MATH 309

Practice problems for the midterm exam

– SUMMARY OF THE SOLUTIONS–

1., 2., 3. See the notes.

4. It is indeed possible, since there is only one linear relation between the points: $(2,3)+(1,3) = (2,5)$, and T preserves this relation: $T((2,3)) + T((1,3)) = T((2,5))$. On the other hand, T is determined by the images of these points, and by the routine calculation, we obtain that

$$T = \begin{pmatrix} 8 & -3 \\ 5 & -1 \end{pmatrix}$$

5. No, since T does not preserve the linear relation between the points:

$$T((2,3)) + T((1,3)) \neq T((2,5))$$

6. Write up the matrix of the linear transformation f for which

$$f((1,2)) = (5,1)\ ,\ \ f((-1,2)) = (3,3)\,.$$

Let the matrix of f be

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\,.$$

Then

$$\begin{aligned} a + 2b &= 5 \\ c + 2d &= 1 \\ -a + 2b &= 3 \\ -c + 2d &= 3\,. \end{aligned}$$

The first and third equations give $a = 1$, similarly, they imply $c = -1$, and then

$$T = \begin{pmatrix} 1 & 2 \\ -1 & 1 \end{pmatrix}$$

7. Let the matrix of f be A and the matrix of g be B. Then

$$A = \begin{pmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}$$

and

$$B = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}\,.$$

The matrix of $f \circ g$ is

$$A\,B = \begin{pmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{pmatrix}\,.$$

The matrix of $g \circ f$ is

$$B\,A = \begin{pmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{pmatrix}.$$

From the matrix form we see that these are reflections: $f \circ g$ is the reflection to the line of angle $2\pi/3$, and $g \circ f$ is the reflection to the line of angle $\pi/3$.

8.

$$\begin{aligned} T_\beta \cdot R_\alpha &= \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix} \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos 2\beta\cos\alpha + \sin 2\beta\sin\alpha & -\cos 2\beta\sin\alpha + \sin 2\beta\cos\alpha \\ \sin 2\beta\cos\alpha - \cos 2\beta\sin\alpha & -\sin 2\beta\sin\alpha - \cos 2\beta\cos\alpha \end{pmatrix} \\ &= \begin{pmatrix} \cos(2\beta-\alpha) & \sin(2\beta-\alpha) \\ \sin(2\beta-\alpha) & -\cos(2\beta-\alpha) \end{pmatrix} = T_{\beta-\alpha/2} \end{aligned}$$

9. See the notes; there are six reflections and six rotations.

10. An example for a basis with vectors of length at least 4 is $(3,3),(5,4)$. An example for basis vectors with large angle is $(-2,-1),(3,3)$.