Weakly representable relation algebras form a variety

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Abstract. We prove that the class of weakly representable relation algebras is closed under homomorphic images, hence it is a variety. As a corollary we classify the subdirectly irreducible algebras in this class.

Introduction

Abstract relation algebras were introduced by Tarski in [17], as a class of algebras satisfying a finite system of equations. A *representation* of a relation algebra is an injective homomorphism into a *proper relation algebra* (the elements of which are binary relations, and the operations are the ordinary intersection, union, complement, relation inverse and relation composition). A relation algebra is said to be *representable* if it has a representation, that is, if it is isomorphic to a proper relation algebra. It turned out that the class of representable relation algebras (`RRA`'s, for short) is not finitely axiomatizable (cf. [14]), so Tarski's finitely many axioms could not be enough to capture exactly the proper relation algebras.

A *weak representation* of a relation algebra is an injective map into a proper relation algebra, preserving the operations intersection, relation inverse and relation composition, but not necessarily the union and the complement. The exact definition is in Def.1.1 below.

It was Jónsson in [11] who defined weakly representable relation algebras (`wRRA`'s for short), in a slightly different form. In this paper we adopt an equivalent form of the definition given in [2], which is still equivalent to the original one, cf. [8] p.164. In [11] Theorem 3, Jónsson proved that the class of `wRRA`'s form a *quasi-variety* by writing up an infinite set of quasi-equations axiomatizing it. He raised the problems whether `wRRA`'s can be defined by a *finite* set of axioms, or, by *equations*. The first question was answered negatively in e.g. [6, 10]. Here we answer the second question: **Yes**, `wRRA`'s do form a variety.

It is easy to see that subalgebras and products of `wRRA`'s are again `wRRA`'s (this fact also follows from Jónsson's theorem mentioned above). In Theorem 3.1 we give a direct proof that the class of `wRRA`'s is also closed under taking homomorphic images, by introducing the concept of the *trace operator* (see Section 2, Def.2.4), which is also interesting on its own right: There is a "weakly represented" Boolean

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algebra for each weakly represented relation algebra $\mathfrak{A}$ with the same congruence lattice, whose elements are the *traces* of elements of $\mathfrak{A}$. (See also Cor. 2.8).

As a corollary we can classify the subdirectly irreducible wRRA's (Cor. 3.8).

In section 4 we sketch more proofs of the analogous theorem for RRA's, and discuss the difficulties in adopting them to wRRA's.

Surprisingly, if we exchange intersection and union in the definition of 'weak representation', we get that *all* relation algebras have such a weak representation, cf. [12] Theorem 4.22.

1. Basic notation and definitions

We denote binary relations by Greek letters: $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}, \ldots$
We use both X^2 and $X \times X$ for the set of ordered pairs $\langle x, y\rangle$ where $x, y \in X$.
We use the infix notation $x\boldsymbol{\alpha}y$ for $\langle x, y\rangle \in \boldsymbol{\alpha}$.
We denote the relation composition by $\circ$, which acts from left to right, i.e. $x(\boldsymbol{\alpha}\circ\boldsymbol{\beta})z \Leftrightarrow \exists y : (x\boldsymbol{\alpha}y$ and $y\boldsymbol{\beta}z)$.
The inverse relation of $\boldsymbol{\alpha}$ is denoted by $\boldsymbol{\alpha}^{\smile}$, that is, $x\boldsymbol{\alpha}y \;\Leftrightarrow\; y(\boldsymbol{\alpha}^{\smile})x$.
We denote algebras by German capitals: $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{M} \ldots$ and their underlying sets by the corresponding Latin letters: $A, B, C, M \ldots$
The set operations are denoted by $\cap, \cup, ^{\mathsf{C}}, \emptyset$ with partial order $\subseteq$.
We let $A \setminus B := A \cap B^{\mathsf{C}}$.
For abstract Boolean operations we use $\cdot, +, -, 0$ with partial order $\leqslant$.
We let $a - b := a\cdot(-b)$.
$\mathcal{P}(X)$ denotes the power set of a set X.
If $\boldsymbol{\alpha}$ is a relation then $\mathrm{dom}\boldsymbol{\alpha}$ denotes its *domain.*
We use the notation $\mathrm{im}f$ and $\ker f$ for the *image* and *kernel* of a homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ between algebras with Boolean reduct. That is, $\mathrm{im}f := \{f(a) \mid a \in A\} \subseteq B$ and $\ker f := \{a \in A \mid f(a) = 0\}$.

Definition 1.1. An algebra $\mathfrak{A} = \langle A, \cap, +, -, \circ, ^{\smile}, \mathrm{id}, \boldsymbol{\emptyset}\rangle$ is a *weakly represented relation algebra*, if $\langle A, \cap, +, -, \boldsymbol{\emptyset}\rangle$ is a Boolean algebra, $\langle A, \circ, \mathrm{id}\rangle$ is a monoid, the elements of A are binary relations, and the operations $\cap, \circ, ^{\smile}, \boldsymbol{\emptyset}$ are the set theoretic intersection, relation composition, relation inverse and the empty relation, respectively, while the constant operation id is the identity relation of some set.

An algebra of the same similarity type is called a *weakly representable relation algebra* (wRRA for short), if it is isomorphic to a weakly represented relation algebra.

Remark. In [11] and [2] wRRA's were explicitly defined to be relation algebras. Here we did not require this, however, Def.1.1 is equivalent to the original one, since all axioms of relation algebras are consequences.[42]

Proposition 1.2. *Let $\mathfrak{A}$ be a weakly represented relation algebra, and let $\boldsymbol{E}$ be its greatest element (i.e. $\boldsymbol{E} = -\boldsymbol{\emptyset}$). Then we have*

[42]In a weakly represented algebra of relations, we always have $(\boldsymbol{\alpha} \cap \boldsymbol{\beta})^{\smile} = \boldsymbol{\alpha}^{\smile} \cap \boldsymbol{\beta}^{\smile}$, so the involution $^{\smile}$ is monotone, and therefore the relation algebra axiom $(\boldsymbol{\alpha} + \boldsymbol{\beta})^{\smile} = \boldsymbol{\alpha}^{\smile} + \boldsymbol{\beta}^{\smile}$ holds. The Peircean law $[(\boldsymbol{\alpha}\circ\boldsymbol{\beta}) \cap \boldsymbol{\gamma}^{\smile} = \boldsymbol{\emptyset} \Leftrightarrow (\boldsymbol{\beta}\circ\boldsymbol{\gamma}) \cap \boldsymbol{\alpha}^{\smile} = \boldsymbol{\emptyset}]$ also holds. Thus, composition is a *conjugated* operation (cf. [8]), which guarantees the additivity $[(\boldsymbol{\alpha} + \boldsymbol{\beta})\circ\boldsymbol{\gamma} = (\boldsymbol{\alpha}\circ\boldsymbol{\gamma}) + (\boldsymbol{\beta}\circ\boldsymbol{\gamma})]$.

(1) *The partial order of* $\mathfrak{A}$ *(given by* $\boldsymbol{\alpha} \leqslant \boldsymbol{\beta}$ *iff* $\boldsymbol{\alpha} \cap \boldsymbol{\beta} = \boldsymbol{\alpha}$*) is the ordinary inclusion of sets.*
(2) $\boldsymbol{E}$ *is an equivalence relation of some set* X*, and* $\mathrm{id} = \mathrm{id}_X = \{\langle x, x\rangle : x \in X\}$*.*
(3) $\boldsymbol{E} = \bigcup_{i \in H} {X_i}^2$ *for some index set* H *and pairwise disjoint nonempty sets* $(X_i)_{i \in H}$*.*
(4) *For any* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A$*:* $\boldsymbol{\alpha} + \boldsymbol{\beta} \supseteq \boldsymbol{\alpha} \cup \boldsymbol{\beta}$ *and* $-\boldsymbol{\alpha} \subseteq \boldsymbol{\alpha}^{\complement}$*.*

Proof. (1): Trivial.
(2): Let $X := \mathrm{dom}\boldsymbol{E} = \{x \mid \exists y : x\boldsymbol{E}y\}$. Then, by $\boldsymbol{E} \circ \boldsymbol{E} \subseteq \boldsymbol{E}$ and $\boldsymbol{E}^{\smile} \subseteq \boldsymbol{E}$, we conclude that $\boldsymbol{E}$ is transitive, symmetric, and it is also reflexive on X.
By definition $\mathrm{id} = \mathrm{id}_Y$ for some set Y, then $Y \subseteq X$ since $\mathrm{id} \subseteq \boldsymbol{E}$, and $X \subseteq Y$ because $\boldsymbol{E} = \boldsymbol{E} \circ \mathrm{id}$.
(3): Choose $(X_i)_{i \in H}$ to be the equivalence classes of $\boldsymbol{E}$. (We note that there is a natural bijection between the index set H and the quotient set $X/_{\boldsymbol{E}}$.)
(4) follows from (1). □

Since the Boolean operations $+$, $-$ play special roles, we introduce the following concept.

Definition 1.3. Let $\mathfrak{A}$ and $\mathfrak{C}$ be algebras with Boolean reduct. By a *semimorphism* we mean a map $\varphi : A \to C$ which preserves all the operations (including the Boolean intersection), but not necessarily the Boolean $+$ and $-$ operations.
If φ is a semimorphism, we simply draw $\varphi : \mathfrak{A} \dashrightarrow \mathfrak{C}$.

N.B. Semimorphisms are called *weak homomorphisms* in [1].

Let $\mathfrak{P}(\boldsymbol{E})$ denote the *full proper* relation algebra of relations included in a given equivalence relation $\boldsymbol{E}$, with ordinary set theoretic operations. That is, $\mathfrak{P}(\boldsymbol{E}) = \langle \mathcal{P}(\boldsymbol{E}), \cap, \cup, {}^{\complement}, \circ, {}^{\smile}, \mathrm{id}, \emptyset \rangle$.
We note that a relation algebra $\mathfrak{A}$ is weakly representable if and only if there is an injective semimorphism $\mathfrak{A} \dashrightarrow \mathfrak{P}(\boldsymbol{E})$ for some equivalence relation $\boldsymbol{E}$.

A *full* `wRRA` would be a weakly represented relation algebra with greatest element $\boldsymbol{E}$, whose underlying set is the whole $\mathcal{P}(\boldsymbol{E})$. In [1], Alm asks whether every `wRRA` embeds (by an injective relation algebra morphism) in a full `wRRA`. Since Andréka in [2] and Alm in [1] gave examples of nonrepresentable `wRRA`'s, the next proposition answers this question negatively:

Proposition 1.4. *The abstract Boolean operations* $+$ *and* $-$ *of a full* `wRRA` *must be the set theoretic* $\cup$ *and* ${}^{\complement}$*, respectively. So, full* `wRRA`*'s are the same as full proper relation algebras, and thus, a* `wRRA` *embeds in a full* `wRRA` *iff it is an* `RRA` *(i.e. representable).*

Proof. Assume that $\mathfrak{A} = \langle \mathcal{P}(\boldsymbol{E}), \cap, +, -, \circ, {}^{\smile}, \mathrm{id}, \emptyset \rangle$. The underlying partial order of $\mathfrak{A}$ is the set inclusion by Proposition 1.2.(1). The reduct $\langle \mathcal{P}(\boldsymbol{E}), \cap, + \rangle$ is clearly a lattice, so $+$ must be the *lowest upper bound* operation corresponding to the partial order $\subseteq$. In the full $\mathcal{P}(\boldsymbol{E})$ it is none other than the set theoretic union. This quickly implies that the negation $(-)$ must be the set complement $({}^{\complement})$, too. □

We recall that a subset $I \subseteq A$ of an abstract Boolean algebra $\mathfrak{A}$ is an *ideal*, if in case of $\boldsymbol{\alpha}, \boldsymbol{\beta} \in I$ and $\boldsymbol{\delta} \leqslant \boldsymbol{\alpha}$ we have $\boldsymbol{\alpha} + \boldsymbol{\beta} \in I$ and $\boldsymbol{\delta} \in I$. The dual notion is *filter*, and an *ultrafilter* is a maximal filter not containing 0. We sometimes refer to ideals of Boolean algebras as *Boolean ideals*.
We also recall (from e.g. [9], Thm.6.2.1.), that a subset $C \subseteq A$ extends to an ultrafilter if and only if it has the *finite intersection property*, that is, every finite intersection of elements $c_1, c_2, \dots, c_n \in C$ is nonzero: $c_1 \cdot c_2 \cdot \ldots \cdot c_n \neq 0$.

Lemma 1.5. *Let I be an ideal of the Boolean algebra $\mathfrak{A}$, and $a \in A \setminus I$. Then there exists an ultrafilter $\mathcal{U} \subseteq A$ such that $I \cap \mathcal{U} = \emptyset$ and $a \in \mathcal{U}$.*

Proof. It is enough to prove that the set $\{-b \mid b \in I\} \cup \{a\}$ has the finite intersection property. For that end, let $b_1, \dots, b_n \in I$ and $b := b_1 + \dots + b_n$. Then we have $b \in I$, too, and

$$(-b_1) \cdot \ldots \cdot (-b_n) \cdot a = (-b) \cdot a = a - b$$

If it were 0, then it would mean $a \leqslant b$, by Boolean arithmetic, and therefore $a \in I$ would hold, contradicting our assumption on a. □

We call $I \subseteq A$ a *relational ideal*, if it is a kernel of a homomorphism $\mathfrak{A} \to \mathfrak{B}$ between relation algebras. It is an excersize to check that I is a relational ideal iff it is a Boolean ideal closed under the unary operations $\breve{}$ and $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha} \circ \boldsymbol{E}$ (cf. e.g. [16], Prop.7.4 or [8], Def.2.36).

2. The trace operator

Let us fix an arbitrary `wRRA` $\mathfrak{A}$ for this whole section. Then we have an injective semimorphism $\mathfrak{A} \dashrightarrow \mathfrak{P}(\boldsymbol{E})$ for some equivalence relation $\boldsymbol{E}$. Without loss of generality we can assume that A is included in $\mathcal{P}(\boldsymbol{E})$, and that the greatest element of A is $\boldsymbol{E}$ itself. We will denote the equivalence classes of $\boldsymbol{E}$ by $(X_i)_{i \in H}$. Thus, $\boldsymbol{E} = \bigcup\limits_{i \in H} {X_i}^2$.

We start with an important observation: the map $\boldsymbol{\alpha} \mapsto \boldsymbol{E} \circ \boldsymbol{\alpha} \circ \boldsymbol{E}$ is a *closure operator* on A. Later we will see that there is a natural one-to-one correspondence between certain subsets of the index set H and the closed elements according to this closure operator.

Definition 2.1. We call a relation $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{E})$ *squarefull*, if $\boldsymbol{E} \circ \boldsymbol{\alpha} \circ \boldsymbol{E} = \boldsymbol{\alpha}$.

Proposition 2.2. *The following assertions are equivalent for an $\boldsymbol{\alpha} \in \mathcal{P}(\boldsymbol{E})$:*

a) *$\boldsymbol{\alpha}$ is squarefull.*
b) *There is a $\boldsymbol{\beta} \in \mathcal{P}(\boldsymbol{E})$ such that $\boldsymbol{\alpha} = \boldsymbol{E} \circ \boldsymbol{\beta} \circ \boldsymbol{E}$.*
c) *$\boldsymbol{\alpha} = \bigcup\limits_{j \in H'} {X_j}^2$ for some subset $H' \subseteq H$.*
d) *For any $i \in H$ and any elements $x, y, x', y' \in X_i$, we have $x \boldsymbol{\alpha} y \Leftrightarrow x' \boldsymbol{\alpha} y'$.*

The proof is left to the reader.
In cylindric algerbas the analogous notion is called *zero dimensional element* (cf. [7]).

Throughout the section let B denote the set of all squarefull elements in $\mathfrak{A}$:

$$B := \{\boldsymbol{\beta} \in A \mid \boldsymbol{E}\circ\boldsymbol{\beta}\circ\boldsymbol{E} = \boldsymbol{\beta}\}.$$

We note that any squarefull relation $\boldsymbol{\beta} \in B$ satisfies $\boldsymbol{\beta}\circ\boldsymbol{\beta} = \boldsymbol{\beta} = \boldsymbol{\beta}^{\smile}$.

Proposition 2.3. *B is closed under the Boolean operations of $\mathfrak{A}$.*

Proof. Clearly, $\boldsymbol{\emptyset}$ and $\boldsymbol{E}$ are squarefull.
The fact that B is closed under intersection, follows easily from Proposition 2.2.c).
For the negation $(-)$, observe that $-\boldsymbol{\alpha}$ is the greatest element *of* A which makes $\boldsymbol{\alpha} \cap (-\boldsymbol{\alpha}) = \boldsymbol{\emptyset}$, or equivalently, which makes $-\boldsymbol{\alpha} \subseteq \boldsymbol{\alpha}^{\complement}$.
On the other hand, if $\boldsymbol{\alpha}$ is squarefull, so is $\boldsymbol{\alpha}^{\complement}$ by Proposition 2.2.c), and hence $-\boldsymbol{\alpha} \subseteq \boldsymbol{\alpha}^{\complement}$ implies $\boldsymbol{E}\circ(-\boldsymbol{\alpha})\circ\boldsymbol{E} \subseteq \boldsymbol{\alpha}^{\complement}$. But $\boldsymbol{E}\circ(-\boldsymbol{\alpha})\circ\boldsymbol{E}$ still lies in A and $\boldsymbol{E}\circ(-\boldsymbol{\alpha})\circ\boldsymbol{E} \supseteq -\boldsymbol{\alpha}$.
Therefore $\boldsymbol{E}\circ(-\boldsymbol{\alpha})\circ\boldsymbol{E} = -\boldsymbol{\alpha}$ because of the maximality of $-\boldsymbol{\alpha}$.
If $\boldsymbol{\alpha}, \boldsymbol{\beta} \in B$ then $-\boldsymbol{\alpha}, -\boldsymbol{\beta} \in B$, too, and thus $\boldsymbol{\alpha} + \boldsymbol{\beta} = -(-\boldsymbol{\alpha} \cap -\boldsymbol{\beta}) \in B$. □

We will denote the Boolean algebra $\langle B, \cap, +, -, \boldsymbol{\emptyset}\rangle$ by $\mathfrak{B}$.

Definition 2.4. For each relation $\boldsymbol{\alpha} \in A$ we collect those indices $i \in H$ where $\boldsymbol{\alpha}$ "acts", i.e. let

$$\tau(\boldsymbol{\alpha}) := \{i \in H \mid \boldsymbol{\alpha} \cap {X_i}^2 \neq \boldsymbol{\emptyset}\}.$$

We call $\tau(\boldsymbol{\alpha}) \subseteq H$ the *trace* of $\boldsymbol{\alpha}$, and the mapping $\tau : A \to \mathcal{P}(H)$ the *trace operator* of $\mathfrak{A}$. For a subset $I \subseteq A$ we write $\tau(I)$ for the set $\{\tau(\boldsymbol{\alpha}) \mid \boldsymbol{\alpha} \in I\}$.

Note that $i \in \tau(\boldsymbol{\alpha})$ holds if and only if there are $x, y \in X_i$ such that $x\boldsymbol{\alpha}y$.
We reserve the letter C for denoting the range of τ (that is, $C := \mathrm{im}\tau$).

Proposition 2.5. *The following statements hold for τ:*

1. *For any $\boldsymbol{\alpha} \in A$, we have $\tau(\boldsymbol{\alpha}) = \tau(\boldsymbol{E}\circ\boldsymbol{\alpha}\circ\boldsymbol{E})$.*
2. *The restriction $\tau|_B$ is a bijection onto C and preserves intersection.*

Proof. (1) follows directly from the definition.
(2): $C = \mathrm{im}\tau = \mathrm{im}(\tau|_B)$ by (1).
By Proposition 2.2.c) each element of B can be written in the form $\bigcup_{j\in H'} {X_j}^2$ for some $H' \subseteq H$. Applying τ we get

$$\tau\left(\bigcup_{j\in H'} {X_j}^2\right) = H' .$$

This shows that $\tau|_B$ is injective and preserves intersection. □

Corollary 2.6. *We can equip C with operations $+$ and $-$, such that $\mathfrak{C} := \langle C, \cap, +, -, \emptyset\rangle$ shall be a Boolean algebra, isomorphic to $\mathfrak{B}$ (via $\tau|_B$).*

Theorem 2.7. *The trace operator gives rise to a one-to-one correspondence between the relational ideals of $\mathfrak{A}$ and the Boolean ideals of $\mathfrak{C}$. Namely, if I is a relational ideal of $\mathfrak{A}$, then $\tau(I)$ is a Boolean ideal of $\mathfrak{C}$, which uniquely determines I.*

Proof. Let us consider the mappings $\varphi : I \mapsto I \cap B$ and $\psi : J \mapsto \tau(J)$, where I runs over all relational ideals of $\mathfrak{A}$, and J runs over all ideals of $\mathfrak{B}$. Then $\psi(\varphi(I)) = \tau(I \cap B) = \tau(I)$ by Proposition 2.5.(1), and ψ is a bijection, since $\tau|_B : \mathfrak{B} \to \mathfrak{C}$ is an isomorphism by 2.6. So, it is enough to prove that φ is bijective, and $\varphi(I)$ is an ideal for each I.
It is clear that $I \cap B$ and I determine each other, since $\boldsymbol{\alpha} \in I$ iff $\boldsymbol{E}\circ\boldsymbol{\alpha}\circ\boldsymbol{E} \in I$. It is also clear that $I \cap B$ remains an ideal in $\mathfrak{B}$. For the other way around, if J is an ideal of $\mathfrak{B}$, then the downset $\langle J\rangle_{\mathfrak{A}} := \{\boldsymbol{\alpha} \in A \mid \exists\boldsymbol{\beta} \in J : \boldsymbol{\alpha} \subseteq \boldsymbol{\beta}\}$ will be a relational ideal of $\mathfrak{A}$, satisfying $\langle J\rangle_{\mathfrak{A}} \cap B = J$. □

Corollary 2.8. *For a class K of algebras let* Con$K$ *denote the class* $\{\mathrm{con}\mathfrak{A} \mid \mathfrak{A} \in K\}$ *of congruence lattices of algebras in K. Then we have*

$$\mathrm{Con}\{\texttt{wRRA}\text{'s}\} = \mathrm{Con}\{\texttt{RRA}\text{'s}\} = \mathrm{Con}\{\texttt{Boolean algebras}\}.$$

Proof. By the previous theorem, and since any `RRA` is also a `wRRA`, we have
$\mathrm{Con}\{\texttt{RRA}\text{'s}\} \subseteq \mathrm{Con}\{\texttt{wRRA}\text{'s}\} \subseteq \mathrm{Con}\{\texttt{Boolean algebras}\}$.
And, if $\mathfrak{W}$ is a Boolean algebra, it can be represented on some set X. Then $\mathfrak{W}$ embeds in the full `RRA` $\mathfrak{P}(\mathrm{id}_X)$ (whose operations satisfy $\boldsymbol{\alpha}\circ\boldsymbol{\beta} = \boldsymbol{\alpha} \cap \boldsymbol{\beta}$, $\boldsymbol{\alpha}^{\smile} = \boldsymbol{\alpha}$ and $\mathrm{id} = \mathrm{id}_X$). □

3. The main result

Theorem 3.1. *The class of* `wRRA`*'s is closed under taking homomorphic images.*

Let $\mathfrak{A}$ be a `wRRA`. For a given relational ideal I we will construct homomorphisms (by means of semimorphisms) $\vartheta_U : \mathfrak{A} \to \mathfrak{M}_U$, such that each $\mathfrak{M}_U$ is a `wRRA`, and that $\bigcap_U \ker\vartheta_U = I$. Thus, the kernel of the induced homomorphism $\langle\vartheta_U\rangle : \mathfrak{A} \to \prod_U \mathfrak{M}_U$ will be exactly I. Then, using the fact that the class of `wRRA`'s is closed under products and subalgebras (and applying the first isomorphism theorem), we get that the quotient algebra $\mathfrak{A}/_I$ is again a `wRRA`.

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We keep the notation introduced in section 2. Thus, $\mathfrak{A}$ is an arbitrary `wRRA` with largest element $\boldsymbol{E}$, whose equivalence classes are collected in $(X_i)_{i\in H}$. We will use the trace operator τ (cf. Definition 2.4), and its range, the set $C = \mathrm{im}\tau \subseteq \mathcal{P}(H)$ of *traces* of elements of A. We equipped C with a Boolean structure, keeping the ordinary intersection of sets (the operations $+$ and $-$ were taken from $\mathfrak{A}$).

Lemma 3.2. *Let $\mathcal{U}$ be an ultrafilter of the Boolean algebra $\mathfrak{C}$, and set*

$$\bar{\mathcal{U}} := \{S \in \mathcal{P}(H) \mid \exists G \in \mathcal{U} : S \supseteq G\}.$$

Then $\bar{\mathcal{U}}$ is a filter in $\mathcal{P}(H)$ and $\mathcal{U} = \bar{\mathcal{U}} \cap C$.

The proof is left to the reader.

Let us fix an ultrafilter $\mathcal{U}$ of $\mathfrak{C}$ for the most part of the section. Let $Z_{\mathcal{U}}$ denote the reduced product $\prod_{i\in H} X_i/_{\bar{\mathcal{U}}}$. We consequently write the corresponding bold letters $\mathbf{x}$,

$\mathbf{y}, \ldots$ for the equivalence classes of sequences $\langle x_i \rangle_{i\in H}$, $\langle y_i \rangle_{i\in H}, \ldots$ in $\prod_{i\in H} X_i$. Then $\mathbf{x} \in Z_{\mathcal{U}}$, and $\mathbf{x} = \mathbf{y} \Leftrightarrow \{i \in H \mid x_i = y_i\} \in \bar{\mathcal{U}}$.

Now we are ready to define the map $\vartheta_{\mathcal{U}} : A \to \mathcal{P}({Z_{\mathcal{U}}}^2)$: For an $\boldsymbol{\alpha} \in A$, let $\vartheta_{\mathcal{U}}(\boldsymbol{\alpha})$ be the relation on $Z_{\mathcal{U}}$ defined by

$$\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha})\ \mathbf{y} \overset{def}{\Leftrightarrow} \{i \in H \mid x_i \boldsymbol{\alpha} y_i\} \in \bar{\mathcal{U}}\ .$$

It is not hard to check that $\vartheta_{\mathcal{U}}(\boldsymbol{\alpha})$ is well defined on $Z_{\mathcal{U}}$, i.e. if $\mathbf{x} = \mathbf{x}'$ and $\mathbf{y} = \mathbf{y}'$ then $\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha})\ \mathbf{y} \ \Leftrightarrow\ \mathbf{x}'\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha})\ \mathbf{y}'$.

For brevity, we introduce the notation

$$[\![x_i \boldsymbol{\alpha} y_i]\!] := \{i \in H \mid x_i \boldsymbol{\alpha} y_i\} \ \big(\subseteq H\big).$$

Proposition 3.3. *The mapping $\vartheta_{\mathcal{U}}$ is a semimorphism $\mathfrak{A} \dashrightarrow \mathfrak{P}({Z_{\mathcal{U}}}^2)$, where $\mathfrak{P}({Z_{\mathcal{U}}}^2)$ is the full proper relation algebra of all binary relations on the set $Z_{\mathcal{U}}$.*

Proof. $\vartheta_{\mathcal{U}}(\boldsymbol{\emptyset}) = \boldsymbol{\emptyset}$ and $\vartheta_{\mathcal{U}}(\boldsymbol{E}) = {Z_{\mathcal{U}}}^2$ follows immediately.

$\cap$: $\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha} \cap \boldsymbol{\beta})\ \mathbf{y} \ \Leftrightarrow\ [\![x_i(\boldsymbol{\alpha} \cap \boldsymbol{\beta})y_i]\!] \in \bar{\mathcal{U}} \ \Leftrightarrow$ $\big([\![x_i \boldsymbol{\alpha} y_i]\!] \in \bar{\mathcal{U}}$ and $[\![x_i \boldsymbol{\beta} y_i]\!] \in \bar{\mathcal{U}}\big) \ \Leftrightarrow\ (\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha})\ \mathbf{y}$ and $\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\beta})\ \mathbf{y})$ $\Leftrightarrow\ \mathbf{x}\ (\vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) \cap \vartheta_{\mathcal{U}}(\boldsymbol{\beta}))\ \mathbf{y}$.

$^{\smile}$: $\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha}^{\smile})\ \mathbf{y} \ \Leftrightarrow\ [\![x_i \boldsymbol{\alpha}^{\smile} y_i]\!] \in \bar{\mathcal{U}} \ \Leftrightarrow\ [\![y_i \boldsymbol{\alpha} x_i]\!] \in \bar{\mathcal{U}} \ \Leftrightarrow\ \mathbf{y}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha})\ \mathbf{x} \ \Leftrightarrow$ $\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha})^{\smile}\ \mathbf{y}$.

id : $\mathbf{x}\ \vartheta_{\mathcal{U}}(\mathrm{id})\ \mathbf{x}' \ \Leftrightarrow\ [\![x_i = x_i']\!] \in \bar{\mathcal{U}} \ \Leftrightarrow\ \mathbf{x} = \mathbf{x}'$.

$\circ$: If $\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha} \circ \boldsymbol{\beta})\ \mathbf{z}$, then $G := [\![x_i(\boldsymbol{\alpha} \circ \boldsymbol{\beta})z_i]\!] \in \bar{\mathcal{U}}$, so, for $i \in G$ there is an $y_i \in X_i$ such that $x_i \boldsymbol{\alpha} y_i \boldsymbol{\beta} z_i$. Extending $\langle y_i \rangle_{i\in G}$ arbitrarily to H, we get an $\mathbf{y}$ such that $[\![x_i \boldsymbol{\alpha} y_i]\!] \supseteq G$ and $[\![y_i \boldsymbol{\alpha} z_i]\!] \supseteq G$, therefore $\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha})\ \mathbf{y}\ \vartheta_{\mathcal{U}}(\boldsymbol{\beta})\ \mathbf{z}$. Conversely, if $\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) \circ \vartheta_{\mathcal{U}}(\boldsymbol{\beta})\ \mathbf{z}$ then there is an $\mathbf{y}$ such that $\mathbf{x}\ \vartheta_{\mathcal{U}}(\boldsymbol{\alpha})\ \mathbf{y}\ \vartheta_{\mathcal{U}}(\boldsymbol{\beta})\ \mathbf{z}$, thus $[\![x_i(\boldsymbol{\alpha} \circ \boldsymbol{\beta})z_i]\!] \supseteq [\![x_i \boldsymbol{\alpha} y_i]\!] \cap [\![y_i \boldsymbol{\beta} z_i]\!] \in \bar{\mathcal{U}}$.

□

Lemma 3.4. *Let $\boldsymbol{\alpha} \in A$ be arbitrary and let $\boldsymbol{\beta} \in A$ be a squarefull relation, and suppose that $\langle x_i \rangle_{i\in H}$ and $\langle y_i \rangle_{i\in H}$ are arbitrary sequences of $\prod_{i\in H} X_i$, then*

$$[\![x_i \boldsymbol{\alpha} y_i]\!] \subseteq \tau(\boldsymbol{\alpha}) \quad \textit{and} \quad [\![x_i \boldsymbol{\beta} y_i]\!] = \tau(\boldsymbol{\beta}).$$

Proof. If $j \in [\![x_i \boldsymbol{\alpha} y_i]\!]$ then $x_j \boldsymbol{\alpha} y_j$, implying $j \in \tau(\boldsymbol{\alpha})$.
If $j \in \tau(\boldsymbol{\beta})$ then there are elements $x', y' \in X_j$ such that $x' \boldsymbol{\beta} y'$, but then also $x_j \boldsymbol{\beta} y_j$ by Proposition 2.2.d). □

Corollary 3.5. *For an $\boldsymbol{\alpha} \in A$ we have*

$$\vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) = \boldsymbol{\emptyset} \quad \textit{if and only if} \quad \tau(\boldsymbol{\alpha}) \notin \mathcal{U}.$$

Proof. Set $\boldsymbol{\beta} := \boldsymbol{E} \circ \boldsymbol{\alpha} \circ \boldsymbol{E}$, it is squarefull. We have $\tau(\boldsymbol{\beta}) = \tau(\boldsymbol{\alpha})$ ($\in C$), and $\vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) = \boldsymbol{\emptyset} \ \Leftrightarrow\ \vartheta_{\mathcal{U}}(\boldsymbol{\beta}) = \boldsymbol{\emptyset}$ because $\vartheta_{\mathcal{U}}$ is a semimorphism. By Lemma 3.4, for arbitrary elements $\mathbf{p}, \mathbf{q} \in Z_{\mathcal{U}}$, $[\![p_i \boldsymbol{\beta} q_i]\!] = \tau(\boldsymbol{\beta})$, so, if $\tau(\boldsymbol{\beta}) \in \mathcal{U}$ then $\mathbf{p}\ \vartheta_{\mathcal{U}}(\boldsymbol{\beta})\ \mathbf{q}$ holds for all $\mathbf{p}, \mathbf{q}$, while if $\tau(\boldsymbol{\beta}) \notin \mathcal{U}$ then $\mathbf{p}\ \vartheta_{\mathcal{U}}(\boldsymbol{\beta})\ \mathbf{q}$ fails for all $\mathbf{p}, \mathbf{q} \in Z_{\mathcal{U}}$. □

Recall that $\boldsymbol{\alpha} - \boldsymbol{\beta}$ means $\boldsymbol{\alpha} \cap (-\boldsymbol{\beta})$.

Lemma 3.6. *For any* $\boldsymbol{\alpha}, \boldsymbol{\beta} \in A$ *we have*

$$\vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) = \vartheta_{\mathcal{U}}(\boldsymbol{\beta}) \;\Leftrightarrow\; \vartheta_{\mathcal{U}}(\boldsymbol{\alpha} - \boldsymbol{\beta}) = \boldsymbol{\emptyset} = \vartheta_{\mathcal{U}}(\boldsymbol{\beta} - \boldsymbol{\alpha})$$

Proof. $\Rightarrow$: $\vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) = \vartheta_{\mathcal{U}}(\boldsymbol{\beta})$ implies $\vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) \cap \vartheta_{\mathcal{U}}(-\boldsymbol{\beta}) = \vartheta_{\mathcal{U}}(\boldsymbol{\beta}) \cap \vartheta_{\mathcal{U}}(-\boldsymbol{\beta})$, that is, since $\vartheta_{\mathcal{U}}$ preserves intersection, $\vartheta_{\mathcal{U}}(\boldsymbol{\alpha} - \boldsymbol{\beta}) = \vartheta_{\mathcal{U}}(\boldsymbol{\beta} - \boldsymbol{\beta}) = \boldsymbol{\emptyset}$.
$\vartheta_{\mathcal{U}}(\boldsymbol{\beta} - \boldsymbol{\alpha}) = \boldsymbol{\emptyset}$ follows by symmetry.
$\Leftarrow$: We prove $\vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) = \vartheta_{\mathcal{U}}(\boldsymbol{\alpha} \cap \boldsymbol{\beta})$. Then, by symmetry, we will have $\vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) = \vartheta_{\mathcal{U}}(\boldsymbol{\alpha} \cap \boldsymbol{\beta}) = \vartheta_{\mathcal{U}}(\boldsymbol{\beta})$.
Let $\boldsymbol{\gamma} := \boldsymbol{\alpha} \cap \boldsymbol{\beta}$ and $\boldsymbol{\delta} := \boldsymbol{E} \circ (\boldsymbol{\alpha} - \boldsymbol{\beta}) \circ \boldsymbol{E}$. Then also $\vartheta_{\mathcal{U}}(\boldsymbol{\delta}) = \boldsymbol{\emptyset}$ holds by hypothesis, so $\tau(\boldsymbol{\delta}) \notin \mathcal{U}$ by Corollary 3.5.

$$\boldsymbol{\gamma} \;=\; \boldsymbol{\alpha} \cap \boldsymbol{\beta} \;\subseteq\; \boldsymbol{\alpha} \;=\; \boldsymbol{\gamma} + (\boldsymbol{\alpha} - \boldsymbol{\beta}) \;\subseteq\; \boldsymbol{\gamma} + \boldsymbol{\delta}.$$

The mapping $\vartheta_{\mathcal{U}}$ is a semimorphism, hence it is monotone, yielding $\vartheta_{\mathcal{U}}(\boldsymbol{\gamma}) \subseteq \vartheta_{\mathcal{U}}(\boldsymbol{\gamma} + \boldsymbol{\delta})$. We show that $\vartheta_{\mathcal{U}}(\boldsymbol{\gamma} + \boldsymbol{\delta}) \subseteq \vartheta_{\mathcal{U}}(\boldsymbol{\gamma})$ holds, too.
We have $\tau(\boldsymbol{\delta}) \in C \backslash \mathcal{U}$ and $\mathcal{U}$ is an ultrafilter of the Boolean algebra $\mathfrak{C}$, so $-\tau(\boldsymbol{\delta}) \in \mathcal{U}$. Using that τ restricted to squarefull elements is a Boolean algebra isomorphism, it yields $\tau(-\boldsymbol{\delta}) = -\tau(\boldsymbol{\delta}) \in \mathcal{U}$.
$-\boldsymbol{\delta}$ is also squarefull, so, by Lemma 3.4, $\vartheta_{\mathcal{U}}(-\boldsymbol{\delta})$ is the whole ${Z_{\mathcal{U}}}^2$. Then $\vartheta_{\mathcal{U}}(\boldsymbol{\gamma}) \subseteq \vartheta_{\mathcal{U}}(\boldsymbol{\gamma} + \boldsymbol{\delta}) = \vartheta_{\mathcal{U}}(\boldsymbol{\gamma} + \boldsymbol{\delta}) \cap \vartheta_{\mathcal{U}}(-\boldsymbol{\delta}) = \vartheta_{\mathcal{U}}((\boldsymbol{\gamma} + \boldsymbol{\delta}) \cap (-\boldsymbol{\delta})) = \vartheta_{\mathcal{U}}(\boldsymbol{\gamma} - \boldsymbol{\delta}) \subseteq \vartheta_{\mathcal{U}}(\boldsymbol{\gamma})$. □

Proposition 3.7. *There is a unique* wRRA *structure* $\mathfrak{M}_{\mathcal{U}}$ *on the range of* $\vartheta_{\mathcal{U}}$ *(included in* $\mathcal{P}({Z_{\mathcal{U}}}^2)$*), such that* $\vartheta_{\mathcal{U}} : \mathfrak{A} \to \mathfrak{M}_{\mathcal{U}}$ *is a* wRRA *homomorphism.*

Proof. We let $\mathfrak{M}_{\mathcal{U}} := \langle \mathrm{im}\vartheta_{\mathcal{U}}, \cap, +, -, \circ, \smile, \mathrm{id}_{Z_{\mathcal{U}}}, \boldsymbol{\emptyset} \rangle$, where
$\vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) + \vartheta_{\mathcal{U}}(\boldsymbol{\beta}) := \vartheta_{\mathcal{U}}(\boldsymbol{\alpha} + \boldsymbol{\beta})$ and $-\vartheta_{\mathcal{U}}(\boldsymbol{\alpha}) := \vartheta_{\mathcal{U}}(-\boldsymbol{\alpha})$. Lemma 3.6 easily implies that $-$ is well defined, then the Boolean law $\boldsymbol{\alpha} + \boldsymbol{\beta} = -(-\boldsymbol{\alpha} \cap -\boldsymbol{\beta})$ and the fact that ϑ_U preserves $\cap$ ensure that also $+$ is well defined. □

Now we are ready for proving our main theorem.

Proof of Theorem 3.1. Let $\mathfrak{A}$ be a wRRA, and let I be a relational ideal of $\mathfrak{A}$. Let $\mathfrak{C}$ denote the Boolean algebra defined on the range of the trace operator τ. Consider all ultrafilters $\mathcal{U}$ of $\mathfrak{C}$ which are disjoint to the ideal $\tau(I)$: let

$$\mathcal{W} := \{\mathcal{U} \text{ ultrafilter of } \mathfrak{C} \mid \mathcal{U} \cap \tau(I) = \emptyset\}.$$

For each $\mathcal{U} \in \mathcal{W}$, we have a wRRA homomorphism $\vartheta_{\mathcal{U}} : \mathfrak{A} \to \mathfrak{M}_{\mathcal{U}}$ by Propositions 3.3 and 3.7, where $\mathfrak{M}_{\mathcal{U}}$ is a wRRA represented on the set $Z_{\mathcal{U}}$ (defined below Lemma 3.2).

We claim that $\bigcap\limits_{\mathcal{U} \in \mathcal{W}} \ker \vartheta_{\mathcal{U}} = I$.

First, if $\boldsymbol{\alpha} \in I$ then $\tau(\boldsymbol{\alpha}) \in \tau(I)$, hence $\tau(\boldsymbol{\alpha}) \notin \mathcal{U}$ for each $\mathcal{U} \in \mathcal{W}$. Then $\boldsymbol{\alpha} \in \ker \vartheta_{\mathcal{U}}$ follows by Corollary 3.5.
Second, if $\boldsymbol{\alpha} \notin I$, then $\tau(\boldsymbol{\alpha}) \notin \tau(I)$ by Prop.2.5(1), so there exists an ultrafilter $\mathcal{U}$ in $\mathfrak{C}$ by Lemma 1.5, such that $\mathcal{U}$ is disjoint to $\tau(I)$ (i.e. $\mathcal{U} \in \mathcal{W}$), but $\tau(\boldsymbol{\alpha}) \in \mathcal{U}$. Then $\boldsymbol{\alpha} \notin \ker \vartheta_{\mathcal{U}}$, again by Corollary 3.5. □

Corollary 3.8. *Let* $\mathfrak{A}$ *be a* wRRA *with greatest element* $\boldsymbol{E}$*. Then the following assertions are equivalent:*

a) $\mathfrak{A}$ *is simple.*
b) $\mathfrak{A}$ *is subdirectly irreducible.*
c) $\mathfrak{A}$ *has a weak representation in which* $\boldsymbol{E} = Z \times Z$ *for some set* Z *(i.e.* $\mathfrak{A}$ *is a square algebra in the sense of* [8]*).*
d) *The squarefull elements of* $\mathfrak{A}$ *are only* $\emptyset$ *and* $\boldsymbol{E}$*.*
e) $\mathfrak{A}$ *has a discriminator term (in the sense of e.g.* [4]*).*

Proof. e)⇒a) and a)⇒b) are trivial.
b)⇒c): Apply the construction of Theorem 3.1 for the trivial ideal $\{\emptyset\}$. Then $\mathfrak{A}$ embeds in $\prod_{\mathcal{U}} \mathfrak{M}_{\mathcal{U}}$, where each $\mathfrak{M}_{\mathcal{U}}$ is of the desired form (with greatest element $Z_{\mathcal{U}} \times Z_{\mathcal{U}}$). Since $\mathfrak{A}$ is subdirectly irreducible, there is an index $\mathcal{U}$, such that $\mathfrak{A}$ embeds in $\mathfrak{M}_{\mathcal{U}}$.
c)⇒d): If $\boldsymbol{E} = Z \times Z$ and $\emptyset \neq \boldsymbol{\alpha} \subseteq \boldsymbol{E}$, then $\boldsymbol{E} \circ \boldsymbol{\alpha} \circ \boldsymbol{E} = \boldsymbol{E}$.
d)⇒e): For such an $\mathfrak{A}$, the term $c(\boldsymbol{\alpha}) = \boldsymbol{E} \circ \boldsymbol{\alpha} \circ \boldsymbol{E}$ makes $c(\boldsymbol{\alpha}) = \boldsymbol{E}$ for each $\boldsymbol{\alpha} \neq \emptyset$ and $c(\emptyset) = \emptyset$. (Such a term is called a *switching term* in [3].) Then we can easily define a discriminator term using c and the Boolean operations. □

Corollary 3.9. *The class of* `wRRA`*'s is a discriminator variety.*

4. Alternative proofs

A) Once we knew that the class of square algebras (`wRRA`'s satisfying condition c) of Corollary 3.8) generates the whole quasivariety of `wRRA`'s, the following theorem of universal algebra (Thm.4.1) would simply imply that it is a variety. This method worked easily for representable relation algebras and cylindric algebras (cf. e.g. [3]), since each representable algebra is included in a full proper algebra, and full algebras arise as product of square algebras. However, this fact is not immediate for `wRRA`'s, because of Prop.1.4.

Let **H**, **S**, **P** and **Up** denote the operators of taking homomorphic images, subalgebras, products and ultraproducts, respectively.

Theorem 4.1. *Let* K *be a class of algebras of the same similarity type, and assume that a term* d *equals to the discriminator term in each member of* K*. Then* $\mathbf{SPUp}K$ *is a discriminator variety, and*

$$\mathbf{SPUp}K = \mathbf{HSP}K$$

We note that [15] (Thm.2.4.3) gives a model theoretic proof for this theorem, translating quasiequations to equations by means of the discriminator term.

B) The first proof that `RRA`'s form a variety is due to Tarski [18] (cf. also [13] Thm.121, p. 166). The first direct construction to represent homomorphic images of `RRA`'s is due to Andréka and Németi (in [7] p.100). Actually, they do it for cylindric algebras, but the case of `RRA`'s can readily be deduced from it. Below we sketch their proof. They use a similar construction to ours in the proof of Thm.3.1: embedding the quotient algebra of an arbitrary `RRA` in an ultraproduct of `RRA`'s. So, they first need the fact that the class of `RRA`'s is closed under taking ultraproducts. We note that this proof also works for `wRRA`'s.

Let us assume that I is a relational ideal of an arbitrary (w)RRA $\mathfrak{A}$. For each $\boldsymbol{\delta} \in I$ set $k(\boldsymbol{\delta}) := -\boldsymbol{E} \circ \boldsymbol{\delta} \circ \boldsymbol{E}$, and let $\mathfrak{B}_{\boldsymbol{\delta}}$ be the (w)RRA on the set $\{\boldsymbol{\beta} \in A \mid \boldsymbol{\beta} \subseteq k(\boldsymbol{\delta})\}$ (the negation of $\boldsymbol{\beta}$ in $\mathfrak{B}_{\boldsymbol{\delta}}$ is defined as $k(\boldsymbol{\delta}) - \boldsymbol{\beta}$).
Since the collection of sets $\langle\{\boldsymbol{\alpha} \in I \mid \boldsymbol{\alpha} \supseteq \boldsymbol{\delta}\}\rangle_{\boldsymbol{\delta} \in I}$ has the *finite intersection property*, there is an ultrafilter $\mathcal{U}$ on I which contains all these subsets of I. Then the mapping $\mathfrak{A} \to \prod_{\boldsymbol{\delta} \in I} \mathfrak{B}_{\boldsymbol{\delta}}/_{\mathcal{U}}$ defined by $\boldsymbol{\alpha} \mapsto \langle \boldsymbol{\alpha} \cap k(\boldsymbol{\delta})\rangle_{\boldsymbol{\delta} \in I}$ is a (w)RRA homomorphism with kernel I.

C) The fact that the class of RRA's is closed under homomorphic images is proved by Hirsch and Hodkinson ([8], Thm.3.37) as well. They show Theorem 4.2 below (but they do not formulate it as a separate claim). If a relation algebra $\mathfrak{A}$ is represented on a set X (i.e. $A \subseteq \mathcal{P}(X^2)$), then for a subset $Y \subseteq X$ we define $\boldsymbol{\alpha}|_Y := \boldsymbol{\alpha} \cap Y^2$ and $A|_Y := \{\boldsymbol{\alpha}|_Y \mid \boldsymbol{\alpha} \in A\}$. It admits naturally the relation algebra operations, yielding the RRA $\mathfrak{A}|_Y$.
The homomorphism $\mathfrak{A} \to \mathfrak{A}|_Y$ (sending $\boldsymbol{\alpha}$ to $\boldsymbol{\alpha}|_Y$) is called the *relativisation* of $\mathfrak{A}$ to Y.

Theorem 4.2. *Each representable relation algebra $\mathfrak{A}$ admits a representation such that every surjective homomorphism $f : \mathfrak{A} \to \mathfrak{B}$ is essentially a relativisation, i.e. there is a set Y and an isomorphism $h : \mathfrak{B} \to \mathfrak{A}|_Y$ with $h(f(\boldsymbol{\alpha})) = \boldsymbol{\alpha}|_Y$ for all $\boldsymbol{\alpha} \in A$.*

Sketch of proof. For a relation algebra $\mathfrak{A}$, we can easily write up a set $\Gamma^{\mathfrak{A}}$ of formulas in a first order language (whose binary relation symbols are just the elements of A), such that $\Gamma^{\mathfrak{A}}$ is consistent iff $\mathfrak{A}$ is representable. In that case, $\Gamma^{\mathfrak{A}}$ also admits a *saturated* model (cf. [5]), say $\mathfrak{X}$. Then $\mathfrak{A}$ is represented on the underlying set X of $\mathfrak{X}$, and, being saturated implies that every ultrafilter of $\mathfrak{A}$ is of the form $\mathcal{U}_{xy} := \{\boldsymbol{\alpha} \in A \mid x\boldsymbol{\alpha}y\}$ for some $x, y \in X$.
For a given surjective homomorphism $f : \mathfrak{A} \to \mathfrak{B}$, we define $Y := X \setminus \{x \mid \exists \boldsymbol{\gamma} \in \ker f : x\boldsymbol{\gamma}x\}$. Then the mapping $h : \mathfrak{B} \to \mathfrak{A}|_Y$ defined by $h(f(\boldsymbol{\alpha})) := \boldsymbol{\alpha}|_Y$ is well defined on B, and it is a surjective homomorphism. For its injectivity we have to use the property mentioned above concerning ultrafilters. □

At first sight, these arguments seem to go through for weak representations instead of representations. The only problem is that, the set $A|_Y$ may not inherit the distinguished Boolean operations $+$ and $-$ from $\mathfrak{A}$.

So, we finish our paper by raising a problem:

Problem 4.3. *Let $\mathfrak{A}$ be a relation algebra, weakly represented on a set X. For which subsets $Y \subseteq X$ will the restriction map $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha}|_Y$ remain a relation algebra morphism?*
Alternatively, let us assume that a Boolean algebra $\mathfrak{A}$ is "weakly represented" on a set X (by an injective function $A \to \mathcal{P}(X)$ which preserves intersection). For which subsets $Y \subseteq X$ will the restriction map $\boldsymbol{\alpha} \mapsto \boldsymbol{\alpha} \cap Y$ keep the Boolean structure (i.e., preserve negation)?

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