CATEGORICAL TREES AS FORMULAS

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1 Introduction

1.1 Abstract.

This master thesis is about a common generalization of several mathematical languages, like Propositional Calculus, Modal Logic, or First Order Logics, but we also pick examples from the geometry of abstract planes, and from a so called diagrammatic language of Category Theory. The basic idea was to work only within the category of models, defining formulas by finite trees or cones (=trees of depth 1) built up of morphisms from a specified subcategory. In papers [1, 2, 6] general theorems are introduced concerning this, among others, analogous to the theorem of Birkhoff which states "a class of algebraic structures is axiomatizable by equalities iff it is closed under homomorphic images, subalgebras and direct products". Here we present a more general situation, starting with two arbitrary categories: one for "abstract situations", one for the models, and with arrows from the first to the second one, as "representations of situations in models". This is the configuration what I distinguished from having simply an embedding functor between the categories, at the study of first order languages, and I named it "branch". Besides the main purpose, we show how to formulate adjoint situations and category equivalences in terms of these "branches".

I am grateful for their helps and inspirations to my supervisors (Marcel van de Vel [*Vrije Universiteit, Amsterdam*] and Ildikó Sain [*Rényi Intézet, Budapest*]), and to professors Endre Makai [*RI, Bp.*], Yde Venema [*UvA, A'dam*], and Aart Blokhuis [*VU, A'dam*].

1.2 About the language of the paper

Mostly we use the "algebraic" notation in which the function symbol is placed to the right side of its argument so that if x is an element and f is a function then x^f or xf will denote the image of x under f. Hence the composition fg of mappings f and g is applied from left to right.
However, we don't insist on this: in order to have perhaps more perspicuous formulas, some further functions will appear, too, acting from the left. [If one wants to indicate the composition of a function $f : A \to B$ acting from the left with $g : B \to A$ acting from the right, use brackets: $(f)g := a \mapsto (fa)g$ and $f(g) := b \mapsto f(bg)$.]

If not indicated otherwise, all the diagrams are assumed to commute throughout the paper. (So, a diagram drawn is a statement of its commutativity.)

If r is a relation between sets [or classes] A and B [ie. $r \subset A \times B$, or in alternative notations $r : A - B$, or $\underset{A-B}{r}$, or $A \overset{r}{-} B$], then arb will denote the proposition that "elements a and b are in the relation r" [that is, $\langle a, b\rangle \in r$]. The opposite relation of an $\underset{A-B}{r}$ will be denoted by r^{op} [so that $r^{op} : B - A$]. We say that $r : A - B$ is <u>full</u> if $\forall a \in A\ \exists b \in B : arb$, and <u>cofull</u> if r^{op} is full. We use the notation $\mathbf{P}S$ for the power set of S, ie. $\mathbf{P}S := \{A \mid A \subset S\}$.

1.3 Category Theory

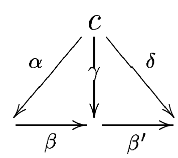
In this section we introduce the category theoretical notion of "branch" to link up categories with morphisms in between, as a generalization of functors, moreover, of all categorical relations.
A category will be considered as the class of its morphisms (denoted by doubled letters $\mathbb{A}, \mathbb{B}$...) identifying the objects with their identities [so that $Ob\mathbb{C} \subset \mathbb{C} := Mor\mathbb{C}$]. We use latin letters a, b, c... for objects, and, in general, greek letters α, β... for morphisms composing them by $\cdot$ or by simply concatenating, and so, instead of 1_a, we will write a again, for an identity morphism.
The domain/codomain of a morphism ϑ will be also called begin/end, denoted by ϑ^B/ϑ^E. Then $\vartheta : a \to b$ or $a \xrightarrow{\vartheta} b$ written would be equivalent to the equalities $\vartheta^B = a \quad \vartheta^E = b$.
Denote the set of morphisms from a to b in a category $\mathbb{C}$ by $[a|b] := [a|b]_{\mathbb{C}}$,

ie. $[a|b]_{\mathbb{C}} := \{\alpha \in \mathbb{C} \,|\, a \xrightarrow{\alpha} b\}$, and the hom-functor (with first variable fixed) by $\underset{\mathbb{C}\to\mathbb{S}\text{et}}{[a|\cdot]} := \underset{x\to y}{\gamma} \mapsto (\underset{[a|x]\to[a|y]}{\vartheta \mapsto \vartheta\gamma})$, where $\mathbb{S}$et is the category of sets.

We will often consider a partially ordered set (*poset*) as a category having exactly one arrow $a \to b$ iff $a \leq b$.

Consider the relation $\underset{Ob\mathbb{C}-Ob\mathbb{C}}{\cong}$ of "being isomorphic to" on the class of objects. Then, by a skeleton of $\mathbb{C}$ we mean a full subcategory $\mathbb{C}_0 \leq \mathbb{C}$ which contains exactly one object from each equivalence class of $\cong$.

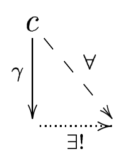
If $\mathbb{B} \leq \mathbb{C}$ is a subcategory, and $c \in Ob\mathbb{C}$, then $c \rightsquigarrow \mathbb{B}$ will denote the *comma category*, the morphisms of which are the commutative triangles $\langle \alpha, \beta, \gamma \rangle$ with $\alpha^B = c, \beta \in \mathbb{B}, \alpha\beta = \gamma$, and composition $\langle \alpha, \beta, \gamma \rangle \cdot \langle \gamma, \beta', \delta \rangle := \langle \alpha, \beta\beta', \delta \rangle$



$(\in \mathbb{B})$

Then the identities are just of the form $\langle \alpha, \alpha^E, \alpha \rangle$ and hence their corresponding objects are considered as arrows (α) from c to $\mathbb{B}$, and then $\underset{\alpha\longrightarrow\gamma}{\langle \alpha, \beta, \gamma \rangle}$.

We call an $O \in Ob\mathbb{C}$ an initial object, if for all $x \in Ob\mathbb{C}$, there is exactly one arrow $O \to x$ (ie. $\#[O|x] = 1$). If exists, it is unique up to isomorphism. An initial object of the comma category $c \rightsquigarrow \mathbb{B}$ is called universal arrow from c to $\mathbb{B}$, and, if such a universal arrow γ exists, we say that c is reflected in $\mathbb{B}$ by γ. $\mathbb{B}$ is a reflective subcategory of $\mathbb{C}$ if every $c \in Ob\mathbb{C}$ is reflected in $\mathbb{B}$. (Note that a $b \in Ob\mathbb{B}$ is always reflected by itself.)



$(\in \mathbb{B})$

Dual notions are *terminal object*, *couniversal arrow*, *coreflective subcategory*.

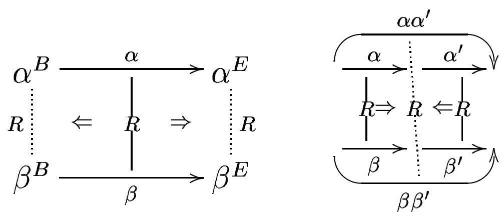
Definition 1.1. *Let us call a category* $\mathbb{H}$ *a* branch *(or directed branch) between categories* $\mathbb{A}$ *and* $\mathbb{B}$ *(in notation* $\mathbb{H} : \mathbb{A} \Rightarrow \mathbb{B}$ *or* $\mathbb{A} \overset{\mathbb{H}}{\Rightarrow} \mathbb{B}$*) if* $\mathbb{A}, \mathbb{B} \leq \mathbb{H}$ *are disjoint subcategories and* $\vartheta \notin \mathbb{A} \cup \mathbb{B} \Rightarrow \vartheta^B \in \mathbb{A}, \vartheta^E \in \mathbb{B}$. ϑ*'s of this kind will be called* diagonal *or* crossing *arrows.*

Sometimes we refer to $\mathbb{A}$ as the "upper" category of $\mathbb{H}$, because in diagrams we direct these diagonal arrows downwards.

For example we have a branch $\mathbb{H} : \mathbb{S}\mathrm{et} \Rightarrow \mathbb{G}\mathrm{r}$ from the category of sets to that of groups, with crossing morphisms just the set-theoretical functions (from a set to an underlying set of a group). This is a particular case of the following construction, inspired by the forgetful functor $\mathbb{G}\mathrm{r} \to \mathbb{S}\mathrm{et}$.

Note that a branch $\mathbb{A} \Rightarrow \mathbb{B}$ is determined if the class of diagonal arrows $\{.., \delta, ..\}$ are given, together with all compositions of the form $\alpha\delta\beta$ ($\alpha \in \mathbb{A}, \beta \in \mathbb{B}$).

Definition 1.2. *A relation* $\mathbb{A} \overset{R}{-} \mathbb{B}$ *between (the morphisms of) categories is said to be categorical, if for all* $\alpha R \beta, \alpha' R \beta'$ *; one has* $\alpha^B R \beta^B \wedge \alpha^E R \beta^E$*, and whenever both* $\alpha\alpha'$ *and* $\beta\beta'$ *exist,* $\alpha\alpha' R \beta\beta'$*.*

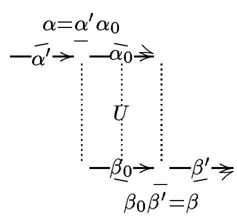


In particular, a functor is always categorical, and, by symmetry, so is an opposite relation of a functor what we will call an opfunctor. In addition, any relation $R : Ob\mathbb{A} - Ob\mathbb{B}$ (defined only between objects) is also categorical.

Once a categorical relation $U : \mathbb{A} - \mathbb{B}$ is given, a branch $\mathbb{U} : \mathbb{A} \Rightarrow \mathbb{B}$ can be obtained in a natural way: we add a "basic" crossing morphism $a \overset{{}_a\nu_b}{\to} b$ to the disjoint union $\mathbb{A} \sqcup \mathbb{B}$ whenever aUb and then identify $\alpha \cdot {}_{\alpha^E}\nu_{\beta^E}$ with ${}_{\alpha^B}\nu_{\beta^B} \cdot \beta$ for $\alpha U \beta$ pairs.

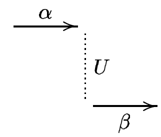
Formally, $\mathbb{U} := {}^{\mathbb{A} \sqcup \{\langle \alpha, \beta \rangle \in \mathbb{A} \times \mathbb{B} \,|\, \alpha^E U \beta^B\} \sqcup \mathbb{B}}/_{\Theta}$, with compositions $\alpha \cdot \langle \alpha', \beta \rangle := \langle \alpha\alpha', \beta \rangle$ and $\langle \alpha, \beta \rangle \cdot \beta' := \langle \alpha, \beta\beta' \rangle$; where Θ is the smallest categorical congruence on $\mathbb{U}$ for which

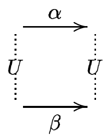
$$\langle \alpha, \beta' \rangle \Theta \langle \alpha', \beta \rangle \iff \exists \alpha_0, \beta_0 : [\alpha = \alpha'\alpha_0,\ \alpha_0 U \beta_0,\ \beta_0\beta' = \beta]$$



$(\in \mathbb{A})$

$(\in \mathbb{B})$

In diagrams, a typical crossing morphism is pictured as  so

that  commutes iff $\alpha U \beta$, where the vertical dotted U-lines stand always for the ${}_a\nu_b := \langle a, b\rangle$ arrows (with aUb), read downwards, if not indicated otherwise.

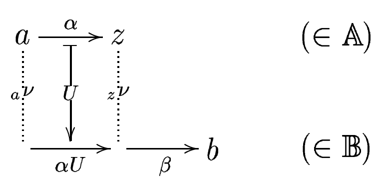
Branches obtained this way from a *functor* will be called functorial, and opfunctorial those from an *opfunctor*. In these cases, one could omit the sign of b [resp. a] from ${}_a\nu_b$, since it is determined.

Proposition 1.1. *For a given branch $\mathbb{U} : \mathbb{A} \Rightarrow \mathbb{B}$ the following assertions are equivalent:*
i) $\mathbb{U}$ is functorial
ii) There is a map $U_0 : Ob\mathbb{A} \to Ob\mathbb{B}$ such that for all $a \in Ob\mathbb{A}$, $[a|b]_{\mathbb{U}} \simeq [aU_0|b]_{\mathbb{B}}$ naturally in b.
iii) $\mathbb{B}$ is a reflective subcategory of $\mathbb{U}$ [ie. $\forall a \in Ob\mathbb{A}$ there is a universal arrow $a \rightsquigarrow \mathbb{B}$.]

Proof. i)⇒ii) Let $\mathbb{U}$ be obtained from a functor $U : \mathbb{A} \to \mathbb{B}$. Then $U_0 := U\restriction_{Ob\mathbb{A}}$ satisfies ii), since in this case every crossing arrow $\langle \alpha, \beta\rangle : a \to b$ can be written uniquely in the form $\langle a, \vartheta\rangle$, namely $\vartheta = (\alpha U)\beta$, for

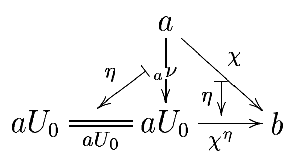
$$\langle \alpha, \beta\rangle = \alpha \cdot {}_z\nu \cdot \beta = {}_a\nu \cdot \alpha U \cdot \beta$$

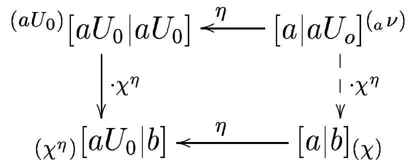
where, as before, ${}_x\nu := \langle x, xU\rangle : x \to xU$ for any $x \in Ob\mathbb{A}$ and $z = \alpha^E$. Thus the desired bijection is $\langle \alpha, \beta\rangle \mapsto (\alpha U)\beta$.



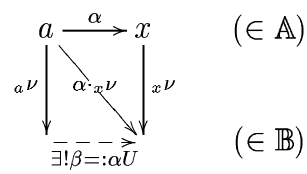
ii)⇒iii) If $a \in Ob\mathbb{A}$ is an arbitrary object, name $\eta : [a|\cdot] \to [aU_0|\cdot]$ the natural isomorphism, and put ${}_a\nu := (aU_0)^{\eta^{-1}}$ the correspondent of the identity morphism $aU_0 \in [aU_0|aU_0]$. We claim that ${}_a\nu$ is a universal arrow $a \rightsquigarrow \mathbb{B}$:

for a diagonal $\chi : a \to b$, χ^η is the only $\mathbb{B}$-arrow which satisfies ${}_a\nu \cdot \chi^\eta = \chi$ (apply naturality to $\chi^\eta : aU_0 \to b$, and that η is invertible):





iii)⇒i) Fix an initial object ${}_a\nu$ of $a \rightsquigarrow \mathbb{B}$ for each $a \in Ob\mathbb{A}$. Construct then a functor U as follows: on objects $aU := ({}_a\nu)^E \in Ob\mathbb{B}$, and for an arrow $\underset{a\to x}{\alpha} \in \mathbb{A}$ by hypothesis, there is exactly one $\beta \in \mathbb{B}$ such that $\alpha \cdot {}_x\nu = {}_a\nu \cdot \beta$.



This gives the mapping $U : \alpha \mapsto \beta$ which is indeed a functor:

$$ {}_a\nu \cdot (\alpha\alpha')U = \alpha\alpha' \cdot {}_{(\alpha'^E)}\nu = \alpha \cdot {}_x\nu \cdot (\alpha')U = {}_a\nu \cdot \alpha U \cdot (\alpha')U $$

□

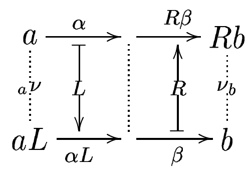
In other words, a branch $\mathbb{A} \Rightarrow \mathbb{B}$ is functorial iff the diagonal arrows can be reflected in $\mathbb{B}$ (just as in the first picture above).
Using this terminology, a branch includes an adjoint-functor situation iff it is both functorial and opfunctorial, ie. the diagonal arrows can be reflected in $\mathbb{B}$ and coreflected in $\mathbb{A}$.

Theorem 1.2 (Adjoint Branch theorem). *A functor $L : \mathbb{A} \to \mathbb{B}$ has a right adjoint iff its branch, $\mathbb{L}$ is opfunctorial [ie. $\mathbb{A}$ is coreflective in $\mathbb{L}$].*
And dually, a functor $R : \mathbb{B} \to \mathbb{A}$ (written on the left) has a left adjoint iff the branch of its opposite, $\mathbb{R}^{\mathbb{op}}$ is functorial.

Proof. Similar arguments as above prove that any branch $\mathbb{L}$ is opfunctorial iff $\exists R : \mathbb{B} \to \mathbb{A} \quad \forall b \in Ob\mathbb{B} \quad [\cdot|Rb]_{\mathbb{A}} \simeq [\cdot|b]_{\mathbb{L}}$. But then using that $\mathbb{L}$ is constructed from L, for all $a \in Ob\mathbb{A}$,

$$ [aL|b]_{\mathbb{B}} \simeq [a|b]_{\mathbb{L}} \simeq [a|Rb]_{\mathbb{A}} $$

naturally in both a and b.



For the other direction, if L is the left adjoint of R, then the natural isomorphisms $[aL|b]_{\mathbb{B}} \xrightarrow{\eta} [a|Rb]_{\mathbb{A}}$ for $a \in Ob\mathbb{A}$, $b \in Ob\mathbb{B}$ induce natural bijections $[a|b]_{\mathbb{L}} \to [a|b]_{\mathbb{R}^{\mathbb{op}}}$ mapping ${}_a\nu \cdot \beta \mapsto \beta^{\eta} \cdot \nu_b$, which constitute together a category isomorphism $\mathbb{L} \to \mathbb{R}^{\mathbb{op}}$ (what is identical in both $\mathbb{A}$ and $\mathbb{B}$). $\square$

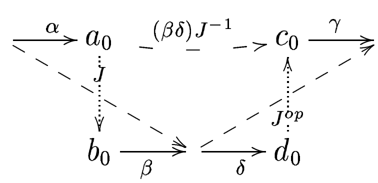
A nice characterization of category equivalence may be given, as well, in terms of (bi-)branches as follows:
Let us call a category $\mathbb{H}$ a <u>bibranch</u> (or nondirected branch) between $\mathbb{A}$ and $\mathbb{B}$, if $\mathbb{A} \sqcup \mathbb{B} \leq \mathbb{H}$ and every outer morphism is from an object of $\mathbb{A}$ to one of $\mathbb{B}$, *or* from $\mathbb{B}$ to $\mathbb{A}$ [ie. crossing morphisms may go also in the other direction].

Theorem 1.3. *Categories $\mathbb{A}$ and $\mathbb{B}$ are equivalent if and only if there exists a bibranch $\mathbb{H}$ between them such that every $a \in Ob\mathbb{A}$ is isomorphic to a $b \in Ob\mathbb{B}$ in $\mathbb{H}$, and conversely, every b is isomorphic to an a in $\mathbb{H}$.*

Note that, in this case, $\mathbb{A} \simeq \mathbb{H} \simeq \mathbb{B}$.

Proof. For the harder part of the proof, take skeletons $\mathbb{A}_0 \leq \mathbb{A}$ and $\mathbb{B}_0 \leq \mathbb{B}$, of the equivalent categories $\mathbb{A}$ and $\mathbb{B}$, and fix an isomorphism $J : \mathbb{A}_0 \to \mathbb{B}_0$, which is, at the same time, a categorical relation $\mathbb{A} - \mathbb{B}$, and $J^{op} = J^{-1}$. Now let $\mathbb{H} := \underset{\mathbb{A}\Rightarrow\mathbb{B}}{\mathbb{J}} \cup \underset{\mathbb{B}\Rightarrow\mathbb{A}}{\mathbb{J}^{\mathbb{op}}}$ with new compositions, such as

$$(\alpha \cdot {}_{a_0}\nu_{b_0} \cdot \beta)(\delta \cdot {}_{d_0}\nu_{c_0} \cdot \gamma) := \alpha \cdot (\beta\delta)J^{-1} \cdot \gamma$$



(∈ $\mathbb{A}$)

(∈ $\mathbb{B}$)

$\square$

1.4 The language induced by a branch

Definition 1.3. *an abstract categorical language is a triple* $\langle \mathbb{M}, \models, F \rangle$ *where* $\mathbb{M}$ *is the "category of models",* $\models$ *is an* $Ob\mathbb{M} - F$ *relation of "validity", while* F *is a set, elements of which are considered "formulas".*

Definition 1.4. *For languages* $\mathcal{L}_F = \langle \mathbb{M}, \models_F, F \rangle$ *and* $\mathcal{L}_S = \langle \mathbb{M}, \models_S, S \rangle$ *with the same model category one can define the relation* ${}_F{\approx}_S$*, as follows:*
$f \approx s \overset{def}{\iff} (\forall m \in Ob\mathbb{M} : m \models_F f \iff m \models_S s)$.
$\mathcal{L}_F$ *and* $\mathcal{L}_S$ *then are regarded equivalent (w.r.t.* $\mathbb{M}$*) if this* $\approx$ *is both full and cofull. (in notation:* $\mathcal{L}_F \underset{\mathbb{M}}{\cong} \mathcal{L}_S$ *or* $\mathcal{L}_F \cong \mathcal{L}_S$*)*

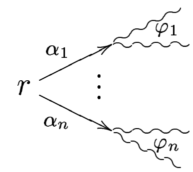
This condition expresses the existence of translator functions $\tau : F \to S$ and $\pi : S \to F$ such that, as relations, $\tau \cup \pi^{op} \subset {}_F{\approx}_S$, ie. both preserve the meaning of formulas.

In the following, we will construct a language for an arbitrary branch, and, in section 3 show for some well-known languages that they are equivalent to one of that kind.

So let us fix a branch $\mathbb{U} : \mathbb{D} \Rightarrow \mathbb{M}$, and consider the objects of $\mathbb{D}$ as some kind of "situations" that can occur in the models, and the diagonal arrows as "presentations" of these situations in the target models; then a finite tree of $\mathbb{D}$, as a sentence, would mean something like "each situation given by the root can be continued in a situation given on the first level such that every continuation of it to the next level of the current subtree leads to one of the following situations such that..."

To formulate this in a mathematical language, we define the finite $\mathbb{D}$-trees, and the semantics on them.

Definition 1.5. *A* $\mathbb{D}$*-tree of length 0 is just an object of* $\mathbb{D}$*, its root is itself, and if trees* $\varphi_1 \ldots \varphi_n$ *are given with roots* $r_1 \ldots r_n$ *then for any* $r \in Ob\mathbb{D}$ *and any n-tuples of arrows* $\langle \underset{r \to r_i}{\alpha_i} \rangle_{i=1}^n$*, the n-tuple of pairs* $\langle < \alpha_i, \varphi_i > \rangle_i$ *is also said to be a* tree*, with root* r.



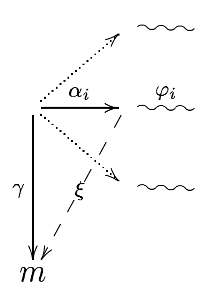
N.B.: one could have defined a $\mathbb{D}$-tree as a functor to $\mathbb{D}$ from a minimal elemented finite poset, which has upper bounds only for comparable pairs.

Denote the set of these $\mathbb{D}$-trees $F_{\mathbb{D}}$ or briefly F.
The set of subtrees of a tree may be defined inductively:
for $d \in Ob\mathbb{D}$, $Sb(d) := \{d\}$; for $\varphi = \langle < \alpha_i, \varphi_i > \rangle$, $Sb(\varphi) := \{\varphi\} \cup [\bigcup_i Sb(\varphi_i)]$.
We use the word vertex (or node) for a root of a subtree, (that is, for images of objects under the corresponding functor); so, in particular, they are also objects of $\mathbb{D}$.

The ALL-EXIST game

There are two fictive players, we call them ALL and EXIST, and they are given a model $m \in Ob\mathbb{M}$ and a tree $\varphi \in F_{\mathbb{D}}$ with root r.

The game sounds quite simple:
The players move by turns: a first move of a *match* is to give an $r \to m$ diagonal morphism (choosing φ to be the "current subtree" for the next move).
A next move after a given γ is to continue "climbing up the tree" choosing an arrow ξ to m from the first level of the current subtree $\langle < \alpha_i, \varphi_i > \rangle_i$, and a branch $< \alpha_i, \varphi_i >$, such that the triangle of γ, α_i, ξ should commute. Then the match continues in the subtree φ_i. The player who cannot make a next move, loses (and then the other one wins).



Definition 1.6 ($\models$)**.** *a tree φ is said to be* <u>*valid*</u> *in the model m (ie. $m \models \varphi$) if* EXIST *can win the game on φ however* ALL *plays. [The game is started by* ALL*.]*

Set $\mathcal{L}(\mathbb{U}) := \langle \mathbb{M}, \models, F_{\mathbb{D}} \rangle$.

Definition 1.7. *an arrow from the root of the tree φ to the model m is called an* <u>*evaluation*</u> *of φ in m.*

Definition 1.8. *a tree φ is said to be* <u>*valid with evaluation*</u> $v : r \to m$ *($m \models \varphi[v]$), if* EXIST *can win the corresponding game with v as the first move by* ALL*.*

Note that $m \models \varphi \iff \forall$ evaluation $v : m \models \varphi[v]$.

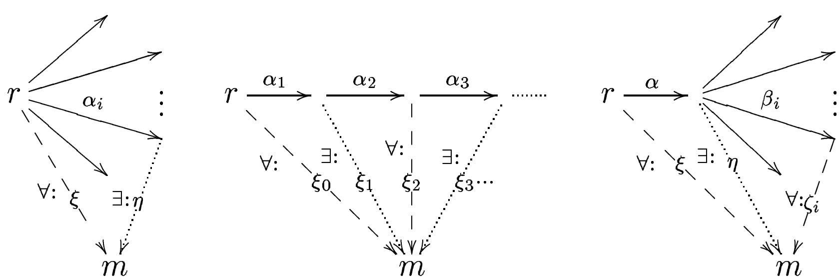
The notion of evaluation enables us to deal with formulas with free variables. (See section 3.3.)

Examples:

If $\varphi = \langle r \rangle$ is a tree of depth 0, called also a *root*, then $m \models \langle r \rangle$ if and only if there is no evaluation, ie. $[r|m] = \emptyset$.

If $\varphi = \langle \alpha_i \rangle_i$ is a cone (tree of length 1), then $m \models \varphi$ is equivalent to the statement

$$\forall_{r \to m} \xi \; \exists i \exists \eta : \xi = \alpha_i \eta$$



If $\varphi = \langle r \xrightarrow{\alpha_1}\xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} \rangle$ is a linear tree, then $m \models \varphi$ iff

$$\forall_{r \to m} \xi_0 \; \exists \xi_1 (\alpha_1 \xi_1 = \xi_0 \land \forall \xi_2 (\alpha_2 \xi_2 = \xi_1 \Rightarrow \exists \xi_3 \ldots))$$

where the last statement is an "$\exists \xi_n (\alpha_n \xi_n = \xi_{n-1})$" if n is odd, and "$\forall \xi_n (\alpha_n \xi_n \neq \xi_{n-1})$" if even.

If $\varphi = \langle \alpha, \langle \beta_i \rangle \rangle$ as in the third picture, $m \models \varphi$ means

$$\forall_{r \to m} \xi \; \exists \eta (\alpha \eta = \xi \land \forall i \forall \zeta_i [\beta_i \zeta_i \neq \eta])$$

Proposition 1.4 (Determinacy).
i) If O is an initial object of $\mathbb{U}$, then for every tree φ and every model m, one has $m \nvDash \varphi \iff m \vDash (O \to \varphi)$.
ii) If r is the root of φ, and $\underset{r\to m}{v}$ is an evaluation, then
$m \nvDash \varphi[v] \iff m \vDash (r \xrightarrow{r} \varphi)[v]$.

Proof. i) The first move on the tree $O \to \varphi$ is unique now, so ALL has no choice, and leaves the subtree φ for EXIST, that is, nothing else happened but the roles of the players exchanged, so EXIST can win on $O \to \varphi$, if and only if ALL can win on φ. The same argument holds for ii). □

Similarly, a unique possible move (*pass*) occurs if the subtree begins with one arrow what is invertible. Two successive passes extinguish each other.

2 Logics

2.1 Łoŝ lemma

Keep $\mathbb{U} : \mathbb{D} \Rightarrow \mathbb{M}$ fixed.

Definition 2.1. *An object $d \in Ob\mathbb{D}$ is called strongly small in $\mathbb{U}$ if the hom-functor $[d|\cdot]_{\mathbb{U}}$ preserves every* directed colimit *in $\mathbb{M}$, ie. for each directed poset J, and each functor $\underset{J\to\mathbb{M}}{T}$:* $\underset{j\in J}{\overrightarrow{\lim}}[d|jT] = [d|\underset{j\in J}{\overrightarrow{\lim}}(jT)]$.

The object m is strongly small with respect to a single category $\mathbb{M}$ if it is so in the branch $\mathbb{M} \Rightarrow \mathbb{M}$ induced by the identity functor.
Being strongly small usually coincides with some kind of finiteness, for example, at the first order models, with the property of being finitely presented. (cf. [9, 1, 2, 4])

Definition 2.2. *Let a filter $\mathcal{F} \subset \mathbf{P}S$ be given on the set S, and a collection of models: $(m_s)_{s\in S} \in (Ob\mathbb{M})^S$ such that $\forall H \in \mathcal{F}$: the category theoretical product $\prod_{s\in H} m_s =: m_H$ exists. Now, the projections $m_H \to m_K$ (for $H \supset K$) give a functor $P : (\mathcal{F}, \supset) \to \mathbb{M}$. Then, the $\mathcal{F}$-reduced product of m_s's is defined by $\prod_s m_s/_{\mathcal{F}} := \overrightarrow{\lim} P$, if it exists. If $\mathcal{F}$ happens to be an ultrafilter, the limit is called ultraproduct.*

Remark. There is another, more general definition of reduced product in categories, which avoids direct products (hence their existence is not required). (cf. [3, 5])

For a property $\mathcal{P}$, denote $\left\|\left[\mathcal{P} \text{ holds for } m_\bullet\right]\right\| := \{s \in S \mid \mathcal{P} \text{ holds for } m_s\}$. If $\mathcal{U}$ is a fixed ultrafilter, we will write "$\mathcal{P}$ holds for enough m_s's" to mean $\left\|\left[\mathcal{P} \text{ holds for } m_\bullet\right]\right\| \in \mathcal{U}$.

Theorem 2.1 (Łoś lemma). *Let φ be a $\mathbb{D}$-tree whose objects are all strongly small, and let $\mathcal{U} \subset \mathbf{P}S$ be an ultrafilter on the set S, and $m := \prod_{s\in S} m_s/_{\mathcal{U}}$ the ultraproduct of models $m_s \in Ob\mathbb{M}$. Then $m \models \varphi \iff \left\|\left[m_\bullet \models \varphi\right]\right\| \in \mathcal{U}$.*

Proof. First of all, a morphism σ from a s.small object a to the ultraproduct m is presented by a family $\langle \underset{a\longrightarrow m_h}{\sigma_h} \rangle_{h\in H}$ for some $H \in \mathcal{U}$, and vice versa, since

$$[a|m] \cong \varinjlim_{H\in\mathcal{U}}[a|m_H] = \coprod_{H\in\mathcal{U}}[a|m_H]/_\sim = \coprod_{H\in\mathcal{U}}(\prod_{h\in H}[a|m_h])/_\sim$$

where $\langle\alpha_h\rangle_{h\in H} \sim \langle\beta_k\rangle_{k\in K}$ iff $\exists L \subset H\cap K,\ L\in\mathcal{U} : \forall l \in L(\alpha_l = \beta_l)$.

Now assume that a player T ($\in$ {All,Exist}) can win any φ-game in *enough* m_s's. Then we have to provide him with a winning strategy in m. Let $H_0 := \left\|\left[\text{T can win in } m_\bullet\right]\right\| \in \mathcal{U}$. We will construct a finite chain $H_0 \supset H_1 \supset \ldots$ of $H_n \in \mathcal{U}$ for each match, such that the n^{th} move of T in m is determined by his n^{th} moves in m_h's ($h \in H_n$).
So suppose, at last, T's opponent chose the arrow $\underset{a\to m}{\sigma}$, which is presented by $\langle \underset{a\longrightarrow m_h}{\sigma_h} \rangle_{h\in H}$. Then, for every $h \in H \cap H_{n-1}$, T has a good respond $\varrho_h : b_h \to m_h$ from the next level of the a-rooted subtree of φ. Since the tree is finite, for *enough* h's the chosen vertex b_h is the same (using here the property of ultrafilters, that for any finite union, $A_1 \cup A_2 .. \cup A_k \in \mathcal{U} \Rightarrow \exists i : A_i \in \mathcal{U}$). This way chosen the new, smaller H_n (so for which $H_n \subset H \cap H_{n-1}$ and $h, k \in H_n \Rightarrow b_h = b_k =: b$), a respond in m ($\varrho : b \to m$) is generated by $\langle\varrho_h\rangle_{h\in H_n}$. □

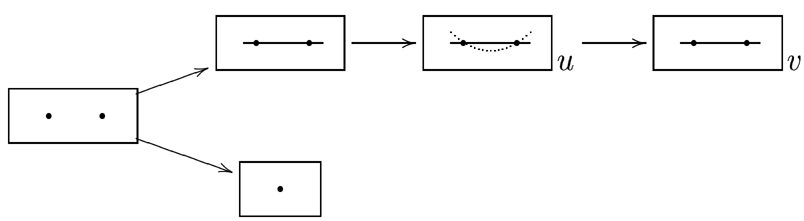
Remark. All the finiteness conditions are well-used in the previous proof. Note that an infinite branching number at a node would mean infinite conjunction or disjunction of the (negated) next subtrees, depending on which player "owns" that node [ie. on the parity of the level of the node]. If a vertex was not strongly small, it would mean something like allowing infinite number of "variables" satisfying an infinite number of "atomic predicates". And finally, if the length of a tree was infinite [ie. if it contained an infinite path of its arrows], we would have troubles (or rather, a variety of possibilities) how to define validity, since then a match could be infinite. A simplest solution is to say that, for instance, EXIST wins every infinite match; but then we may lose "negation". Read more about infinite games in [10].

2.2 Geometrical examples

Definition 2.3. *An abstract (or* Beukenhout-*)* $\overline{\textit{plane}}$ *is simply a binary relation* $\underset{P-L}{\iota}$ *where* P *is said to be the set of "points",* L *the set of "lines", and* ι *stands for "incidence". (cf. [11])*

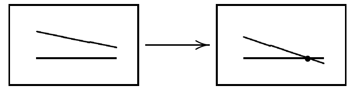
A morphism between such planes will be an incidence-preserving pair of functions mapping points to points and lines to lines. Now we form some axiom formulas in the language of trees given by the identity branch of this category.

⋆ "Every pair of distinct points lies on a unique line."



Adding ∀ and ∃ before the pictures one by one, it would read as: "For each pair of points: either they coincide or there exists a line through them such that all other line also containing both, is in reality not different".
Here the pictures stand for some simple abstract planes. For example the one named u has two points and two lines, each incident to each ($\#P_u = \#L_u = 2$, $\iota_u = P_u \times L_u$), while v has only one line (and two points, both on the line; $\#P_v = 2$, $\#L_v = 1$, $\iota_v = P_v \times L_v$). The morphism arriving to u maps the line into the solid one, and the one from u to v maps both lines to the single line of v.

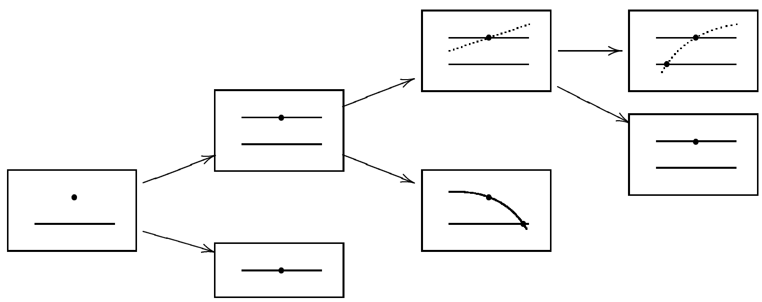
⋆<u>*proj*</u> "Every pair of lines intersects in a point."



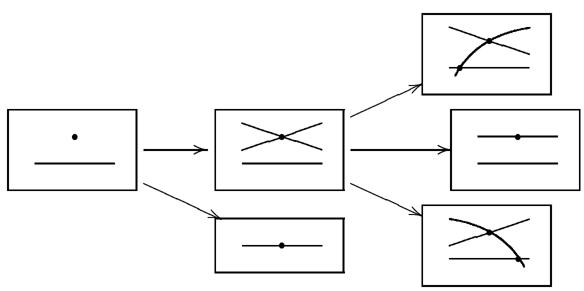
Note that this sentence also includes that every line has at least one point (applying it for the pair of equal lines). But, adding a model with a single line (and no points) below the second picture, one would get "Every pair of *distinct* lines intersect".

The next examples can be interpretted similarly:

⋆<u>*Eucl*</u> "For a line and a non-incident point there is a unique line on the point not intersecting the line."

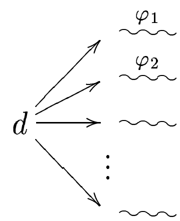


⋆<u>*hyp*</u> "For a line and a non-incident point there are at least two lines on the point not intersecting the line."



3 Tree-translations

In this section we recall the ordinary formulas and models for the current logics, define suitable branches, and show that they induce the original languages by constructing translator functions τ : Formulas $\to$ Trees and π : Trees $\to$ Formulas. For the "upper" category will be a *poset* in each case, such a tree is uniquely determined by its vertices, and therefore we may and will use the notation $\langle d; \varphi_1, \varphi_2, \ldots \rangle$ for



3.1 Propositional Calculus (PC)

Let us fix an alphabet (a set of elements called *proposition letters*) $\Sigma = \{p, q, r, s \ldots\}$. A model of PC($\Sigma$) then consists of a nonvoid set W of "states" equipped with the "truth" what is a function $T : \Sigma \to \mathbf{P}W$ of Σ to the power set of W, so that p^T is considered to be the set of those states which satisfy the proposition p. This is the same as to have a fixed $\Vdash$: $W - \Sigma$ relation (where $w \Vdash p$ holds exactly when p is true in w, that is, $w \in p^T$ in the above sense).

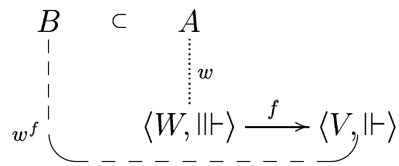
If $m = \langle W, \Vvdash \rangle$ and $n = \langle V, \Vdash \rangle$ are models, then a function $f : W \to V$ is a <u>morphism</u> $m \to n$ iff for all $p \in \Sigma$: $w \Vvdash p \;\Rightarrow\; w^f \Vdash p$.

Let $\mathbb{M}^{PC}$ denote this category, and F^{PC} the set of ordinary formulas (ie. those built up by the connectives $\wedge, \vee, \to, \neg, \top, \bot$ and the proposition letters). This leads to the familiar language $\mathcal{L}^{PC}$.
We will use $\bigwedge P := p_1 \wedge \ldots \wedge p_n$ for $P = \{p_1 \ldots p_n\}$, and $\bigwedge \emptyset := \top$, the "definitely true" formula. We similarly define $\bigvee P$, with $\bigvee \emptyset := \bot$, the "falsum".

Define a branch $\mathbb{U} := \mathbb{U}^{PC}$ over $\mathbb{M}^{PC}$ as follows: the upper category (of "situations") would be the partial order ($\subset$) of finite subsets of proposition letters, while certain elements of models would serve as diagonal morphisms:

$\mathbb{D} := \mathbb{D}^{PC} := \big\langle \{A \subset \Sigma \,|\, \#A < \omega\}, \subset \big\rangle$, then let $\langle A, w, m \rangle$ (written also as $\underset{A \to m}{w}$) be a crossing morphism $A \to m = \langle W, \Vvdash \rangle$, if $w \in W$ and

$\forall q \in A : w \Vdash q$. The composition comes naturally:

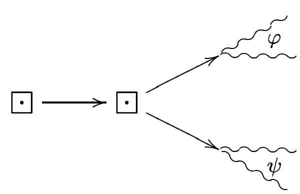


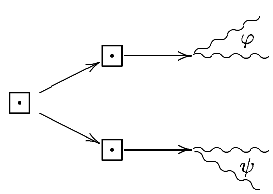
Note that $\mathbb{D}$ is equivalent to the full subcategory of one-stated finitary models (ie. those having $\#W = 1$, $\#(\Vdash) < \omega$), and $\mathbb{U}$ is to the branch given by this inclusion functor.

In trees we draw $\boxdot$ for the empty set $\emptyset$, because a move from it is just picking a state; and $\boxed{p_1 p_2 p_3 \cdots}$ for the $\mathbb{D}$-object $\{p_1, p_2, p_3, \ldots\}$.

Theorem 3.1. $\mathcal{L}(\mathbb{U}^{PC}) \cong \mathcal{L}^{PC}$

Proof. i) First we translate the formulas into trees: each formula has an equivalent in which negation appears only before the propositional letters.
For $p \in \Sigma$, $p^\tau := \boxdot \longrightarrow \boxed{p}$ and $(\neg p)^\tau := \boxed{p}$
The conjunction and disjunction can be realized always in a way like this: branching the tree at ALL's or EXIST's vertex.
For $\mathbb{D}^{PC}$-trees φ and ψ, define

$\varphi \wedge \psi \;:=$ 

$\varphi \vee \psi \;:=$ 

Then simply take $(\phi \wedge \theta)^\tau := \phi^\tau \wedge \theta^\tau$, etc.
Note that if the condition of an implication $\phi =$ "$\chi \to \theta$" is a conjunction of proposition letters ($\chi = \bigwedge_i p_i$) (ie. its negated is equivalent to a root), then ϕ could be translated also as $\boxed{\cdot\cdot p_i \cdot\cdot} \longrightarrow \boxed{\cdot\cdot p_i \cdot\cdot} \cup (\neg\theta)^\tau$ [where the union is

meant nodewise].

ii) For the other way around, observe that, in any match, ALL will select a state in the given model for the first move, what is never going to change. (That is, evaluation here would mean to choose a state with perhaps some initial assumptions given in the root.)
Signify $\varphi\backslash X$ the tree where every vertex A of φ is replaced with $A\backslash X$.
For a root, $\boxed{p_1p_2..p_n}^{\pi} := \neg(p_1 \wedge p_2 \wedge ..p_n)$ (because ALL cannot make a move iff there is no state where all proposition letters p_i are true).
For a tree $\varphi = \langle X; \varphi_1..\varphi_n\rangle$, take

$$\varphi^{\pi} := \bigwedge X \to \bigvee_i (\neg(\varphi_i\backslash X)^{\pi})$$

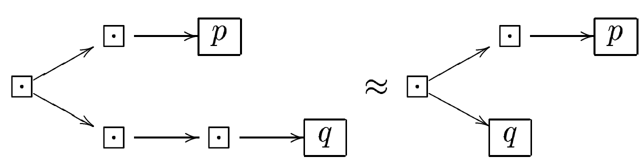
The first step of ALL is to choose a state w satisfying $\bigwedge X$, if he succeeds, EXIST selects one of the subtrees, say φ_i, whose root is valid at w and they continue playing. EXIST can win now iff ALL can win on $\varphi_i\backslash X$ starting at w, ie. when $m \nvDash (\varphi_i\backslash X)[w]$ ($\iff$ $w \Vdash \neg(\varphi_i\backslash X)^{\pi}$). $\square$

Remark. 1. It is important that ALL should start. To see this, take the one vertexed tree $\boxed{p}$ and the model $m = \boxed{\overset{a}{\cdot} \quad \overset{b}{\cdot}_p}$ (ie. where we have two states, a and b, and p is true in b [and not in a]). If EXIST starts, he can pick b, so the tree should be valid, though neither p, nor $\neg p$ cannot be so in m.

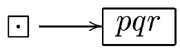
2. As it is seen in part i) of the proof, it would be enough to restrict $\mathbb{D}$ to subsets of Σ with *at most one element*, ie. $\mathbb{D}$ would be isomorphic to the partial order on the set $\{0\} \sqcup \Sigma$ given by $< := \{0\} \times \Sigma$.
If $\mathbb{D}$ was just the discrete category of Σ (ie. $\mathbb{D} = Ob\mathbb{D} = \Sigma$), then one would have $F_{\mathbb{D}} \underset{\mathbb{M}}{\cong} \{\neg p | p \in \Sigma\} \cup \{\top\} \subset F^{PC}$.

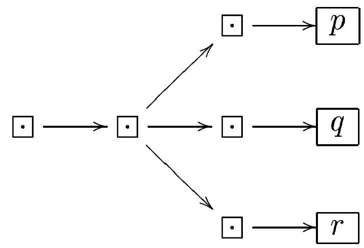
Examples.

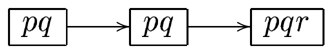
⋆ $\boxed{p} \to \boxed{pq}$ $\approx$ "$p \to q$" $\approx$ "$\neg p \vee q$" $\approx$

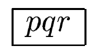


⋆ $\boxed{\cdot} \to \boxed{pq} \to \boxed{pqst}$ $\approx$ "$p \wedge q \wedge \neg(s \wedge t)$"

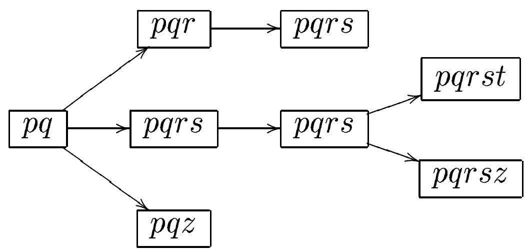
⋆ 

≈ "$p \wedge q \wedge r$" ≈ 

⋆ "$p \wedge q \to \neg r$" ≈ 

≈ 

≈ "$\neg(p \wedge q \wedge r)$"

⋆ Set $\varphi :=$ 

Here the first subtree [after subtracting $\{p, q\}$] translates as "$r \to s$", the last one as "$\neg z$", and the middle one as "$(r \wedge s) \to \neg(t \vee z)$": since its next subtree [also, after subtracting the convenient $\{p, q, r, s\}$] means $(t \vee z)$. So the next formula is obtained:

$$\varphi^\pi = (p \wedge q) \to \big[\neg(r \to s) \vee \neg[(r \wedge s) \to \neg(t \vee z)] \vee \neg(\neg z)\big] \approx$$

$$\approx (p \wedge q) \to \big[(r \wedge \neg s) \vee [r \wedge s \wedge (t \vee z)] \vee z\big]$$

3.2 Modal Logic

Keep Σ fixed as the set of proposition letters. A modal formula is constructed then with elements of Σ, with the ordinary logical operations $\wedge, \vee, \neg, \to, \top, \bot$, and, in addition, with two unary ones, namely $\Box, \Diamond$, read as "box" and "diamond", or, "necessarily" and "maybe". It would be enough picking only one of them, say $\Diamond$, and introducing the other one as $\Box := \neg\Diamond\neg$.
This leads us to the set of modal formulas F^{ML}.

A model (also called a *Kripke-structure*) is a triple $\langle W, \underset{W-W}{R}, \underset{W-\Sigma}{\Vdash} \rangle$ with W nonvoid. Elements of W are still called "states" or "worlds", and tRs reads "t sees (or reaches) s". A function between the sets of worlds will be a morphism in $\mathbb{M}^{ML}$ if preserves these relations R and $\Vdash$.
The classical validity is defined extending $\Vdash$ from Σ to F^{ML} by formula

induction, just as in propositional calculus: $s \Vdash \neg\phi$, $[s \Vdash \phi \vee \theta$ etc.$]$ should hold iff $s \nVdash \phi$, $[s \Vdash \phi \;\vee\; s \Vdash \theta$ etc.$]$, while

$$s \Vdash \Diamond\phi \overset{def}{\iff} \exists t : sRt \Vdash \phi$$

(and hence $s \Vdash \Box\phi \iff \forall t(sRt \Rightarrow t \Vdash \phi)$). Then, a model validates a formula iff every state of it does. This presents the modal language $\mathcal{L}^{ML}$.

Now, a "situation" is going to be a chain of conditions:
$\mathbb{D} := \mathbb{D}^{ML} := \big\langle \{\langle P_0, P_1, \dots P_n\rangle \,|\, n \in \omega \,, P_i \overset{fin}{\subset} \Sigma\}, \;\leq \big\rangle$
defining $\langle P_0 \dots P_n\rangle \leq \langle Q_0 \dots Q_m\rangle$ iff $n \leq m$; $i{<}n \Rightarrow P_i = Q_i$; $P_n \subset Q_n$.
Let $\mathbb{U} := \mathbb{U}^{ML}$ be the branch $\mathbb{D} \Rightarrow \mathbb{M}$ with certain R-chains of states as crossing arrows: $\langle w_0 \dots w_n\rangle : \langle P_0 \dots P_n\rangle \to \langle W, R, \Vdash\rangle$ whenever $W \ni w_i \Vdash \bigwedge P_i$ and $w_1 R w_2 \dots R w_n$. For brevity, use simply P for $\bigwedge P$ in modal formulas, since they will occur always in this context.
For $\langle P_0, \dots P_n\rangle =: \vec{P} \leq \vec{Q} =: \langle P_0, \dots P_{n-1}, Q_n, Q_{n+1} \dots Q_m\rangle$ pairs one introduces a subtraction $\vec{Q}\backslash\vec{P} := \langle Q_n \backslash P_n, Q_{n+1} \dots Q_m\rangle$.
Note that $\mathbb{D}$ is a partially ordered semigroup with operation
$\langle P_0.., P_n\rangle \star \langle Q_0, ..Q_m\rangle := \langle P_0.., P_n \cup Q_0, ..Q_m\rangle$, and that
$\vec{P} \leq \vec{R} \iff \exists\vec{Q} : \vec{P} \star \vec{Q} = \vec{R}$, and in that case, $\vec{R}\backslash\vec{P}$ is one of such $\vec{Q}$'s [ie. $\vec{P} \star (\vec{R}\backslash\vec{P}) = \vec{R}$]. (Remark that the notation ${}_{\vec{P}}\backslash\vec{R}$ would be more convenient instead of $\vec{R}\backslash\vec{P}$ here.)

In trees we draw $\boxed{P_0{\sim}P_1{\sim}\dots{\sim}P_n}$ for $\langle P_0, P_1, \dots P_n\rangle$, omitting the occurent unnecessary { } brackets, (ie. instead of $\{p, q, ..s\}$ use simply $p, q, ..s$), and $\boxdot$ for $\langle\emptyset\rangle$. (A dot denotes that there is no condition there.) So again, as in PC, a move from $\boxdot$ is just to pick a state; and, from $\boxed{\cdot{\sim}\cdot}$ is to pick a pair of states [such that the first sees the second one], etc.

Theorem 3.2. $\mathcal{L}(\mathbb{U}^{ML}) \cong \mathcal{L}^{ML}$

Proof. i) By induction construct a translator τ from formulas to trees: first, as before, without loss of generality, we can assume that negation occurs only before proposition letters. It is possible to get all the translated trees with root $\boxdot$, so that, an evaluation in a model could be always just pointing a state w, and therefore, $m \models \phi^\tau[w] \iff w \Vdash_m \phi$.

For a letter $p \in \Sigma$, as in PC, $p^\tau := \boxdot \longrightarrow \boxed{p}$

For its negated, $(\neg p)^\tau := \boxed{p} \approx \boxdot \longrightarrow \boxdot \longrightarrow \boxed{p}$
For conjunction and disjunction, also the same as above.
Denote $\boxdot\varphi$ the tree that we obtain by adding an extra, empty initial point to each node of φ (in other words, for a node $\vec{P}$, $\boxdot\vec{P} := \langle\emptyset,\emptyset\rangle \star \vec{P}$), or

$\boxdot\boxed{P_0{\sim}P_1{\sim}\dots} := \boxed{\cdot{\sim}P_0{\sim}P_1{\sim}\dots}$).

By that, depending on who is to choose the successor state, τ should be extended this way:

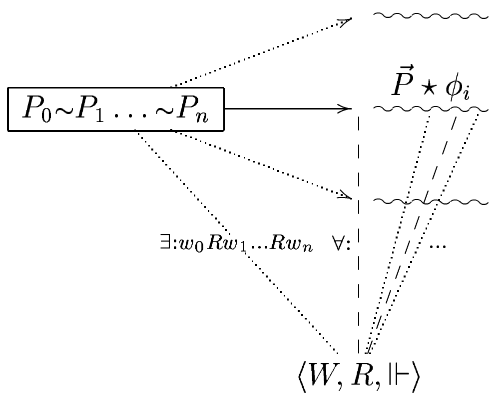
$$(\Box\phi)^\tau := \boxdot\phi^\tau \approx \boxdot \longrightarrow \boxdot \longrightarrow \boxdot\phi^\tau$$

$$(\Diamond\phi)^\tau := \boxdot \longrightarrow \boxed{\cdot{\sim}\cdot} \longrightarrow \boxdot\phi^\tau$$

ii) For a $\vec{P} = \boxed{P_0{\sim}\dots{\sim}P_n} \in Ob\mathbb{D}$ and a tree ψ with root $\geq \vec{P}$, define $\psi\backslash\vec{P}$ again nodewise (exchange each node $\vec{S}$ to $\vec{S}\backslash\vec{P}$).
Now, for a tree $\varphi = \langle\vec{P};..\varphi_i..\rangle$, assume inductively that $(\varphi_i\backslash\vec{P})^\pi =: \phi_i$ are already defined, and set

$$\varphi^\pi := \neg\big(P_0 \wedge \Diamond[P_1 \wedge \Diamond(..\Diamond\langle P_n \wedge \bigwedge_i \phi_i\rangle..)]\big)$$

Note that even for a root (where the conjunction of ϕ_i's is empty, hence meant to be true), it results the modal formula $\neg\big(P_0 \wedge \Diamond[P_1 \wedge \Diamond(..\Diamond P_n..)]\big)$, giving the first step of the induction.
To prove $\varphi^\pi \approx \varphi$, or rather $\neg\varphi^\pi \approx \neg\varphi$, change the role of the players, and imagine Exist starts a match on φ. Then he is to choose a chain $w_0Rw_1\dots Rw_n$ so that it should satisfy $w_0 \Vdash P_0, w_1 \Vdash P_1 \dots, w_n \Vdash P_n$; and, that he could win on all of the following subtrees φ_i, which means not else that $w_n \Vdash \bigwedge_i \phi_i$, since the previous states and assumptions on them (ie. w_j and P_j for $j<n$) would never change any more, and the continuation on φ_i is started by All.

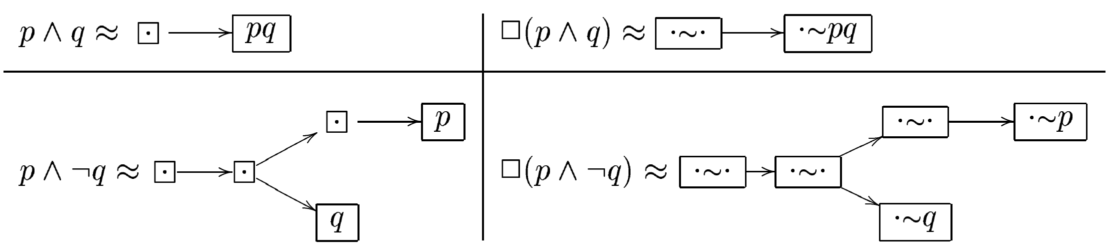


□

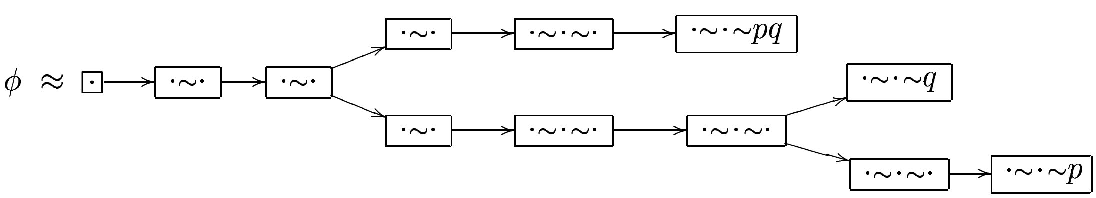
Remark. Again, it would be enough to deal only with chains of the form $\boxed{\cdot\sim\cdot\sim\ldots\sim(p)}$ (with at most one proposition letter at the end). In this case $\mathbb{D}$ would be isomorphic to a partial order on $(\omega \sqcup \omega \times \Sigma)$.

Examples.

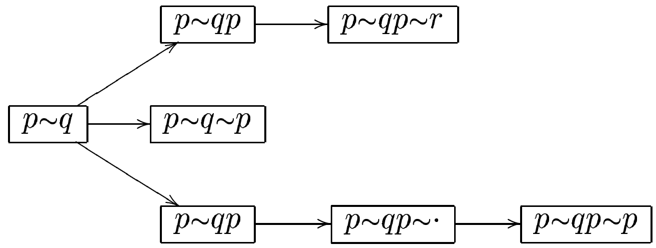
- ⋆ $\boxed{\cdot\sim s} \longrightarrow \boxed{\cdot\sim st} \approx \neg[\top \wedge \Diamond(s \wedge \neg t)] \approx \Box(s \to t)$
- ⋆ $\boxed{\cdot\sim\cdot\sim s} \longrightarrow \boxed{\cdot\sim\cdot\sim st} \approx \Box\Box(s \to t)$
- ⋆ $\phi := \Diamond[\Box(p \wedge q) \vee \Box(p \wedge \neg q)]$ translate the contents this way:



and then, linking these at Exist's branch (for $\vee$) and adding an extra initial $\cdot\sim$ (for $\Diamond$), finally we get:



$\star \quad \varphi :=$



Follow the procedure given in ii): after subtraction, the first subtree becomes $\boxed{p} \longrightarrow \boxed{p\sim r}$ which translates as $\neg(p\wedge\neg\Diamond r) \approx p \to \Diamond r$; the second as $\neg\Diamond p$; and the third one goes like

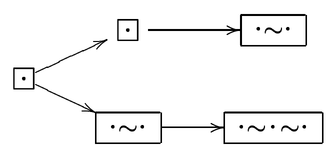
$$\boxed{p} \longrightarrow \boxed{p\sim\cdot} \longrightarrow \boxed{p\sim p} \approx \neg\left(p \wedge \left[\boxed{\cdot\sim\cdot} \longrightarrow \boxed{\cdot\sim p}\right]^{\pi}\right) \approx$$

$$\approx \neg\,(p \wedge \ \Box[\top \to p]) \approx \neg(p \wedge \Box p) \approx p \to \Diamond\neg p.$$

So, all in all, $\varphi^{\pi} \approx \neg\left(p \wedge \Diamond\left[q \wedge (p \to \Diamond r) \wedge \neg\Diamond p \wedge (p \to \Diamond\neg p)\right]\right)$.

$\star$ The modal formula of Gödel's second incompleteness theorem (reading $\Box$ as "provable") translates like:

$$\neg\Box\bot \to \neg\Box\neg\Box\bot \approx \Box\bot \vee \Diamond\Box\bot \approx$$



3.3 First order Logics

Let us fix a type $t : Q \sqcup R \to \omega$ (the arity function itself, cf. [12]), where elements of Q will be referred as function symbols of t, while those of R as relation symbols. ($r^t \geq 1$ is assumed for all $r \in R$.) $\mathbb{M} := \mathbb{M}^{FOL_t} := \mathbb{M}\mathrm{od}_t$ will be the category of the ordinary first order models (ie. sets equipped with q^t-ary operations and r^t-ary relations) with the homomorphisms between them (ie. functions which preserve all of these operations and relations).
For simplicity, one may suppose to have the added 2-ary relation symbol $(=)$ with its usual meaning in models. Denote $At(X)$ the set of atomic formulas with variables of X. Formulas (of $F^{FOL_t} = F$) are obtained by the atomic

formulas, combined with the previously rehearsed logical connectives ($\wedge, \vee$, etc.), and quantifiers $\forall, \exists$ for variables.

Definition 3.1 **($\mathbb{U}^{FOL_t}$).** *$\mathbb{D} := \left\langle \{\langle X, \Gamma\rangle \mid X \text{finite set}, \Gamma \overset{fin}{\subset} At(X)\}, \leq \right\rangle$, where $\langle X, \Gamma\rangle \leq \langle Y, \Delta\rangle \overset{def}{\Longleftrightarrow} X \subset Y, \Gamma \subset \Delta$.*
A mapping $\varrho : X \to m$ will be a diagonal morphism $\langle X, \Gamma\rangle \to m$ if each atomic predicate of Γ is true in m after substituting the variables x to x^{ϱ}.

Note that this $\mathbb{U} = \mathbb{U}^{FOL_t}$ just defined is the branch given by the *free functor* $Fr : \langle X, \Gamma\rangle \mapsto \mathbf{F}X/_{\Gamma}$, which produces exactly the finitely presented models, and that $\mathbb{D}$ is "almost" equivalent to this subcategory of $\mathbb{M}^{FOL_t}$ in the sense of

Lemma 3.3. *For a given finite presentation $\langle X|\Gamma\rangle$ of a model m, a finitely presented n, and a homomorphism $\underset{m\to n}{\vartheta}$, there exists a $\mathbb{D}$-arrow δ:*

$$\begin{array}{ccc} \langle X, \Gamma\rangle & \overset{\delta}{\hookrightarrow} & \langle Y, \Delta\rangle \\ {\scriptstyle Fr}\vdots & & \vdots{\scriptstyle Fr} \\ m & \underset{\vartheta}{\longrightarrow} & n \end{array}$$

Proof. Take any finite presentation $\langle Y_0|\Delta_0\rangle$ of n, put $Y := X \sqcup Y_0$, pick a term τ_x with variables from Y_0 for each $x \in X$ such that $x^{\vartheta} = \tau_x$ holds in n, and then set

$$\Delta := \Gamma \sqcup \{x = \tau_x | x \in X\} \sqcup \Delta_0$$

□

In papers [1, 2, 6] instead of $\mathbb{U}$, the embedding functor (from the full subcategory of strongly small objects) was used. Proposition 3.4 asserts that it is essentially the same.

Proposition 3.4. *The language given by the inclusion functor of the full subcategory of strongly small [ie. finitely presented] models $J : \mathbb{S}\mathrm{m} \hookrightarrow \mathbb{M}$, is equivalent to the one induced by our $\mathbb{U}$:*

$$\mathcal{L}(\mathbb{J}) \cong \mathcal{L}(\mathbb{U}^{FOL_t})$$

Proof. One translator comes from $Fr : \mathbb{D} \to \mathbb{S}\mathrm{m}$, translating the trees arrow by arrow, and the other one can be obtained by the lemma, building an equivalent tree starting from the root. ☐

Theorem 3.5. $\mathcal{L}(\mathbb{U}^{FOL_t}) \cong \mathcal{L}^{FOL_t}$

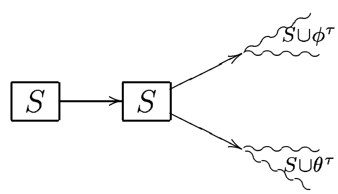
Proof. i) For a given formula ϕ, one can assume again that negations appear only at atomic levels, and that each variable is bound at most once, moreover, a bound variable may occur only within the quantified subformula. (Here it is supposed to have inifinitely many variables.) Denote $S := \mathrm{Free}(\phi)$ the set of free variables of ϕ. We could put $\langle S, \emptyset\rangle$ in the root of ϕ^τ, so that an evaluation into a model m in the above sense (a mapping $S \to m$) coincides with that in the original sense. For an object $d = \langle X, \Gamma\rangle \in Ob\mathbb{D}$, define $S \cup d := \langle S \cup X, \Gamma\rangle$ and similarly $d \backslash S := \langle X \backslash S, \Gamma\rangle$, and $d \backslash \langle S, \Delta\rangle := \langle X \backslash S, \Gamma \backslash \Delta\rangle$. Then for a $\mathbb{D}$-tree φ, $S \cup \varphi$ and $\varphi \backslash S$ are defined nodewise (replace each node d of φ to $S \cup d$ [resp. $d \backslash S$, or $d \backslash \langle S, \Delta\rangle$]).

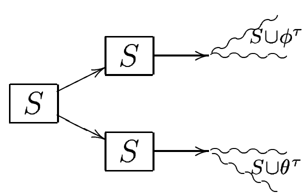
For getting nicer pictures, draw $\boxed{S|_{\gamma_1..\gamma_n}}$ instead of $\langle S, \{\gamma_1..\gamma_n\}\rangle$ in trees. We translate formulas recursively:

For an atomic ϕ, $S := \mathrm{Free}(\phi)$ and $\phi^\tau := \boxed{S} \longrightarrow \boxed{S|_\phi}$.

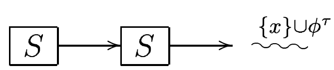
$$(\neg\phi)^\tau := \boxed{S|_\phi} \quad \approx \quad \boxed{S} \longrightarrow \boxed{S} \longrightarrow \boxed{S|_\phi}$$

For conjunction and disjunction $S := \mathrm{Free}(\phi \wedge \theta) = \mathrm{Free}(\phi) \cup \mathrm{Free}(\theta)$,

$(\phi \wedge \theta)^\tau :=$ 

$(\phi \vee \theta)^\tau :=$ 

For quantifiers, $S := \mathrm{Free}(\forall x : \phi) = \mathrm{Free}(\phi) \backslash \{x\}$

$(\forall x : \phi)^\tau := \{x\} \cup \phi^\tau \approx$ 

$$(\exists x : \phi)^\tau := \boxed{S} \longrightarrow \boxed{xS} \longrightarrow \overset{\{x\}\cup\phi^\tau}{\sim\!\sim}$$

Where xS stands for $\{x\} \cup S$.
Here, after the first move (to arbitrarily assign values to the elements of S in the given model), All [respectively, Exist] in turn evaluates also the new variable x. Now it is easy to see that these trees are valid with a given evaluation, iff the corresponding formulas ($\forall x : \phi$, $\exists x : \phi$) are so.

ii) For the other direction, the most convenient is to allow also trees on vertices of the form $\langle X, \Gamma\rangle$ where the elements of Γ are still atomic formulas, but with variables *not necessarily* in X. These variables could be understood free in subformulas, quantified later. However, starting with a $\mathbb{D}$-tree, the process below will finally result in a closed formula, beginning with a universal quantification on the set of variables of the root. So, for the corresponding not necessarily free formula, we could simply erase those quantifiers.
Signify $\forall X := \forall x_1..\forall x_n$ for $X = \{x_1..x_n\}$.
For a one noded tree, $\boxed{X|_\Gamma}^\pi :=$ "$\forall X : \neg \bigwedge \Gamma$".

Then let $\varphi = \boxed{X|_\Gamma}$ with arrows to $\overset{\varphi_1}{\sim\!\sim}, \ldots, \underset{\varphi_n}{\sim\!\sim}$ and $\phi_i := (\varphi_i \backslash \langle X, \Gamma\rangle)^\pi$.

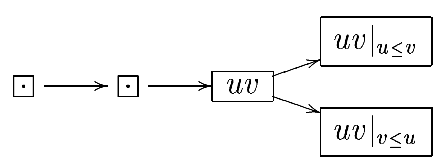
Now $\varphi^\pi :=$ "$\forall X : \bigwedge \Gamma \to \bigvee_i \neg\phi_i$" is a translation. □

Corollary 3.6. $\mathcal{L}(\mathbb{J}) \cong \mathcal{L}^{FOL_t}$.

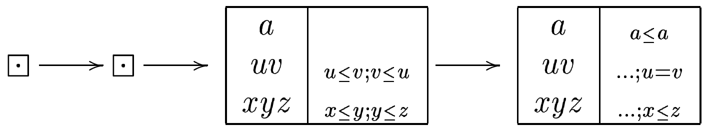
Examples.

Fix a type with binary relation symbols $\leq$, $=$ and operation symbols 0, 1, $\cdot$, $+$ (with arities 0, 0, 2, 2).

⋆ "$\leq$ (as a partial order) is total" as "$\forall u \forall v : u \leq v \vee v \leq u$"

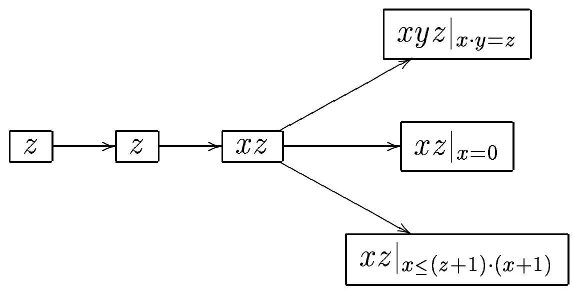


⋆ "$\leq$ is a partial order"

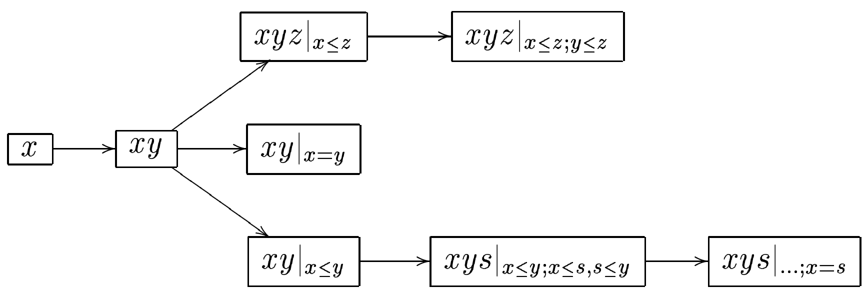


$\{$such as $\forall a, u, v, x, y, z : ([u \leq v \leq u] \wedge [x \leq y \leq z]) \to ([a \leq a] \wedge [u = v] \wedge [x \leq z])\}$
(The three dots stand for repetition of the previous conditions, and the $\boxdot$'s at the root are just to sign that the corresponding formula is closed.)

⋆ "$\forall x : \exists y(x \cdot y = z) \vee (x = 0) \vee (x \leq (z+1) \cdot (x+1))$" $\approx$

$\approx$ 

⋆ Consider the tree

$\varphi :=$ 

Follow the process given in ii):
Now the linear subtrees after subtraction get the forms $\boxed{z|_{x\leq z}} \to \boxed{z|_{x\leq z; y\leq z}}$, $\boxed{\;|_{x=y}}$, and $\boxed{\;|_{x\leq y}} \to \boxed{s|_{x\leq y; x\leq s, s\leq y}} \to \boxed{s|_{\ldots; x=s}}$, translate as "$\forall z(x \leq z \to y \leq z)$", "$x \neq y$"; "$x \leq y \to \neg \left[\!\left[\boxed{s|_{x\leq s, s\leq y}} \to \boxed{s|_{x\leq s, s\leq y; x=s}} \right]\!\right]^{\pi}$ " $\approx$ "$x \leq y \to \neg\, [\forall s(x \leq s \leq y \to x = s)]$ " $\approx$ "$x \leq y \to [\exists s(x \leq s \leq y \;\wedge\; x \neq s)]$".

Hence, altogether:

$\varphi \approx$ "$\exists y \big[(\forall z : x \leq z \to y \leq z) \wedge (x \neq y) \wedge (x \leq y \to \exists s(x \leq s \leq y \;\wedge\; x \neq s))\big]$".

See more examples in [1, 2, 4, 6].

3.4 Diagrammatic language for category theory

Definition 3.2. *A pair of set-functions* $G : V \underset{E}{\overset{B}{\leftleftarrows}} A$ *is called a directed* *graph,* $V =: V_G$ *is the set of vertices,* $A =: A_G$ *of arrows or edges, and* B, E *determine the begin [resp. end] of the arrows.*

A path is a finite sequence of well-connected edges $\langle \alpha_1 \dots \alpha_n \rangle =: \vartheta$ ie. where $\alpha_i^E = \alpha_{i+1}^B$. The begin/end of paths defined straightforward: $\vartheta^B := \alpha_1^B$, $\vartheta^E := \alpha_n^E$. Paths of length 0 shall be the vertices themselves. We call a pair of paths $\langle \xi, \eta \rangle$ with the same begin and same end a commutativity condition (that is, if $\xi^B = \eta^B$, $\xi^E = \eta^E$).

Definition 3.3. *A C-graph is a graph equipped with a set of commutativity conditions.*

Note that the paths of a graph G as morphisms form a so called free category $\mathbf{F}G$ (the objects being the paths of 0 length, ie. the vertices), and, this *free functor* extends also to C-graphs: $Fr : \langle G, C \rangle \mapsto \mathbf{F}G/_C$.

Definition 3.4. $\langle G, C \rangle$ *is a sub-C-graph of* $\langle H, D \rangle$ *(in symbols:* $\langle G, C \rangle \leq \langle H, D \rangle$*) if* $C \subset D$ *and* G *is a subgraph of* H *[ie.* $V_G \subset V_H$, $A_G \subset A_H$, $B_G = B_H \restriction_{A_G}$, $E_G = E_H \restriction_{A_G}$*]*

Denote the category of this partial order $\mathbb{CG}\mathrm{r}^{\leq}$, and $\mathbb{U} := \mathbb{U}^{\mathbb{C}\mathrm{at}} : \mathbb{CG}\mathrm{r}^{\leq} \Rightarrow \mathbb{C}\mathrm{at}$ the branch induced by Fr. Then, consider a C-graph as an abstract diagram, and a crossing morphism as its realization in a category (this is the notion called "diagram" usually in the literature).

An elementary property of categories is a first order formula of a 2 sorted language (sorts of $Objects$ and $Morphisms$), in which the atomic predicates are compositions, equality, and source and target (ie. begin/end) assertions (ie. we have $\underset{Mor^2 \to Mor}{\circ}$, $\underset{Mor \to Ob}{B, E}$ operation symbols beside the relation symbol of equality $\langle = \rangle$).

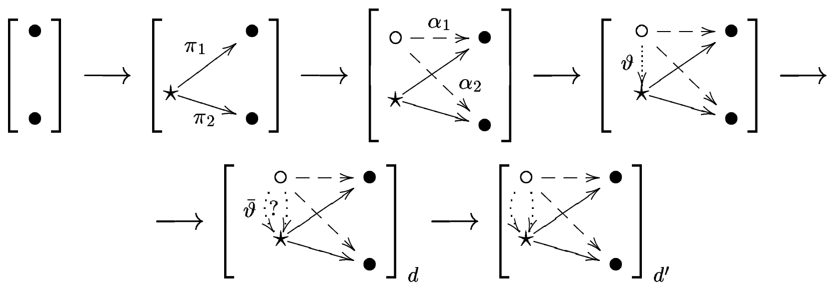
Theorem 3.7. *An elementary property is invariant on category equivalence if and only if it can be translated to a* $\mathbb{CG}\mathrm{r}^{\leq}$*-tree.*

A sketch of the proof can be found in [8].

Examples.

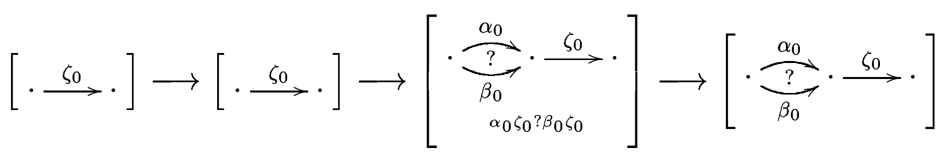
In the following diagrams, in order to use less symbols (to make them easier to overview), instead of indicating the commutativity conditions, we mark by [?] those pairs of paths which are not supposed to commute, (either in the circle they surround or simply written below). Since in the trees these graphs are always included in the next ones, if one wishes to name the edges, it is sufficient to name only those that appear for the first time in the current picture.

⋆ "Binary products do exist"



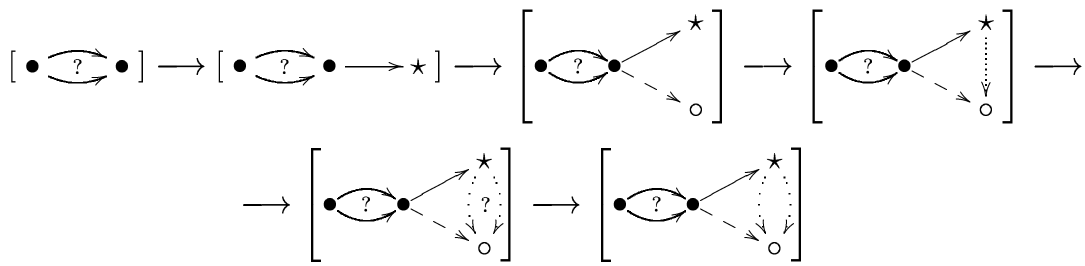
For instance, the C-graph denoted by d has conditions $\langle\vartheta\pi_1,\alpha_1\rangle$; $\langle\vartheta\pi_2,\alpha_2\rangle$; $\langle\bar\vartheta\pi_1,\alpha_1\rangle$; $\langle\bar\vartheta\pi_2,\alpha_2\rangle$, since these triangles are not marked, and d' in addition $\langle\bar\vartheta,\vartheta\rangle$.
So, one could phrase this linear tree as: "For every pair of objects there exist an obj.(⋆) with π_1 and π_2, such that for every situation of α_1,α_2 there is a *unique* ϑ satisfying $(\vartheta\pi_i=\alpha_i)_{i=1,2}$."
Note that replacing the root with an arbitrary abstract diagram g (and, of course, carrying it through the whole tree) changes the corresponding sentence to "Every g-diagram has limit".

⋆ "$\zeta\in\mathbb{C}$ is a *constant morphism*"



(With fixed evaluation $\zeta_0\mapsto\zeta$ in $\mathbb{C}$.) Now one would get the sentence: "Whenever $\alpha^B=\beta^B$ and $\alpha^E=\beta^E=\zeta^B$, $\alpha\zeta=\beta\zeta$ holds."

⋆ "Coequalizers do exist"



4 Problems

1. What algebraic structure does (may) the class of (bi-)branches carry? (How to compose branches?)

2. Formulate the results of [1, 2, 6] in the present terminology.

3. Find necessery and sufficient conditions for an abstract language to be equivalent to one induced by a branch.

4. Which formulas can be represented by
   a) linear trees?
   b) trees of depth $\leq n$?

5. How to handle infinite matches?

6. What if we replace homomorphisms with other morphisms such as partial homomorphisms or structure preserving relations, etc. in $\mathbb{M}$?

7. Give concrete languages induced by a branch over geometrical categories (cat. of topological spaces, differentiable manifolds, etc.).

8. Continue with second or higher order logics, intuitionistic logics, etc.

Index of notations

ω	:	the set of natural numbers
$\mathbb{S}$et, $\mathbb{G}$r, $\mathbb{M}$od$_t$, $\mathbb{CG}$r	:	the categories of sets, groups, first order models of type t, C-graphs
$\#S$	:	the cardinality of S
$\mathbf{P}S$	:	the power set of S, ie. $\mathbf{P}S := \{A \mid A \subset S\}$
A^S	:	the set of S-sequences in A
$A \overset{fin}{\subset} S$	:	A is a *finite* subset of S
$\prod_s m_s,\ a \times b$	:	categorical product of objects m_s [resp. of a and b]
$\coprod_s m_s,\ a \sqcup b$	:	coproduct (disjoint union)
$A \leq B$	:	A is a substructure of B (subcategory, sub-C-graph, etc.)
$\vec{P}$	:	$\langle P_0, P_1, ..P_n \rangle$
$\mathbb{A} \overset{\mathbb{H}}{\Rightarrow} \mathbb{B}$	:	branch from $\mathbb{A}$ to $\mathbb{B}$
$\mathcal{L}(\mathbb{U})$	:	the language induced by the branch $\mathbb{U}$
$\mathcal{L}^K$	:	the language of the logic K ($K \in \{PC,\ ML,\ FOL_t\}$)
$\left\|\left[\mathcal{P} \text{ holds for } m_\bullet \right]\right\|$	:	the set of indices s which make $\mathcal{P}$ hold for m_s
$a \cong b$	:	a is isomorphic to b
$\mathcal{L} \cong \mathcal{L}'$	:	languages $\mathcal{L}$ and $\mathcal{L}'$ are equivalent
$\mathbb{A} \simeq \mathbb{B}$	:	cat. $\mathbb{A}$ is equivalent to $\mathbb{B}$
$F \simeq G$	:	functors F and G are naturally isomorphic
$\varphi \approx \theta$	:	φ is equivalent to θ in meaning
$\Box, \Diamond, \wedge, \vee, \top, \bot$	:	necessary, maybe, and, or, true, false
${}_a\nu_b$	:	basic crossing arrow $a \to b$
$c \rightsquigarrow \mathbb{B}$	:	comma category
$f\restriction_H$	:	mapping f restricted to H
$\langle X \mid \Gamma \rangle$	:	a presentation of a first order model (generated by X)
Free(ϕ)	:	the set of free variables of ϕ
$\mathbf{F}X$	:	the universal algebra generated by variables of the set X
$\mathbf{F}G$	:	free category on the graph G
Fr	:	the free functor ($\langle Q, \Upsilon \rangle \mapsto \mathbf{F}Q/_{\Upsilon}$)

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