 0$ for all $y \in C_l \setminus \{O\}$ and $v \cdot y > 0$ for all $y \in (-C_j) \setminus \{O\}$. (Here $v \cdot x$ denotes the scalar product of $v, x \in \mathbb{R}^3$.) Then $v \cdot z < 0$ for all $z \in (C_i + C_j) \setminus \{O\}$ proving that $(C_i + C_j)$ is pointed.

Now we prove that $C_i + C_j$ is closed. Assume it is not, then there are elements $x_m \in C_i$ and $y_m \in C_j$ with $x_m, y_m \in S^2$ and positive numbers $\alpha_m, \beta_m$ such that $z_m = -x_m x_m + \beta_m y_m$ is in $(C_i + C_j) \cap S^2$ but $z - \lim z_m$ is not. By the compactness of $S^2$ we may assume that $x = \lim x_m$ and $y = \lim y_m$ exists. Then $\alpha_m$ and $\beta_m$ must tend to infinity and so $z_m \in S^2$ is possible only if $x + y = 0$. This implies that $C_i + C_j$ contains the line through $x$ and $-x = y$ which is impossible because it is a pointed cone.

We define, for a closed pointed cone $C \subseteq \mathbb{R}^3$ and for $\alpha > 0$ the set

\[
C^\alpha = \{x \in \mathbb{R}^3 : \text{there is } y \in C \text{ with } \langle x, Oy \rangle = \alpha\},
\]

where $\langle x, Oy \rangle$ denotes the angle of the triangle $xOy$ at vertex $O$. $C^\alpha$ is clearly a convex, pointed cone with nonempty interior provided $\alpha$ is small enough.

Condition $(i, j; k, e)$ says that the two closed and pointed cones $C_i + e(i, j)C_j$ and $eC_k$ are disjoint (except for the common apex). Then there is $\alpha(i, j; k, e) > 0$ such that for $0 < \alpha < \alpha(i, j; k, e)$

\[
(eC_k^\alpha) \cap (C_i^\alpha + e(i, j)C_j^\alpha) = \{O\};
\]

and $C_i^\alpha, C_j^\alpha, C_k^\alpha, C_i^\alpha + e(i, j)C_j^\alpha$ are all pointed, convex, closed cones. Set $\beta = \min \alpha(i, j; k, e)$ and take a closed polyhedral cone $B_i$ with nonempty interior satisfying

\[
C_i \subseteq B_i \subseteq C_i^\beta \quad \text{for } i = 1, \ldots, n.
\]

We may choose the finitely many halflines generating the cones $B_i$ to be in general position. We will clarify later what is meant by general position here.

This is what we have now: The cones $B_i$ are convex, closed, pointed and polyhedral with nonempty interior, and they satisfy condition $(i, j; k, e)$. Moreover, for each plane $P$ through the origin $P \cap \text{int } B_i \neq \{O\}$ for some $i = 1, \ldots, n$. 

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Consider now the polars $D_i = B_i^*$ of $B_i$ defined as

$$D_i = \{ x \in \mathbb{R}^3 : x \cdot y \leq 0 \text{ for } y \in B_i \}.$$ 

The $D_i$'s are convex, closed, pointed, polyhedral cones in $\mathbb{R}^3$ with nonempty interior. We claim now that condition $(i, j; k, e)$ implies the following condition:

$$(i, j; k, e)^* \quad (-eD_k) \cap D_i \cap (e(i, j)D_j) \neq \{0\},$$

and the last condition in the theorem implies this one: For each $p \in \mathbb{R}^3 \setminus \{0\}$ there is an $i \in \{1, \ldots, n\}$ such that

$$(*) \quad p \notin D_i \quad \text{and} \quad p \notin -D_i.$$ 

We prove this claim using standard techniques from the theory of convex polyhedral cones (cf. [4] or [5]). Condition $(i, j; k, e)$ for the cones $B_i$ is of the form $B_k \cap (B_i + B_j) = \{0\}$ (here we dropped the signs) that has polar form $D_k + (D_i \cap D_j) = \mathbb{R}^3$. Assume now that $(-D_k) \cap (D_i \cap D_j) = \{0\}$, then the cones $-D_k$ and $(D_i \cap D_j)$ can be separated, i.e., there is $v \in \mathbb{R}^3 \setminus \{0\}$ such that $v \cdot x \leq 0$ for all $x \in -D_k$ and $v \cdot y > 0$ for all $y \in D_i \cap D_j$. But then $v \cdot z > 0$ for all $z \in D_k + (D_i \cap D_j)$, a contradiction. Let us see now the last condition:

$$P \cap \text{int } B_i \neq \{0\},$$

and consider $q \in P \cap \text{int } B_i$ with $q \neq 0$. Write $p$ for a normal of the plane $P$. Then $q \cdot p = 0$ and $q \cdot x < 0$ for all $x \in B_i \setminus \{0\} = D_i \setminus \{0\}$, so indeed, $\pm p \notin D_i$.

(As a matter of fact, from now on we will give the proof of Theorem 3 in the case when the sets $D_i$ are polyhedral cones in $\mathbb{R}^3$ with nonempty interior. The general case follows by a standard continuity argument.)

Choose a point $d_i \in \text{int } D_i$ now for $i = 1, \ldots, n$ and shrink each set $D_i$ to the point $d_i$ linearly and simultaneously with a parameter $t \in [0, 1]$, so that the shrinking set $D_i(t)$ equals $D_i$ when $t = 1$ and $d_i$ when $t = 0$. Write $I$ for the set of indices $i, j, k, e_i, e_j, e_k$ and set

$$D_i(t) = (e_iD_i(t)) \cap (e_jD_j(t)) \cap (e_kD_k(t))$$

when $t \in [0, 1]$. We assume that the cones $B_i$ and the points $d_i$ are in general position to ensure that $D_i(1) \neq \{0\}$ implies that int $D_i(1)$ is nonempty. Moreover, as the cones $D_i(t)$ shrink, the cones $D_i(t)$ shrink as well and $D_i(t) = \{0\}$ for $t < t_0(I)$ where $t_0(I)$ is the smallest $t$ for which $D_i(t)$ is different from $\{0\}$. (If, for some, $D_i(1) = \{0\}$ already, then $t_0(I)$ is not defined.) We assume that the cones $B_i$ and the points $d_i$ are in general position to ensure that $D_i(t)$ is a halfline when $t = t_0(I)$ and that int $D_i(t) \neq \emptyset$ for $t > t_0(I)$.

As $t$ decreases, condition $(*)$ remains true because the cones $D_i$ get smaller and smaller. But conditions $(i, j; k, e)^*$ will fail for each $(i, j; k, e)$ for some $t$ because $D_i(t) = \{0\}$ for all $I$. The condition $(i, j; k, e)^*$ holds for all $t > t(i, j; k, e)$ and fails for all $t \leq t(i, j; k, e)$ where $t(i, j; k, e)$ is uniquely determined. Write $t_0$ for the largest $t(i, j; k, e)$, then $t_0 = t(i, j; k, e)$ for some $(i, j; k, e)$. We may assume with-
out loss of generality that \(i=1, j=2, k=3\) and \(e(1, 2)=1\) and \(e=-1\). So condition \((1, 2; 3, -1)^*\) fails, i.e.,

\[
D_1(t_0) \cap D_2(t_0) \cap D_3(t_0) = K
\]

where \(K\) is a halfline of the form \(\{xv : a \geq 0\}\) with \(v \in \mathbb{R}^8 \setminus \{O\}\). We know that \(D_1(t) \cap D_2(t) \cap D_3(t) = \{O\}\) for \(t < t_0\) and has nonempty interior for \(t > t_0\). We claim now that for each \(j=1, 2, \ldots, n\), \(v \in D_j(t_0) \) or \(v \in -D_j(t_0)\). This will contradict condition \((\ast)\) and so prove the theorem.

The claim is evident when \(j=1, 2, 3\). We are going to prove it with notation \(j=4\). There are two cases to consider.

1st case. When the intersection of two of the cones \(D_j(t_0)\) \((j=1, 2, 3)\) is equal to \(K\), \(D_1(t_0) \cap D_2(t_0) = K\), say. From condition \((2, 4; 1, e=-1)\) we get for \(t=t_0\) that

\[
D_1(t_0) \cap D_2(t_0) \cap \{e(2, 4)D_4(t_0)\} \neq \{O\}.
\]

But \(K=D_4(t_0)\cap D_2(t_0)\) and so \(v \in K \subset e(2, 4)D_4(t_0)\) indeed.

2nd case. When the intersection of any two cones \(D_j(t_0)\) have nonempty interior \((j=1, 2, 3)\). Then, by a wellknown theorem (see [3], for instance), there are vectors \(a_j \in \mathbb{R}^8\) such that \(a_j \cdot x \leq 0\) for all \(x \in D_j(t_0)\) \((j=1, 2, 3)\) and \(O\) is in the convex hull of \(a_1, a_2\) and \(a_3\). The case when some \(a_j\) is parallel with some other \(a_i\) has been dealt with in the first case. So we assume that every \(a_j\) is nonzero and \(0 = a_1 + a_2 + a_3\) and every \(a_j > 0\). Then \(a_j \cdot x \leq 0 \) \((j=1, 2, 3)\) implies that \(x = \beta v\) for some real number \(\beta\). Moreover, \(a_j \cdot v = 0\) for \(j=1, 2, 3\).

Assume now that \(\pm v \in D_4(t_0)\). Then \(L\), the line through \(v\) and \(-v\) can be separated from \(D_4(t_0)\), i.e., there exists a nonzero \(a_4 \in \mathbb{R}^8\) such that \(a_4 \cdot x < 0\) when \(x \in D_4(t_0) \setminus \{O\}\) and \(a_4 \cdot x = 0\) when \(x \in L\). This shows that the vectors \(a_i\) \((i=1, 2, 3, 4)\) are all orthogonal to \(v\) and so \(a_4 = \beta_1 a_1 + \beta_2 a_2\) for some real numbers \(\beta_1\) and \(\beta_2\). We show now that \(\beta_1\) and \(\beta_2\) are both different from zero. Assume that \(\beta_4 = 0\), say. Then \(a_1\) and \(a_4\) are parallel and, then \(D_1(t_0)\) is separated either from \(D_4(t_0)\) or from \(-D_4(t_0)\), contradicting condition \((1, j; 4, \pm 1)^*\).

Consider now condition \((1, 2; 4, e)^*\): there exists an \(x \in \mathbb{R}^8 \setminus L\) such that

\[
x \in -(eD_4(t_0)) \cap D_1(t_0) \cap D_2(t_0).
\]

Then \(-e a_1 \cdot x < 0\), \(a_1 \cdot x \leq 0\) and \(a_2 \cdot x \leq 0\). This implies that \(\beta_1\) and \(\beta_2\) cannot be of the same sign. We may assume that \(\beta_1 > 0\) and \(\beta_2 < 0\).

Suppose now that \(e(3, 4) = 1\) and consider condition \((3, 4; 2, -1)^*\). In the same way as above this implies the existence of an \(x \in \mathbb{R}^8 \setminus L\) with \(a_3 \cdot x \leq 0\), \(a_4 \cdot x = 0\) and \(a_2 \cdot x \leq 0\). Now \(a_3\) is a positive linear combination of \(a_2\) and \(a_4\), so \(a_3 \cdot x = 0\). But \(a_1 \cdot x = 0\), \(a_2 \cdot x \leq 0\), \(a_3 \cdot x \leq 0\) is impossible. Assume now that \(e(3, 4) = -1\) and consider condition \((3, 4; 1, -1)^*\). Again, this implies the existence of an \(x \in \mathbb{R}^8 \setminus L\) with \(a_3 \cdot x \leq 0\), \(a_4 \cdot x \geq 0\) and \(a_1 \cdot x \leq 0\). Now \(a_2\) is a positive linear combination of \(a_1\) and \(-a_4\), so \(a_2 \cdot x = 0\). But \(a_1 \cdot x \leq 0\), \(a_3 \cdot x > 0\), \(a_2 \cdot x \leq 0\) is impossible.

We mention finally that it is possible to extend these results to higher dimensional spaces but, unfortunately, the conditions in the theorems become rather unintelligible.
References


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