MATH 309

Platonic solids

In the previous lecture, we have seen symmetries of planar objects. In particular, we witnessed that regular polygons possess many symmetries. What happens, if we leave the plane and go for the 3–dimensional space? Isometries, that serve as symmetries, can be shown to be compositions of rotations around an axis, reflections to a plane, and translations. The question that we are interested in is what is the analogue of regular polygons. Thus, we are looking for three dimensional polytopes - solids bound by planar faces - which possess a large amount of symmetry.

A regular (or Platonic) solid is a convex polytope which satisfies the following:
1. all of its faces are congruent regular polygons;
2. all vertices have the same number of faces adjacent to them.

We will characterise the Platonic solids with the aid of Euler's formula. Let K be a Platonic solid, and let v be the number of vertices, e be the number of edges, and f be the number of faces of K. Euler's formula states that

$$v - e + f = 2. \tag{1}$$

All the faces are congruent; assume that they have n vertices (and, thus, n edges). Let us assume moreover that each vertex is adjacent to m faces (and, thus, it has m edges adjacent to it). Since each edge is adjacent to exactly two faces,

$$2e = n\,f\,. \tag{2}$$

Moreover, each edge is adjacent to two vertices, and one vertex belongs to m edges, thus

$$m\,v = 2e\,. \tag{3}$$

Expressing v and f in terms of e, and substituting to Euler's formula (1), we obtain that

$$\frac{2e}{m} - e + \frac{2e}{n} = 2\,.$$

Rearranging, we arrive at

$$\frac{1}{m} + \frac{1}{n} = \frac{1}{2} + \frac{1}{e}\,. \tag{4}$$

Note that since K is a 3–dimensional polytope, each of its faces is a polygon and thus has at least 3 vertices: that is, $n \geqslant 3$. Moreover, at each vertex, there are at least three faces meeting: $m \geqslant 3$. On the other hand, since $e \geqslant 1$, we must have

$$\frac{1}{m} + \frac{1}{n} > \frac{1}{2}\,.$$

These conditions do not leave too much leeway; there are only five possible (n, m) pairs for which the above inequality holds. These are $(3, 3), (3, 4), (3, 5), (4, 3), (5, 3)$.

For each of these pairs, there belongs a Platonic solid. We list them below.

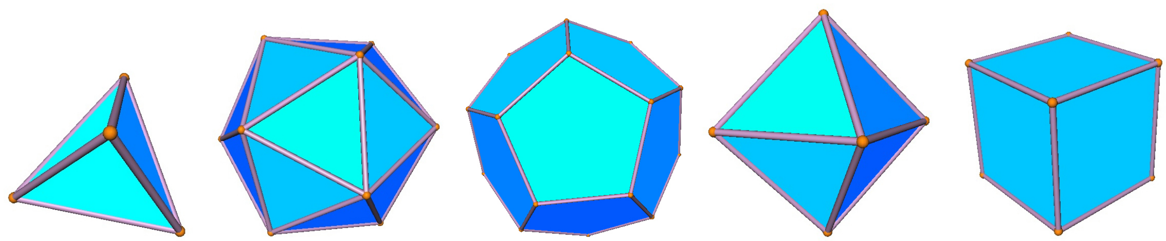
Tetrahedron. Here $n = 3$ and $m = 3$. Thus, (4) yields that $e = 6$. Thus, by (3), $v = 4$, and by (2), $f = 4$. There are 4 vertices and 4 faces of the tetrahedron; the faces are regular triangles, and the vertices are adjacent to 3 edges.

Octahedron. Here $n = 3$ and $m = 4$. Thus, (4) yields that $e = 12$. Thus, by (3), $v = 6$, and by (2), $f = 8$. There are 6 vertices and 8 faces of the octahedron; the faces are regular triangles, and the vertices are adjacent to 4 edges.

Icosahedron. Here $n = 3$ and $m = 5$. Thus, (4) yields that $e = 30$. Thus, by (3), $v = 12$, and by (2), $f = 20$. There are 12 vertices and 20 faces of the icosahedron; the faces are regular triangles, and the vertices are adjacent to 5 edges.

Cube. Here $n = 4$ and $m = 3$. Thus, (4) yields that $e = 12$. Thus, by (3), $v = 8$, and by (2), $f = 6$. There are 8 vertices and 6 faces of the cube; the faces are squares, and the vertices are adjacent to 3 edges.

Dodecahedron. Here $n = 5$ and $m = 3$. Thus, (4) yields that $e = 30$. Thus, by (3), $v = 20$, and by (2), $f = 12$. There are 20 vertices and 12 faces of the octahedron; the faces are regular pentagons, and the vertices are adjacent to 3 edges.



[From Wikipedia:] *The ancient Greeks studied the Platonic solids extensively. In particular, the Platonic solids feature prominently in the philosophy of Plato for whom they are named. Plato associated each of the four classical elements (earth, air, water, and fire) with a regular solid. Earth was associated with the cube, air with the octahedron, water with the icosahedron, and fire with the tetrahedron. There was intuitive justification for these associations: the heat of fire feels sharp and stabbing (like little tetrahedra). Air is made of the octahedron; its minuscule components are so smooth that one can barely feel it. Water, the icosahedron, flows out of one's hand when picked up, as if it is made of tiny little balls. By contrast, a highly un-spherical solid, the hexahedron (cube) represents earth. These clumsy little solids cause dirt to crumble and break when picked up, in stark difference to the smooth flow of water. Moreover, the solidity of the Earth was believed to be due to the fact that the cube is the only regular solid that tesselates Euclidean space. The fifth Platonic solid, the dodecahedron, Plato obscurely remarks, "...the god used for arranging the constellations on the whole heaven". Aristotle added a fifth element, aither (aether in Latin, "ether" in English) and postulated that the heavens were made of this element, but he had no interest in matching it with Plato's fifth solid.*