SHORT COMMUNICATION

BORSUK'S THEOREM THROUGH COMPLEMENTARY PIVOTING

Imre BÁRÁNY

OTSZK, KTT, Budapest, Hungary

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In this short note a simple and constructive proof is given for Borsuk's theorem on antipodal points. This is done through a special application of the complementary pivoting algorithm.

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In this paper we give a new proof of Borsuk's theorem on antipodal points [2]. This proof may be of some interest because it is simple and constructive. The need for such a combinatorial proof emerged in connection with a surprising application of Borsuk's theorem in graph theory [1, 5]. We shall make use of the so-called complementary pivoting algorithm (see e.g. [4, 6]). The reader is supposed to be familiar with this technique. We mention that our treatment is based mostly on [4].

Let $x_i$ denote the $i$-th coordinate of $x \in \mathbb{R}^n$ for $i = 1, \ldots, n$. Write $\|\|$ and $\cdot$ for the Euclidean resp. max norm. Put $S^n = \{x \in \mathbb{R}^{n+1}: \|x\| = 1\}$ and $C^n = \{x \in \mathbb{R}^{n+1}: |x| = 1\}$. If $\delta > 0$ and $A \subseteq \mathbb{R}^n$, then $\delta A = \{\delta x \in \mathbb{R}^n: x \in A\}$. A function $f: A \rightarrow \mathbb{R}^n$ is said to be odd if $x \in A$ implies $-x \in A$ and $f(-x) = -f(x)$ (here $A \subseteq \mathbb{R}^m$ for some $m$). We write $x < y$ for $x, y \in \mathbb{R}^n$ if $x$ is lexicographically less than $y$. If $K$ is a triangulation, then $K^i$ denotes its $i$-dimensional simplices, in particular, $K^0$ is the set of vertices of $K$. Finally, $e_i$ denotes the $i$-th basis vector of $\mathbb{R}^{n+1}$ for $i = 1, \ldots, n+1$.

Theorem 1 (Borsuk [2]). If $f: S^n \rightarrow \mathbb{R}^n$ is continuous and $n \geq 1$, then there exists a point $x \in S^n$ with $f(x) = f(-x)$.

It is clear that this theorem is equivalent to the following one.

Theorem 2. If $f: C^n \rightarrow \mathbb{R}^n$ is an odd continuous map and $n \geq 1$, then there exists a point $x \in C^n$ with $f(x) = 0$. 

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We shall prove this second theorem. Now we need some preparations.

First we shall define a special triangulation, $L$, of $\mathbb{R}^{n+1}$ as follows ($L$ is the same as the triangulation $K_1$ of [6, page 29]). $L^0$ is the set of all integer lattice points of $\mathbb{R}^{n+1}$, and a set $\{y_1, y_2, \ldots, y_{n+2}\} \subset L^0$ with $y_1 < y_2 < \cdots < y_{n+2}$ is the set of vertices of an $(n+1)$-simplex of $L$ if there exists a permutation $\pi$ of the numbers $1, 2, \ldots, n+1$ such that for $i = 1, 2, \ldots, n+1$

$$y_{i+1} = y_i + e_{\pi(i)}.$$  

(1)

It is shown in [6] that $L$ is indeed a triangulation of $\mathbb{R}^{n+1}$. Here we claim that $L$ is symmetric with respect to the origin, i.e., $\sigma \in L$ implies $-\sigma \in L$. The proof of this fact is quite easy and is, therefore, omitted.

Now if $t \in \mathbb{R}^1$, then $\lfloor t \rfloor$ denotes the vector $(1, t, \ldots, t^{n-1}) \in \mathbb{R}^n$. Let $0 < t_1 < t_2 < t_3 < \cdots < t_{n+1} < 1$ and for $u \in L^0$ let $m(u)$ be the integer for which

$$m(u) = \sum_{i=0}^{n} 2^i u_{i+1} \text{ mod } 2^{n+1}. \tag{2}$$

Clearly, $m(u)$ is well-defined. Now let $h : (L^0 \setminus \{0\}) \to \mathbb{R}^n$ be defined in the following way

$$h(u) = \begin{cases} \lfloor t_{m(u)} \rfloor & \text{if } 0 < u, u \in L, \\ -\lfloor t_{m(u)} \rfloor & \text{if } u < 0, u \in L. \end{cases}$$

It is evident that $h$ is odd. We shall need one more property of $h$: if $u_1, \ldots, u_n \in L^0 \setminus \{0\}$ are the vertices of any $\sigma \in L^{n-1}$, then

$$\det[h(u_1), \ldots, h(u_n)] \neq 0. \tag{3}$$

Indeed, if $v_1, \ldots, v_{n+2}$ are the vertices of an $(n+1)$-simplex of $L$, then it is easy to check by (1) and (2) that $m(v_1), \ldots, m(v_{n+2})$ are pairwise different integers. Then, a fortiori, $m(u_1), \ldots, m(u_n)$ are again pairwise different and

$$\det[h(u_1), \ldots, h(u_n)] = (-1)^\gamma \det[[t_{m(u_1)}], \ldots, [t_{m(u_n)}]],$$

where $\gamma$ is the number of $u_i$'s with $u_i < 0$. Clearly, this last determinant is not equal to zero. This proves (3).

**Proof of Theorem 2.** In what follows a complementary pivoting routine will take place on a (vector labelled) finite triangulation of the set $H = H_k = \{x \in \mathbb{R}^{n+1} : 1 - 1/k \leq |x| \leq 1\}$ where $k \geq 2$ is an integer. This triangulation is defined to be $K = K_k = \{(1/k)\sigma : \sigma \in L \text{ and } (1/k)\sigma \subset H\}$. It is easy to check that $K$ is a triangulation of $H$, $K$ is finite and symmetric with respect to the origin, further, $K^0 \subset \partial H$ and $\text{diam } \sigma \leq 1/k$ for every $\sigma \in K$ (diam is meant in the max norm).

Clearly, $\partial H = B \cup C$ where $B = (1 - 1/k)C_n$ and $C = C_n$. Choose a vector $v \in \mathbb{R}^n$ such that $|v| < 1 - 1/k$ and $(v_1, \ldots, v_n, 1 - 1/k) \in \text{relint } \sigma_0$ for some $\sigma_0 \in K^n$,
of course, $\sigma_0 \subset B$. Now we define a map $g : B \to \mathbb{R}^n$ by

$$g(x) = g(x_1, \ldots, x_{n+1}) = (x_1, \ldots, x_n) - \frac{k}{k-1} x_{n+1}(v_1, \ldots, v_n).$$

Clearly, $g$ is odd and $g(x) = 0$ if and only if $x = \pm (v_1, \ldots, v_n, 1 - 1/k)$.

Now let us define the vector labelling $l : K^0 \to \mathbb{R}^n$ by

$$l(x) = l_e(x) = \begin{cases} f(x) + e h(kx) & \text{if } x \in K^0 \cap C, \\ g(x) + e h(kx) & \text{if } x \in K^0 \cap B, \end{cases}$$

where $e > 0$. Extend this labelling rule to a piecewise linear $l : H \to \mathbb{R}^n$ map. $l$ is odd. Now we claim that there exists a positive $\delta \leq 1/k$ such that for $0 < e < \delta$ we have

(i) there are exactly two solutions, $x_0$ and $-x_0$, of the equation $l(x) = 0$ satisfying $x \in B$, and one of them, say $x_0$, lies in $\text{relint } \sigma_0$;

(ii) $0 \in l(\sigma)$, $\sigma \in K$ implies $\sigma \subset K^n \cup K^{n+1}$.

Indeed, $|l_e(x) - g(x)| < e$ for every $x \in B$. Clearly, for some $\eta > 0$ $|g(x)| \geq \eta$ if $x \in B \setminus (\text{relint } \sigma_0 \cup \text{relint } -\sigma_0) = D$ whence $|l_e(x)| \geq \eta - e$ for every $x \in D$. Thus $l_e(x) = 0$ has no solution with $x \in D$ if $e < \eta$. Further, $g$ and $l_e$ are linear maps on $\sigma_0$ and $g$ has exactly one zero in $\sigma_0$. So if $g$ and $l_e$ are sufficiently near, i.e., $e < \eta'$ for some $\eta' > 0$, then $l_e$ too, has exactly one zero in $\sigma_0$, $x_0$. As we have seen $x_0$ cannot be on the relative boundary of $\sigma_0$ if $e < \eta$. So for $0 < e < \min(\eta, \eta')$ (i) holds true.

Suppose now that $0 \in l_e(\sigma)$ for some $\sigma \subset K^{n-1}$. This means that for some $\alpha_i \geq 0, i = 1, \ldots, n$,

$$\sum_{i=1}^n \alpha_i l_e(u_i) = 0 \land \sum_{i=1}^n \alpha_i = 1,$$  \hspace{1cm} (4)

where $u_1, \ldots, u_n$ are the vertices of $\sigma$. Writing $l_e(u_i) = a_i + e h(ku_i)$ (here either $a_i = f(u_i)$ or $a_i = g(u_i)$) we have from (4),

$$P(e) = \det[a_1 + e h(ku_1), \ldots, a_n + e h(ku_n)] = 0.$$  

$P(e)$ is a polynomial of $e$ and the coefficient of $e^n$, $\det[h(ku_1), \ldots, h(ku_n)]$ is different from zero by (3). This implies that $P(e) \neq 0$ for $e \in (0, \delta_\sigma)$ for some $\delta_\sigma > 0$, i.e., (4) cannot be true for $0 < e < \delta_\sigma$. This implies that for $0 < e < \delta$ with $\delta = \min(1/k, \eta, \eta', \min_{\sigma \subset K^{n-1}} \delta_\sigma)$ (i) and (ii) hold true.

We mention that in the terminology of [4], the condition (ii) means that 0 is a regular value of the piecewise linear map $l_e$. Now fix $e$ with $0 < e < \delta$.

Put $M = \{z \in H : l(z) = 0\}$. We define a graph $G$ as follows. Its nodes are the points $x \in M$ with $x \in \sigma$ for some $\sigma \subset K^n$ and two different nodes, $x$ and $y$, form an edge of $G$ iff $x, y \in \tau$ for some $\tau \subset K^{n+1}$. The degree of a node of $G$ is the number of edges adjacent to this node. We write $[u, v]$ for the line segment connecting $u \in \mathbb{R}^{n+1}$ and $v \in \mathbb{R}^{n+1}$. Due to the implication (ii) the following facts are true (for the proofs see [4] or [6]).
The degree of a node of $G$ is 1 or 2 according to whether it is contained in $\partial H$ or in $\text{int} \ H$. Further,

$$M = \{z \in H : z \in [x, y] \text{ for some edge } (x, y) \text{ of } G\}.$$  

Together these facts imply that $M$ is a 1-manifold (for the definition of 1-manifold, see [4]) and so it consists of a finite number of pairwise disjoint polygonal paths, and there are two types of these paths, the first type going from a boundary node to a boundary node without cycles and the second type being a single cycle lying entirely in $\text{int} \ H$.

The main step in the proof of this facts is that for a node $x$ of $G$ with $x \in \sigma \subset \tau \in K^{n+1}$ and $\sigma \in K^n$ there exists exactly one node, $x'$, of $G$ with $x' \in \sigma' \subset \tau$ for which $(x, x')$ is an edge of $G$. To determine $x'$ and $\sigma'$ from $x$, $\sigma$ and $\tau$ is easy, it is done through a linear programming pivot step (see [4] or [6]).

Our algorithm follows a path $l(x) = 0$.

Start with the triple $(x_0, \sigma_0, \tau_0)$ where $\tau_0 \in K^{n+1}$ is the only $(n + 1)$-simplex containing $\sigma_0$.

Step $j$ for $j = 0, 1, 2, \ldots$. For the triple $(x_j, \sigma_j, \tau_j)$ determine $x_{j+1} \in \tau_j$ as the only node of $G$ adjacent to $x_j$ and $\sigma_{j+1} \in K^n$ with $x_{j+1} \in \sigma_{j+1}$. If $x_{j+1} \in \partial H$, then stop, else determine $\tau_{j+1} \in K^{n+1}$ as the only $(n + 1)$-simplex containing $\sigma_{j+1}$ and different from $\tau_j$. (The rules for this end are given in [6, p. 35]). Proceed to step $j + 1$ with the triple $(x_{j+1}, \sigma_{j+1}, \tau_{j+1})$.

We know that this algorithm produces a path through the nodes $x_0, x_1, \ldots, x_p$ from the boundary node $x_0$ to the boundary node $x_p$ ($p \geq 1$). We claim that $x_p \in C$. Suppose, on the contrary, that $x_p \in B$, then, by property (i) of $l$, we must have $x_p = -x_0$. Starting now the algorithm with the triple $(-x_0, -\sigma_0, -\tau_0)$, we shall get the polygonal path through the points $-x_0, -x_1, \ldots, -x_p$ because $l$ is odd. These two paths are not disjoint for $x_p = -x_0$ and so they coincide: $x_{p-1} = -x_1, x_{p-2} = -x_2, \ldots, x_0 = -x_p$. Let $z$ be the “middle point” of the first path, i.e.,

$$z = \begin{cases} x_{p/2} & \text{if } p \text{ is even}, \\ \frac{1}{2}(x_{(p+1)/2} + x_{(p-1)/2}) & \text{if } p \text{ is odd}. \end{cases}$$

It is easy to check that $z \in H$ and $z = -z$. Consequently $z = 0$ and $0 \in H$. This contradicts to the definition $H$ for $x \in H$ implies $|x| \geq 1 - 1/k$.

As we have seen $x_i \in C$. Then condition (ii) implies that $\sigma_p \subset C$. Writing $y_1, \ldots, y_{n+1}$ for the vertices of $\sigma_p$ we have for some $\alpha_i \geq 0$ ($i = 1, \ldots, n + 1$) that

$$x_p = \sum_{i=1}^{n+1} \alpha_i y_i, \quad \sum_{i=1}^{n+1} \alpha_i = 1, \quad \sum_{i=1}^{n+1} \alpha_i l(y_i) = 0.$$  

This implies that

$$\left| \sum_{i=1}^{n+1} \alpha_i f(y_i) \right| = \left| \sum_{i=1}^{n+1} \alpha_i e h(ky_i) \right| \leq \epsilon \sum_{i=1}^{n+1} \alpha_i \leq 1/k. \quad (5)$$
For each $k = 2, 3, \ldots$ we have an $n$-simplex $\sigma_p(k) \subseteq C$ whose vertices satisfy (5). There is a subsequence of $\sigma_p(k)$ that converges to a point $y \in C$ because $C$ is compact and $	ext{diam } \sigma_p(k)$ tends to zero. By continuity and (5) we must have $f(y) = 0$. And this is what we wanted to prove.

Remarks. This proof is not “quite constructive” because $\delta$ and so $\epsilon$ in the perturbation $e\delta(ku)$ is not determined constructively. One may hope that, as usual (see e.g., [4] or [6]), a lexicographic scheme could be used to produce a path between $B$ and $C$. However, it is not difficult to find an example showing that this is not the case, i.e., an example when the lexicographic scheme produces a path between the two solutions $g(x) = 0$, $x \in B$. In connection with this we mention the following theorem which is similar to Browder’s theorem (see [3]).

Theorem 3. Suppose $n \geq 1$ and $f : C^n \times [0, 1] \to \mathbb{R}^n$ is a continuous map with $f(-x, t) = -f(x, t)$ for $(x, t) \in C^n \times [0, 1]$. Then there exists a connected set $K \subseteq C^n \times [0, 1]$ meeting both $C^n \times \{0\}$ and $C^n \times \{1\}$ such that $f(x, t) = 0$ for every $(x, t) \in K$.

This theorem can be proved combining the ideas of the proof of Browder’s theorem in [4, p. 129] and this paper. We omit details.

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References