MATH 309

Lattices in the plane

When tiling the plane with squares or with triangles, we see that the centres form a highly regular and symmetric pattern. We see a similar image when observing crystals with a microscope. Such a pattern will be called a *lattice in the plane.* Here is the formal definition:

Definition 1. *Let u and v be two independent non-zero vectors in the plane. The set Λ of points*

$$a\,u + b\,v \ : \ a, b \in \mathbb{Z}$$

is called a lattice. The pair (u, v) is called a basis of Λ.

Thus, the lattice Λ is the set of linear combinations of u and v with integer coefficients. In particular, we see that if $x, y \in \Lambda$, then $x + y \in \Lambda$, and $ax \in \Lambda$ for any $a \in \mathbb{Z}$. Writing it shortly,

$$\Lambda + \Lambda \subset \Lambda \ , \text{ and } \ a\Lambda \subset \Lambda \text{ for any } a \in \mathbb{Z}.$$

If $u = e_1$ and $v = e_2$ the coordinate vectors, then Λ is the integer lattice $\mathbb{Z}^2$: it consists of the points of the plane, both of whose coordinates are integers.

We have seen, that for any $u, v \in \mathbb{R}^2$, there exists a linear transformation H, so that $H(e_1) = u$ and $H(e_2) = v$. The matrix of H is given by

$$H = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix}$$

where $u = (u_1, u_2)$ and $v = (v_1, v_2)$, understood as column vectors. Then, it is straightforward that

$$H(\mathbb{Z}^2) = \lambda,$$

so, every planar lattice is the image of $\mathbb{Z}^2$ by a linear transformation, whose matrix is formed by the basis vectors of the lattice. The determinant of H is called the *determinant of the lattice*:

$$\det \Lambda = \det H = \begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix}$$

It is not only the pair of vectors (u, v) that generate Λ; indeed, there are infinitely many such pairs. Any pair of vectors $(\tilde{u}, \tilde{v})$ in Λ, which generate exactly the same lattice Λ, is called a *basis of* Λ.

The parallelogram with vertices $0, u, v, (u + v)$ is called the *fundamental parallelogram* of Λ. As we see by a simple calculation, the area of the fundamental parallelogram is $|\det \Lambda|$.

The absolute value of the determinant of a lattice Λ is independent of the basis. This is the content of the following theorem.

Theorem 1 (Lattice bases theorem)**.** *The area of the fundamental parallelogram of Λ, which is $|\det \Lambda|$, is independent of the choice of the basis of Λ.*

Proof. Let (u, v) and $(\tilde{u}, \tilde{v})$ be two bases for the lattice Λ, and let H and $\tilde{H}$ be the corresponding base matrices. Since $\tilde{u}$ and $\tilde{v}$ are points of Λ, there exist integer numbers a_1, a_2, b_1, b_2, so that

$$\tilde{u} = a_1\, u + a_2\, v$$
$$\tilde{v} = b_1\, u + b_2\, v\,.$$

The linear transformation T with matrix

$$T = \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix}$$

satisfies that

$$HT = \begin{pmatrix} u_1 & v_1 \\ u_2 & v_2 \end{pmatrix} \begin{pmatrix} a_1 & b_1 \\ a_2 & b_2 \end{pmatrix} = \begin{pmatrix} \tilde{u}_1 & \tilde{v}_1 \\ \tilde{u}_2 & \tilde{v}_2 \end{pmatrix} = \tilde{H},$$

the matrix of the basis $(\tilde{u}, \tilde{v})$. Thus, T is matrix of the change of basis of Λ. By the same argument (by changing the role of (u, v) and $(\tilde{u}, \tilde{v})$), there exists another matrix S with integer entries, so that

$$\tilde{H}S = H.$$

Remember that $\det(AB) = \det A \det B$. We deduce that

$$\det H = \det(\tilde{H}S) = \det(HTS) = \det H \det T \det S.$$

Since u and v are linearly independent, $\det H \neq 0$, and we can divide the above equation by it. This leads to

$$\det T \det S = 1.$$

Now we refer to the fact that both T and S have integer entries, and thus, their determinants are integer numbers as well! But there are only two possibilities to write 1 as a product of two integers:

$$1 = 1 \cdot 1\ ,\ 1 = (-1) \cdot (-1).$$

In both cases, $|\det T| = 1$, and thus, $|\det H| = |\det \tilde{H}|$. □

Thus, the matrices belonging to changing the basis of Λ have integer coordinates and determinant ± 1.

The most interesting question here is: given two vectors $(\tilde{u}, \tilde{v}) \in \Lambda$, how can we decide if they form a basis? Luckily, it is a very easy task:

The vectors $u, v \in \Lambda$ form a basis of Λ, if and only if

$$\begin{vmatrix} u_1 & v_1 \\ u_2 & v_2 \end{vmatrix} = \pm \det \Lambda.$$

There is also another method:

The vectors $u, v \in \Lambda$ form a basis of Λ, if and only if the triangle with vertices $0, u$, and v does not contain any other point of the lattice Λ.

Choosing the "best" basis is not straightforward. Sometimes, we impose conditions on the length of the vectors, sometimes, on the angle between them, and so on. We are going to practice such problems in class.