SELFDUALITIES OF SERIAL RINGS, REVISITED

PHẠM NGỌC ÁNH

Abstract. A description is given of serial rings whose maximal quotient rings are quasi-Frobenius (QF). Every serial ring is a factor of a serial ring whose maximal quotient ring is a QF-ring. This result is used to give a new, conceptual proof for the selfduality of serial rings emphasising the importance of weakly symmetric rings.

1. Introduction

The question as to whether serial rings are selfdual was put by Haack [5], and was answered in positive by Dischinger and Müller [4]. Waschbüsch [16] noticed that the result had been claimed (without proof) earlier by Amdal and Ringdal ([2] Remark 5(c)). In [16] he presents a proof which uses Kupisch's classification of serial rings described in [1], [2], [10]. All these proofs, however, are of highly technical nature; furthermore, using the Kupisch classification for this purpose seems to us like using a sledge hammer to crack an almond. It is therefore quite reasonable to look for a conceptual proof, and it is not surprising that several authors (see e.g. [6], [7], [8], [9], [13], [14][1]) are still working on this fascinating topic.

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[1]The author is indebted to the referee for calling his attention to the works [7], [13] and [14] where Theorem 3.5 is also proved.

In this paper we present such a conceptual proof. We describe serial rings as factors of serial rings whose maximal quotient rings are quasi-Frobenius (QF), and then show that the latter rings admit weakly symmetric selfduality. By an observation of Haack, however, such selfdualities carry over to factor rings.

2. Basic facts, notations

For the benefit of the reader we present some easy, but basic results and notation from [3], [10] in such an order that their proofs can be easily deduced.

The *radical*, the *length* and the *injective hull* of a module M is denoted by $J(M)$, $c(M)$ and $I(M)$, respectively, and module homomorphisms will be written opposite the scalars. J will be the radical of a ring R. An artinian ring R is called *selfdual* if there is a ring isomorphism $\varphi : R \longrightarrow \mathrm{End}(E)$ for some injective cogenerator ${}_RE$. Note that E is in general not $I(R/J)$. The question of selfduality is probably the most intriguing puzzle in the theory of Morita duality. It turns out that even for the class of serial rings — the best–understood class of non-semisimple rings — it is not simple to check a selfduality. Since minimal injective cogenerators are quite complicated, with a few exceptions when a ring is commutative or hereditary with some additional properties, it is not an easy job to find out a way of embedding a ring into the endomorphism ring of an appropriate injective cogenerator. The isomorphism φ induces *a weakly symmetric selfduality* if $E\varphi(e) \cong I(Re/Je)$ for all $e^2 = e \in R$. In particular, a QF-ring R is called *weakly symmetric* if $Re \cong I(Re/Je)$ for every $e^2 = e \in R$. A selfduality φ is called a *good duality* if $\varphi(K) = r_R r_E(K)$ for every ideal K of R where $r_X(Y)$ denotes the right annihilator of Y in X with respect to the multiplication under consideration. A good duality obviously induces selfdualities for factor rings. A *serial* ring is an artinian ring over which each module is a direct sum of uniserial modules (that is, modules with chain for subomdule lattices). Avoiding triviality we will consider only serial rings which are not uniserial: that is not local ones.

In what follows, with one exception in Proposition 4.1, R is an indecomposable, basic, serial ring with a basic set $\{e_i \mid i = 1, ..., n\}$ of idempotents such that there are projective covers $Re_{i-1} \longrightarrow Je_i$ for $i = 2, \dots, n$ and $Re_n \longrightarrow Je_1$ for the case $Je_1 \neq 0$. This means that the associated quiver of R is A_n for the case $Je_1 = 0$ or $\tilde{A}_{n-1}$ for $Je_1 \neq 0$. Let $[k]$ be the least positive integer congruent to $k \in \mathbb{Z}$ modulo n and

$$S_i = Re_i/Je_i, \;\; P_i = Re_i, \;\; R_i = e_iRe_i, \;\; c_i = c({}_RP_i).$$

Fix $a_i \in e_iRe_{[i+1]} \setminus J^2 \;\; (i = 1, ..., n)$ with $a_n = 0$ in case $Je_1 = 0$ and for $k \in \mathbb{N}$ let $\overset{k}{a}_i := a_{[i-k+1]} \dots a_{[i-1]}a_i$. Then we have $\overset{1}{a}_i = a_i$ and

$$\overset{n}{a}_{[i-1]} \in R_i, \;\; \overset{n}{a}_i \in R_{[i+1]}, \;\; \overset{n}{a}_{[i-1]}a_i = \overset{n+1}{a}_i = a_i\overset{n}{a}_i, \;\; R\overset{k}{a}_i = J^k e_{[i+1]} \tag{1}$$

and

$$e_{[i-k]}J^k e_{[i+1]} = e_{[i-k]}R\overset{k}{a}_i = R_{[i-k]}\overset{k}{a}_i = \overset{k}{a}_iR_{[i+1]}. \tag{2}$$

If, starting from the top, $N_1, N_2, ..., N_c \;\; (c = c(M))$ are the composition factors of a uniserial module M, then

$$N_i \cong N_j \iff [i] = [j]. \tag{3}$$

Since a simple R-module S is isomorphic to S_k iff $e_kS \neq 0$, we obtain

$$e_kJ^je_i \neq e_kJ^{j+1}e_i \iff [i-j] = k, \;\; c_i > j. \tag{4}$$

Observing that the proof of implication (d) $\Longrightarrow$ (a) in Theorem 32.2 in [3] works also for semiprimary rings, we get

Proposition 2.1. *A semiprimary ring R is serial iff R/J^2 is serial.*

Proposition 2.2. *Every serial QF-ring R admits a weakly symmetric selfduality $\Phi : R \longrightarrow R$ such that $\Phi(e_i) = e_{[i+1-c]}$ where $c = c(P_i) \;\; (i = 1, ..., n)$.*

Proof. By assumption all c_i are equal, say, to c. Then $[c] = 1$ if and only if R is weakly symmetric by (3). Therefore it is sufficient to prove the case $[c] \geq 2$. This implies $1 \leq [c] - 1 = [c-1]$. The length of the module ${}_{R_i}e_iRe_{[i+1]}$ is the number of the simple factors of ${}_RP_{[i+1]}$ isomorphic to S_i which is precisely $l + 1$ if l is the greatest positive integer satisfying $ln \leq c - 1$ by (3). Therefore, as

$1 \leq [c] - 1 = [c-1]$ and again in view of (3), $l+1$ is also the length of ${}_{R_{[i+1]}}R_{[i+1]}$ which is the number of simple factors of $P_{[i+1]}$ isomorphic to $S_{[i+1]}$. Similarly, one obtains that the length of the right $R_{[i+1]}$-module $e_iRe_{[i+1]}$ is equal to the length of ${}_{R_i}R_i$. Since the lengths of the uniserial right $R_{[i+1]}$-module $e_iRe_{[i+1]}$ and the left R_i-module $e_iRe_{[i+1]}$ are equal, we see that $e_iRe_{[i+1]}$ is cyclic and free both as a left R_i- and a right $R_{[i+1]}$-module. Consequently the a_i induce ring isomorphisms

$$g_i : R_{[i+1]} \longrightarrow R_i : x \in R_{[i+1]} \mapsto y \in R_i : ya_i = a_i x.$$

and

$$\overset{k}{g}_i = g_{[i-k+1]} \cdots g_{[i-1]} g_i : R_{[i+1]} \longrightarrow R_{[i-k+1]}, \ \ k \in \mathbb{N}.$$

Thus $\overset{1}{g}_i = g_i$ and in view of (1) we have

$$g_i(\overset{n}{a}_i) = \overset{n}{a}_{[i-1]}, \quad g_i(\overset{nk}{a}_i) = \overset{nk}{a}_{[i-1]} \quad (i = 1, ..., \ n; \ \ k \in \mathbb{N}) \tag{5}$$

We construct an automorphism φ of R satisfying $\varphi(e_i) = e_{[i-1]}$ as follows. If $x \in R_i$, put $\varphi(x) = g_{[i-1]}(x) \in R_{[i-1]}$. For $z \in e_k J^p e_i$ one can assume in view of (2) and (4) that

$$[i-p] = k, \ \ z = x\overset{p}{a}_{[i-1]}, \ \ x \in R_k = R_{[i-p]}$$

and put

$$\varphi(z) = \varphi(x\overset{p}{a}_{[i-1]}) = g_{[k-1]}(x)\overset{p}{a}_{[i-2]} \in e_{[k-1]}Re_{[i-1]}.$$

φ is well-defined on e_kRe_i. For if $z \in e_kJ^qe_i, \ \ [i-q] = k, \ \ z = y\overset{q}{a}_{[i-1]}$ and t is the smallest positive integer with

$$J^{t-1}a_i/J^ta_i = R\overset{t}{a}_i/R\overset{t+1}{a}_i \cong S_k,$$

then $p - t = nn_1, \ \ q - t = nn_2$ for some nonnegative integers $n_1, \ \ n_2$ and

$$0 = z - z = x\overset{p}{a}_{[i-1]} - y\overset{q}{a}_{[i-1]} = (x\overset{nn_1}{a}_{[i-t]} - y\overset{nn_2}{a}_{[i-t]})\overset{t}{a}_{[i-1]}.$$

This implies together with (5)

$$0 = g_{[k-1]}(x\overset{nn_1}{a}_{[i-t]} - y\overset{nn_2}{a}_{[i-t]})\overset{t}{a}_{[i-2]} = g_{[k-1]}(x)\overset{nn_1}{a}_{[i-t-1]}\overset{t}{a}_{[i-2]} -$$

$$g_{[k-1]}(y)\overset{nn_2}{a}_{[i-t-1]}\overset{t}{a}_{[i-2]} = g_{[k-1]}(x)\overset{p}{a}_{[i-2]} - g_{[k-1]}(y)\overset{q}{a}_{[i-2]} = \varphi(x\overset{p}{a}_{[i-1]}) - \varphi(y\overset{q}{a}_{[i-1]}).$$

Since $c({}_{R_{[i-1]}}e_{[i-1]}Re_{[j-1]}) = c({}_{R_i}e_iRe_j)$ by (3), φ is an additive isomorphism between $e_{[i-1]}Re_{[j-1]}$ and e_iRe_j for all $i,j = 1,...,n$. Therefore φ can be extended to an additive automorphism of R. In this case φ is also multiplicative, by which we mean $\varphi(uw) = \varphi(u)\varphi(w)$ for any $u,v \in R$. It obviously suffices to see $\varphi(uw) = \varphi(u)\varphi(w)$ for any $u = e_que_k$ and $w = e_kwe_i(q,k,i = 1,...n)$. Write

$$u = x\overset{j}{a}_{[k-1]},\ \ w = y\overset{p}{a}_{[i-1]};\ \ x \in R_q,\ \ y \in R_k,\ \ [k-j] = q,\ \ [i-p] = k,$$

then

$$uw = x\overset{j}{a}_{[k-1]}y\overset{p}{a}_{[i-1]} = x\overset{j}{g}_{[k-1]}(y)\overset{j}{a}_{[k-1]}\overset{p}{a}_{[i-1]} = x\overset{j}{g}_{[k-1]}(y)\overset{j+p}{a}_{[i-1]}.$$

Thus we have

$$\varphi(uw) = g_{[q-1]}(x\overset{j}{g}_{[k-1]}(y))\overset{j+p}{a}_{[i-2]} = g_{[q-1]}(x)g_{[q-1]}(\overset{j}{g}_{[k-1]}(y))\overset{j+p}{a}_{[i-2]} =$$

$$g_{[q-1]}(x)\overset{j+1}{g}_{[k-1]}(y)\overset{j}{a}_{[k-2]}\overset{p}{a}_{[i-2]} = g_{[q-1]}(x)\overset{j}{a}_{[k-2]}g_{[k-1]}(y)\overset{p}{a}_{[i-2]} = \varphi(u)\varphi(w).$$

Let $l = [c-1] = [c]-1$ and $\Phi = \varphi^l$. If we defines an R-R-bimodule R_Φ by $R_\Phi := R,\ \ r * x * t = rx\Phi(t)\ \ (r,\ \ x,\ \ t \in R)$, then R_Φ induces a weakly symmetric selfduality for R because the socle of $R_\Phi * e_i = R\Phi(e_i) = Re_{[i+1-[c]]}$ is exactly $S_{[i+c-[c]]} = S_i$ for all $i = 1,...,n$. □

Remark 2.0.1. This result was proved by Haack (cf. Theorem 3.3 [5]) though it is also observed earlier by Kupisch (cf. footnote 4 [10]) as a consequence of his classification. The above proof is a simplified version of the much easier proof to Satz 2.1 [10]. Indeed, both proofs are based implicitly on the fact that the $\overset{k}{a}_i$ together with the idempotents $1,0,e_1,...,e_n$ form a semigroup under multiplication. Roughly speaking, this semigroup can be considered as a “multiplicative base” or in the other words, a “Cartan basis”, of a serial ring in a generalized sense that the corresponding images yield a basis of the graded ring associated to the filtration given by powers of the radical. Moreover, in the case of a not weakly symmetric serial QF-ring the permutation sending i to $[i-1]$ induces an automorphism of this multiplicative semigroup which can be extended to a ring automorphism. We do not know about the existence of such a "multiplicative basis" for locally distributive rings and this lack of knowledge of existence is probably also a reason why a corresponding question on selfduality for such rings or even for a narrower class of regular representation-finite rings seems to be more difficult. The advantage of Haack's proof is a quite natural, easily understandable definition of the map φ. This map φ in Haack's proof is immediately multiplicative by observing the obvious equality $a_i x = \varphi(x) a_j$ for all $x \in e_{[i+1]} R e_{[j+1]}$. Furthermore, Proposition 2.2 is equivalent to the statement that the automorphism group of a serial, not weakly symmetric QF-ring contains a cyclic subgroup of order n.

3. Structure of serial rings

Let $P_{l_1}, \dots, P_{l_m}$ $(1 \le l_1 < l_2 < \cdots < l_m \le n)$ be the injective indecomposable projectives. Let $l_0 = 0$ and $\{k\}$ be the least positive integer congruent to k modulo m for each $0 \ne k \in \mathbb{Z}$. Moreover, define $\{0\} = 0$ if $Je_1 = 0$, otherwise $\{0\} = m$. P_{l_k} has exactly δ_k nonzero projective submodules where δ_1 is l_1 if $Je_1 = 0$ or $n - l_m + l_1$ if $Je_1 \ne 0$, and $\delta_k = l_k - l_{\{k-1\}}$ $(k = 2, \dots, m)$. Let

$$e = e_{l_1} + \cdots + e_{l_m}, \;\; I_i = I(P_i), \;\; I = I({}_R R) = \bigoplus_{i=1}^{n} I_i \cong \bigoplus_{k=1}^{m} P_{l_k}^{\delta_k}, \;\; T = \mathrm{End}({}_R I).$$

Here, ${}_R Re$ is a minimal faithful left ideal (that is, a direct sum of isomorphism types of indecomposable injective projectives in the case of serial rings), and T is a serial ring as it is Morita equivalent to eRe. Let ε_i be the projection of I onto I_i and $T_i = T\varepsilon_i$ $(i = 1, \dots, n)$. Put

$$B = \{t \in T \;\; | \;\; Rt = 0\}, \quad A = \{t \in T \;\; | \;\; Rt \subseteq R\}.$$

Proposition 3.1. *If $B = 0$, then T is a QF-ring.*

Proof. Having $B = 0$ implies that R can be identified as a subring of T in the usual way. Since $a_{[l_k+i]}$ $(1 \le i \le \delta_{\{k+1\}} - 1; \;\; k = 1, \dots, m)$ induces an isomorphism between ${}_T T\varepsilon_{[l_k+i]}$ and ${}_T T\varepsilon_{[l_k+i+1]}$, all ${}_T T\varepsilon_{[l_k+1]}, \dots, \;\; {}_T T\varepsilon_{l_{\{k+1\}}}$ are isomorphic and ${}_T T a_{l_k}$ is the radical of ${}_T T\varepsilon_{[l_k+1]}$. Observing that the kernel of a_{l_k} is not zero, we obtain that ${}_T T\varepsilon_{l_k}$ is injective for all $k = 1, \dots, m$, i.e. T is a QF-ring. □

Remark 3.0.2. It is obvious that $B = 0$ if and only if the socle of ${}_RR$ is a sum of simple modules $S_i (i \in \{[l_k + 1] \mid k = 1, ..., m\})$ or equivalently if the e_iR_R $(i \in \{[l_k+1] \mid k = 1, ..., m\})$ are the indecomposable projective injectives. Moreover, $B = 0$ implies $t \in e_iRe_j$ if $t \in e_iTe_j$ and $P_it \subseteq P_j$. Consequently, $e_iRe_j = e_iTe_j$ if P_j is not isomorphic to a submodule of P_i. If P_j is isomorphic to a submodule of P_i, then $P_it \subseteq P_j$ for all $t \in e_iTe_j$ which are not isomorphisms between I_i and I_j, i.e., e_iRe_j is precisely the radical of ${}_{R_i}e_iTe_{j\,R_j}$. Therefore for the sum f_k of idempotents e_i such that P_i is isomorphic to submodule of P_{l_k}, $f_kRf_k = \mathrm{End}(Rf_k)$ is a $\delta_k \times \delta_k$ matrix ring of the form

$$\begin{pmatrix} S & S & \dots & S & S \\ M & S & \dots & S & S \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ M & M & \ddots & S & S \\ M & M & \dots & M & S \end{pmatrix}$$

where $S = R_{l_k}$ and M is its radical. Thus serial rings with $B = 0$ are a common generalization of both serial QF-rings and the so-called $(S : M)$-upper triangular matrix rings that appear in the characterization of semiperfect HNP rings. Moreover, $B = 0$ implies also $Re_{l_k} = Te_{l_k}$ $(k = 1, ..., m)$. For if $t = te_{l_k}$, then $It \subseteq R$ and thus its restriction to R is some $r \in Re_{l_k}$, hence $t = r \in Re_{l_k}$.

Theorem 3.2. $B = 0$ *if and only if the maximal quotient ring of* R *is a QF-ring.*

Proof. If $B = 0$, then R is a subring of T and ${}_RT = I = I({}_RR)$ by the previous remark. Hence T is the biendomorphism ring of I, i.e., the maximal quotient ring of R by its definition given in [12]. Thus the maximal quotient ring of R is a QF-ring by Proposition 3.1.

Conversely, if the maximal quotient ring of R is a QF-ring, then T must be the maximal quotient ring of R and hence $B = 0$ in view of [12] Propositions 4.3.2, 4.3.3 and 4.3.6. □

Remark 3.0.3. For an arbitrary ring R the equality $B = 0$ holds if an only if the maximal left quotient ring is left selfinjective. For, $B = 0$ means that $I_T = 1T$ is a free right T-module and hence $I = T = \mathrm{End}(I_T)$. If the maximal quotient ring of R is QF, then the endomorphism ring of a minimal faithful R-module is obviously also QF, but the converse is not true. For example, if R has a strictly increasing admissible sequence $\{c_1, ..., c_n\}$ $(n > 1)$ such that $[c_1] \neq 1$, then ${}_RRe_n$ is the minimal faithful module and $R_n = e_nRe_n$ is trivially QF, but $B \neq 0$, i.e., the maximal quotient ring of R cannot be QF.

We can characterize serial rings R with $B = 0$ as follows

Theorem 3.3. *For each $k = 1, ..., m$ and $i = 1, ..., \delta_k$, let*

$$W_k^i := \bigoplus_{l=[l_{\{k-1\}}+i]}^{l_k} T_l, \;\; W_k = W_k^1, \;\; \widetilde{W_k^i} = W_k^i \oplus \bigoplus_{l=[l_{\{k-1\}}+1]}^{[l_{\{k-1\}}+i-1]} J({}_TT_l).$$

If $B = 0$, then R is the ring of all endomorphisms of ${}_TT$ satisfying $W_k^i r \subseteq \widetilde{W_k^i}$ for all $i \in \{1, ..., \delta_k\}$ provided r induces an endomorphism of $W_k (k = 1, ..., m)$.

Conversely, let T be a serial QF-ring with a basic set $\{e_i \mid i = 1, ..., n\}$ of idempotents and $e = e_{l_1} + \cdots + e_{l_m}$ $(1 \leq l_1 < l_2 < \cdots < l_m \leq n)$ be such that for $k = 1, ..., m$ ${}_TT_i \cong_T T_{l_k}$ $(i = [l_{\{k-1\}} + 1], ..., l_k)$. Let R be the ring of all endomorphisms r of ${}_TT$ satisfying $W_k^i r \subseteq \widetilde{W_k^i}$ if r induces an endomorphism of W_k. Then R is a serial ring with $T = \mathrm{End}(I({}_RR))$ and $B = 0$.

Proof. The first statement of this theorem is already proved in Remark 3.0.2.

For the second statement it is clear that $e_iRe_j = e_iTe_j$ if ${}_TTe_i$ and ${}_TTe_j$ are nonisomorphic. Moreover, both ${}_RRe_i$ and e_iR_R $(i = l_{[\{k-1\}+1]}, ..., l_k)$ are isomorphic to submodules of ${}_RRe_{l_k}$ and $e_{[l\{k-1\}+1]}R_R$, respectively. Therefore simple T-modules $Te_{l_k}/J(T)e_{l_k}$ and $e_{l_k}T/e_{l_k}J(T)$ $(k = 1, ..., m)$ are uniserial R-modules with socles $Re_{[l_{\{k-1\}}+1]}/Je_{[l_{\{k-1\}}+1]}$ and $e_{l_k}R/e_{l_k}J$, respectively. These facts altogether imply that ${}_RRe_i$ and e_iR_R $(i = 1, ..., n)$ are uniserial and hence R is a serial ring. It is now routine to check that $B = 0$ and $T = \mathrm{End}(I({}_RR))$. $\square$

Remark 3.0.4. Theorem 3.3 shows that in the case $B = 0$ there is a one-to-one correspondence between R and its QF-subring eRe which is the endomorphism ring of a minimal faithful R-module, and R is uniquely determined up to isomorphism by eRe and the numerical invariants δ_k $(k = 1, ..., m)$. For example, using the notation of Theorem 3.3 let $n = 5$, $m = 2$, $l_1 = 2$, $l_2 = 3$ and T is Morita equivalent to an indecomposable, basic serial QF-ring $S = S_1 + S_2 + S_{12} + S_{21}$ where $S_1 = e_1Se_1$, $S_2 = e_2Se_2$, $S_{12} = e_1Se_2, S_{21} = e_2Se_1$, $1 = e_1 + e_2$ with primitive orthogonal idempotents e_1, e_2 and the radicals M_1, M_2 of S_1, S_2, respectively, then R is isomorphic to the generalized matrix ring

$$\begin{pmatrix} S_1 & S_1 & S_{12} & S_{12} & S_{12} \\ M_1 & S_1 & S_{12} & S_{12} & S_{12} \\ S_{21} & S_{21} & S_2 & S_2 & S_2 \\ S_{21} & S_{21} & M_2 & S_2 & S_2 \\ S_{21} & S_{21} & M_2 & M_2 & S_2 \end{pmatrix}$$

and the QF-subring eRe which is the endomorphism ring of the minimal faithful R-module, is isomorphic to S.

In the general case when B is not necessarily zero, we have

Proposition 3.4. *A is a serial ring.*

Proof. If $t = \varepsilon_i t \varepsilon_j \in B$ $(i,\ j = 1, ..., n)$, then t maps I_i in I_j and It cannot contain P_j because $P_i t = 0$ by assumption. Hence $I_i t \subseteq P_j \subseteq R$. Therefore $It \subseteq R$ for every $t \in B$ as $t = (\sum_{i=1}^n \varepsilon_i) t (\sum_{i=1}^n \varepsilon_i)$. Consequently, $B^2 = 0$. Since $B \lhd A$ and $A/B \cong R$, A is a semiprimary ring. In view of Proposition 2.1 it is enough to see $\varepsilon_i t \varepsilon_j \in J(A)^2$ for each $t \in B$. Let $b = \varepsilon_i t \varepsilon_j$ and assume $b \neq 0$, then $P_i \neq I_i$, i.e., P_i is not injective. Since I_i is also projective, every submodule between P_i and I_i is projective. Thus without loss of generality one can assume $P_i = \mathrm{Ker}(t)$. Hence the socle of P_j is contained in J and isomorphic to $S_{[i+1]}$. Consequently there is a nonzero element $r = e_{[i+1]} r e_j$ in the socle of P_j satisfying $e_{[i+1]} r = r$.

Let $u = \varepsilon_i u \varepsilon_{[i+1]}$ and $v = \varepsilon_{[i+1]} v \varepsilon_j$ in T extend a_i and r, respectively. Then we have $\mathrm{Ker}(t) = \mathrm{Ker}(uv)$, i.e., $b \in J(A)^2$. $\square$

Since endomorphisms of I_i send P_i into itself, $\varepsilon_i A \varepsilon_i$ is a factor ring of $\varepsilon_i T \varepsilon_i$ for $i = 1, 2, ..., n$. The equalities $\varepsilon_{l_k} T \varepsilon_{l_k} = \mathrm{End}(P_{l_k}) = \varepsilon_{l_k} A \varepsilon_{l_k}$ $(k = 1, ..., m)$ imply that the maximums of the lengths $c({}_{R_i}R_i)$ and $c({}_{\varepsilon_i A \varepsilon_i}\varepsilon_i A \varepsilon_i)$ $(i = 1, ..., n)$, respectively, are equal, say, to a constant d. Therefore both the lengths of ${}_RR$ and ${}_AA$ are at most dn^2. Moreover, being a factor ring of A, one obtains $c({}_RR) \leq c({}_AA)$. Let $R = A_0$. For $i > 0$ let T_i be the endomorphism ring of the injective module $I({}_{A_{i-1}}A_{i-1})$, and B_i, A_i the subsets of endomorphisms in T_i sending A_{i-1} into 0 and itself, respectively. Then B_i is an ideal in A_i and the factor ring $A_i/B_i \cong A_{i-1}$. Since $c({}_{A_{i-1}}A_{i-1}) \leq c({}_{A_i}A_i) \leq dn^2$ by the previous observation, after finitely many steps, say, N we obtain a QF-ring T_N which is the maximal quotien ring of A_N and R is a factor ring of A_N. One can now contruct A_N from the serial QF-ring T_N in the way suggested in Theorem 3.3. Thus we obtain as a final result

Theorem 3.5. *Every basic indecomposable serial ring R is a factor ring of an indecomposable basic serial ring whose maximal quotient ring is a QF-ring.*

Example 3.0.1. If Q is the factor of the path algebra of the quiver $\tilde{A}_2$ with arrows a_i from i to $[i+1]$ $(i = 1, 2, 3; n = 3)$ by the ideal generated by all paths of length 4, then Q is a weakly symmetric serial QF-ring. Let R be the factor of Q by the ideal generated by $a_3a_2a_1$ and $P_i = Re_i$ where e_i is the idempotent associated to the vertex i. Then $\mathrm{Soc}(P_1) \cong S_3$ and $\mathrm{Soc}(P_i) \cong S_i$ for $i = 2, 3$ where S_i is the simple module associated to i. Moreover, $I = I(R) = I(P_1) \oplus P_2 \oplus P_3$ and $I(P_1) \cong P_3$. Therefore, B is generated by any $\gamma = \varepsilon_1 \gamma \varepsilon_1 : I \longrightarrow I_1$ satisfying $\mathrm{Ker}\gamma = P_1 \oplus P_2 \oplus P_3$, and A is a subring of $T = \mathrm{End}({}_RI)$ isomorphic to Q.

Example 3.0.2. If Q is the factor of the path algebra of $\tilde{A}_2$ by the ideal generated by all paths of length 6, then although Q is a serial QF-ring, Q is not weakly symmetric. Let R be the factor of Q by the ideal generated by $a_2a_1a_3a_2a_1$, and put $P_i = Re_i$. Then $\mathrm{Soc}(P_1) \cong \mathrm{Soc}(P_3) \cong S_2$ and $\mathrm{Soc}(P_2) \cong S_1$. Furthermore, $I = I(R) = I(P_1) \oplus P_2 \oplus P_3$ with $I(P_1) \cong P_3$ and $B = 0$ hold.

Remark 3.0.5. Kupisch (cf. Folgerung 3.9 [11]) showed that every indecomposable (basic) serial ring R satisfying $c_i \neq 1 \pmod{n}$ $(i = 1, ..., n)$ is a factor of a QF-ring. The ring in Example 3.0.1 satisfies $c_2 = c_3 = 1 \pmod{n}$ and is a factor of a weakly symmetric serial QF-ring. However, by Example 3.0.2 there exists a factor of a serial QF-ring with $B = 0$. Haack (cf. Example 4.7 in [5]) gave an example of a serial ring which is not a factor of any serial QF-ring.

As an application of Theorem 3.5 assume that $c_1 = 1$, i.e., $Je_1 = 0$ and R is a serial ring with a simple projective module. Note that this condition is also satisfied by the rings A_i constructed above as it is easy to check. Let now R be a serial such that P_1 is simple and the maximal quotient ring of R is a QF-ring, i.e., $B = 0$. If P_{l_1} is an indecomposable injective projective module with the socle isomorphic to P_1, then a_{l_1} induces a projective cover $P_{l_1} \longrightarrow J(Re_{[l_1+1]})$ with the kernel, say, P_i, $i \neq l_1$ if $l_1 \neq n$. Since all nonzero submodules of P_{l_1} are projective, the condition $l_1 \neq n$ implies that there exists a nonzero homomorphism from I_i into $I_{[l_1+1]}$ sending P_i to 0, i.e., $B \neq 0$. This contradiction show that $l_1 = n$ and hence the maximal quotient ring of R is a matrix ring over a division ring, say F and R is an upper triangular matrix ring over F. Thus we reobtain the well-known result (cf. Theorem 32.7 [3])

Corollary 3.6. *A basic indecomposable serial ring is a factor ring of a serial ring with projective socle if and only if it has a simple projective module. A basic indecomposable serial ring with projective socle is an upper triangular matrix ring over a division ring.*

4. Weakly symmetric selfduality of serial rings

Recall that an artinian ring is *locally distributive* if the lattices of submodules of indecomposable projective left or right modules are distributive. We begin with a basic observation of Haack (cf. Proposition 4.1 [5])

Proposition 4.1. *Every weakly symmetric selfduality of a locally distributive ring is a good duality.*

Proof. Assume that ${}_RE_R$ induces a weakly symmetric selfduality for a basic indecomposable locally distributive ring R. We have to show $K = r_Rr_E(K)$ for every ideal K of R. Let $1 = e_1 + \cdots + e_n$ be a decomposition of 1 as a sum of pairwise orthogonal primitive idempotents e_i and $X = r_Rr_E(K)$, $R_i = e_iRe_i$, $E_i = e_iEe_i$. To complete the proof it is enough to see that $e_iKe_{j\,R_j}$ and $e_iXe_{j\,R_j}$ have the same length for all i, j because the ${}_{R_i}e_iRe_{j\,R_j}$ are uniserial on both sides of the same length in view of the local distributive condition. Since E induces a weakly symmetric selfduality for R, the natural nondegenerate pairing $e_iR \times Ee_i \longrightarrow E_i$ shows that e_iR is the dual of Ee_i with respect to the selfduality of R_i induced by E_i. Consequently, by putting $V_i = Ee_i, W_i = r_E(K)e_i$ we have

$$c(e_iXe_{j\,R_j}) = c({}_{R_j}e_j(V_i/W_i)) = c({}_{R_j}e_jV_i) - c({}_{R_j}e_jW_i) =$$

$$= c(e_iRe_{j\,R_j}) - c(e_iR/Ke_{j\,R_j}) = c(e_iKe_{j\,R_j}),$$

from which the statement follows. $\square$

Now we are able to give a conceptual proof to a well-known result (cf. [4], [16]).

Theorem 4.2. *Every serial ring admits a weakly symmetric selfduality.*

Proof. Using the notation of Sections 2 and 3, in view of Proposition 4.1, Proposition 2.2 and Theorem 3.5 one can assume that R is a basic indecomposable not selfinjective ring with $B = 0$. By Proposition 3.1 $T = \mathrm{End}({}_RI) = \mathrm{End}(I(R))$ is a QF-ring and one can identify ε_i $(i = 1, ..., n)$ with e_i. For simplicity let $g_k = e_{l_k}$ $(k = 1, ..., m)$. There are two cases.

1. T is weakly symmetric. Since the top and the socle of ${}_TTg_k$ $(k = 1, ..., m)$ are isomorphic, they are such as R-modules, too, and hence with composition factors, starting from the bottom, $S_{[l_{\{k-1\}}+1]}, ..., S_{l_k}$. Consequently, if $M_{[l_{\{k-1\}}+1]}, ..., M_{l_k}$ are R-submodules of $P_{[l_{\{k-1\}}+1]}, ..., P_{l_k}$ of lengths $0, 1, ..., \delta_k - 1$, then the factor R-modules I_i/M_i are injective with socle isomorphic to S_i. Otherwise $P_{[l_k+1]}$ would be an epimorphic image of the minimal submodule in $I(I_i/M_i)$ containing I_i/M_i, a contradiction. Since $M = \bigoplus_{i=1}^n M_i$ is a subbimodule of ${}_RT_R$ (as it is easy to check in view of Theorem 3.3), the bimodule ${}_RE_R = T/M = \bigoplus_{i=1}^n T\varepsilon_i/M_i \cong \bigoplus_{i=1}^n I(P_i)/M_i$ induces a weakly symmetric selfduality for R.

2. T is not weakly symmetric. By Proposition 2.2 there is an automorphism Ψ of $eTe = eRe$ fixing the set $\{g_k \mid k = 1, ..., m\}$ such that an $eRe - eRe$-bimodule $eRe_\Psi := eRe$ defined by $a * x * b = ax\Psi(b)$ $(a, x, b \in eRe)$ induces a weakly symmetric selfduality for eRe. Moreover, if $Te_\Psi := Te$ is an $T - eRe$-bimodule defined by $a * x * b = ax\Psi(b)$ $(a \in T,\ x \in Te,\ b \in eRe)$, then

$$Te_\Psi \underset{eRe}{\otimes} - : eRe - \mathrm{Mod} \longrightarrow T - \mathrm{Mod}$$

is an equivalence functor and $V := Te_\Psi \underset{eRe}{\otimes} eT$ is an $T - T$-bimodule. Since the socle of $eTg_k = eRg_k$ $(k = 1, ..., m)$ is isomorphic to $eR\Psi^{-1}(g_k)/eJ\Psi^{-1}(g_k)$, it has an element $0 \neq x = \Psi^{-1}(g_k)x$. Therefore

$$0 \neq e \otimes x = e \otimes \Psi^{-1}(g_k)x = e * \Psi^{-1}(g_k) \otimes x = e\Psi(\Psi^{-1}(g_k)) \otimes x = g_k \otimes x \in Vg_k$$

and hence the socle of ${}_TVg_k$ is isomorphic to the top of Tg_k. Therefore $Tg_k/J(T)g_k$ is the socle of ${}_TVe_i$ for all $i = [l_{\{k-1\}} + 1], ..., l_k$. Thus, as in Case 1, their composition factors as left R-modules, starting from the bottom, are $S_{[l_{\{k-1\}}+1}, ..., S_{l_k}$ and, if $M_{[l_{\{k-1\}}+1]}, ...,\ M_{l_k}$ are R-submodules of $Ve_{[l_{\{k-1\}}+1]}, ..., Ve_{l_k}$ of lengths $0, 1, ..., \delta_{l_k} - 1$, then Ve_i/M_i are injective R-modules having the socle isomorphic to S_i, respectively. Since $M = \bigoplus_{i=1}^n M_i$ is an $R - R$-subbimodule of V as it is easy to check in view of Theorem 3.3, the $R - R$-bimodule $E = V/M$ induces a weakly symmetric selfduality for R. □

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Alfréd Rényi Mathematical Institute, Hungarian Academy of Sciences,, H-1364 Budapest pf.127, Hungary

E-mail address: `anh@renyi.hu`