MATH 309

Symmetries in the plane

These notes are intended to be a helping material; for anything that is not clear here, pictures and illustrations, you should consult the literature listed at the end.

Symmetries are perhaps the most natural geometric transformations. They show up frequently in nature, and for some reason, we usually find object that possess many symmetries aesthetic (sure you can think of a couple of examples). Perhaps the most illustrative manifestation of that phenomenon is in the Moorish ornaments of the Alhambra and other palaces; make sure you check the internet for those in case you have not seen them before.

Linear and affine transformations. We start by defining *linear transformations* of the plane. The points of the plane are represented by 2×1 column vectors; that is, $\mathbf{x} = (x_1, x_2)^\top$. For simplicity, usually we shall omit the transpose sign, and just say $\mathbf{x} = (x_1, x_2)$.

In the coordinate form, a linear transformation $f : \mathbb{R}^2 \to \mathbb{R}^2$ can be written as

$$f(\mathbf{x}) = A\,\mathbf{x},$$

where A is a 2×2 matrix:

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

Note that the image of $\mathbf{0}$ is $\mathbf{0}$; thus, the origin is a fixed point of a linear transformation. Also note that the image of a line through the origin is determined by the image of any point of the line different from $\mathbf{0}$. Moreover, suppose that the image of $\mathbf{x} = (x_1, x_2)$ is $\mathbf{u} = (u_1, u_2)$, and the image of $\mathbf{y} = (y_1, y_2)$ is $\mathbf{v} = (v_1, v_2)$, where $\mathbf{x}$ and $\mathbf{y}$ are not on the same line trough the origin. Then, writing out the matrix multiplication,

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= u_1 \\ a_{21}x_1 + a_{22}x_2 &= u_2 \\ a_{11}y_1 + a_{12}y_2 &= v_1 \\ a_{21}y_1 + a_{22}y_2 &= v_2. \end{aligned}$$

Let's try to determine the matrix A from this information. On one hand, we have the above set of four linear equations. On the other hand, A has four entries. Now, the condition that $\mathbf{x}, \mathbf{y}$, and $\mathbf{0}$ are not on a line means that the system of equations is independent. In that case, we know from linear algebra, that there is exactly one solution for the variables $a_{11}, a_{12}, a_{21}, a_{22}$. We say that a set of n points in $\mathbb{R}^n$ is in *general position*, if the points are linearly independent (so they span the whole space $\mathbb{R}^n$). Thus, *a linear transformation is determined by the images of two points in general position.*

This also gives us a way to find the matrix. Suppose, that we are given the images of two points. Then we can write up the above system of equations, and solve it - notice that the first and the third, and the second and the fourth share the common variables. So it is very easy to solve this system of linear equations.

It is the easiest to read out the images of the coordinate vectors:

$$f((1,0)) = A\,(1,0)^\top = (a_{11}, a_{21})\,,$$
$$f((0,1)) = A\,(0,1)^\top = (a_{12}, a_{22})\,.$$

Thus, the columns of the matrix A are the images of the coordinate vectors.

What happens if the matrix A is degenerate, i.e. its rank is less than 2? Well, there are two cases. The first (and simplest) is when A is the 0 matrix; in this case, every point is mapped to $\mathbf{0}$. In the second case, the rank of A is 1, that is, A has the form

$$A = \left(\begin{array}{cc} a_{11} & \lambda a_{11} \\ a_{21} & \lambda a_{21} \end{array} \right)$$

for some real scalar λ. Then, the image of $\mathbf{x}$ is

$$f(\mathbf{x}) = (a_{11}x_1 + \lambda\, a_{11}x_2, a_{21}x_1 + \lambda\, a_{21}x_2),$$

which is $(x_1 + \lambda\, x_2)$ times the point (a_{11}, a_{21}). Thus, the image of any point lies on the line through $\mathbf{0}$ and (a_{11}, a_{21}). It is easy to check that the transformation is a scaled copy of the projection to this line. (Why?) We note that in general, a characteristic property of projections is that for the matrix determining them, $AA = A$. (This is true in any dimensions; what is the relation between the rank of the matrix and the dimension of the subspace that we project to?)

A nice property of the linear maps is that we can multiply them together: if f and g are linear transformations, then

$$(g \circ f)(\mathbf{x}) = g(f(\mathbf{x})),$$

that is, first we apply f, and then apply g to the image points. It is easy to see that the resulting map is again a linear map, whose matrix is $B\,A$, where A and B are the matrices of f and g. Note a very important property: the multiplication of linear transformations is not commutative! That is, there are linear maps f and g, for which $f \circ g \neq g \circ f$. For an example, consider a rotation and a reflection (the definitions are below).

For symmetries, we will usually consider only linear transformations. However, there is another important class that we will encounter: these are the *affine transformations* of the plane. In coordinate form, an affine transformation $g : \mathbb{R}^2 \to \mathbb{R}^2$ is written as

$$g(\mathbf{x}) = \mathbf{a} + A\,\mathbf{x},$$

where A is again a 2×2 matrix, and $\mathbf{a}$ is a point of the plane, which is called the *shift* or *translation.* Note that the image of the origin is now $\mathbf{a}$. We can think of the affine transformation as first executing the linear transformation given by A, and then translating the picture by $\mathbf{a}$.

We have seen that a linear transformation is determined by the images of two points in general position; with a similar reasoning, we obtain that *affine transformations of the plane are determined by the images of three points in general position.* Similarly, affine transformations of $\mathbb{R}^n$ (defined just in the same way as above) are determined by the images of $n + 1$ points in general position.

Isometries of the plane. Next, we are going to analyse linear transformations of the plane that preserve the distances between points. It turns out that it suffices to assume that they map the unit circle onto itself. (Why?) Can you find out which maps these are? Good news: there are only two types of them! This is due to the fact that, as we have seen, it is enough to track two points. Giving the unit circle the positive (counter-clockwise) order, the order of the image points may be the same as the original - in this case we are talking about an *orientation preserving transformation.* Or, the two points may be "flipped", that is, their order is opposite to the order of their images. This is an *orientation changing transformation.* So here they are:

Rotations. These are the orientation preserving linear isometries. The matrix of the rotation by an angle α is given by

$$R_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}$$

Note that unless $\alpha = 2k\pi$, the only invariant point is the origin.

Reflections. These are the orientation changing linear isometries. Given a line through the origin of angle β, the matrix of the reflection through the line is

$$T_\beta = \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ \sin 2\beta & -\cos 2\beta \end{pmatrix}$$

The line is called the axis of the reflection; note that it is invariant.

Now, what is the product of two rotations? And a reflection and rotation? Plainly, the product of two isometries is again an isometry. Particularly, the product of two linear isometries is a linear isometry. Since there is only two type of these depending on the orientation preservation, we can state that *the product of finitely many linear isometries of the plane is a rotation, if there is an even number of reflections, and a reflection, if there is an odd number of reflections among them.* (You can verify this by using the matrix forms).

Now, what about general isometries? Well, they are the product (or composition) of a linear isometry and a translation. These are also called skew isometries.

Symmetries in the plane. So we get to the question: what do we mean by symmetry? Especially, what does it mean that an object (in the present situation, a planar figure) is symmetric? Here is the definition:

Given a subset S and an isometry f of the plane, we say that f is a symmetry of S (or S is symmetric with respect to f), if S is invariant under f:

$$f(S) = S.$$

This means that for any point $\mathbf{x} \in S$, $f(\mathbf{x}) \in S$, and on the other hand, there is a $\mathbf{y} \in S$, such that $f(\mathbf{y}) = \mathbf{x}$.

Let assume that S is bounded. Then it can only have linear isometries (why?). Based on the above, there are only two types of isometries: rotational and reflectional. The general question will be to determine all the isometries of a given shape K. This is usually not a hard task; we will take the simplest example, which is the symmetries of the regular n-gon K_n. For this, we assume that the regular n-gon is positioned so that its centre is the origin, and one of its vertices lies on the x-axis. First, let us consider the rotational symmetries. The centre of rotations is the centre of K_n, the origin. The image of a vertex must be a vertex as well. Modulo 2π, this gives n rotational symmetries:

$$R\left(k\frac{2\pi}{n}\right),\ k=0,\ldots,n-1.$$

Second, let us list the reflection symmetries. Here, we must separate between the cases when n is even and odd. If n is odd, then the axes for reflection symmetries are the lines going through the origin and the vertices; they bisect the sides opposite to the vertices. Thus, we have n reflection symmetries:

$$T\left(k\frac{2\pi}{n}\right),\ k=0,\ldots,n-1.$$

If n is even, then there are two types of axes: lines which go through opposite vertices ($n/2$ of them), and lines which go through the bisecting points of opposite faces ($n/2$ of them as well). So altogether, we again have n reflection symmetries:

$$T\left(k\frac{\pi}{n}\right),\ k=0,\ldots,n-1.$$

Thus, the regular n-gon has n rotational and n reflectional symmetries.

Further reading.

Wikipedia: Linear map, Affine transformation, Symmetry, Dihedral group.
`http://math.hws.edu/eck/math110_s08/symmetries/index.html`
`http://www.math.cornell.edu/~mec/2008-2009/Victor/part1.htm`