Preprint: September, 2005.
Submitted to Contr. Gen. Alg. 17

BRIDGES IN CATEGORIAL LOGIC

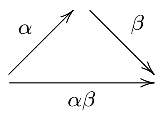
BERTALAN PÉCSI

Abstract. This paper is to introduce a model theoretic approach of mathematical languages, in which sentences are built up not by terms or words, but as trees of "abstract situations" what may be interpreted in the models and are reflected by homomorphisms.

1. Bridges

In this section we introduce a concept that we will call a (directed) **bridge**. Bridges appeared in order to link up two categories with morphisms between them, and it turned out that they are essentially the same as Lawvere's **profunctors** (or categorial **bimodules**). We will see that adjoint situations, natural transformations and category equivalences can be expressed by means of bridges in a simple way. In section 2 we will use a bridge to make abstract logic within.

Throughout the paper, composition of arrows in any category is written in the following order:



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In a category $\mathbb{A}$, the **hom-set** (i.e. set of morphisms) from $a \in Ob\mathbb{A}$ to $b \in Ob\mathbb{A}$ will be denoted by $(a \mid b)_{\mathbb{A}}$.

We will use infix notation for binary relations, that is, if $\mathtt{r} \subseteq A \times B$ is a relation between sets, then instead of $\langle a, b\rangle \in \mathtt{r}$ we will write $a\mathtt{r}b$.

Definition 1.1 (Bridge)**.** *Let $\mathbb{H}$ be a category, and let $\mathbb{A}$ and $\mathbb{B}$ be two disjoint subcategories of $\mathbb{H}$. We call $\mathbb{H}$ a* ***(directed) bridge*** *from $\mathbb{A}$ to $\mathbb{B}$, iff each morphism of $\mathbb{H}$ not belonging to $Mor\mathbb{A} \cup Mor\mathbb{B}$ has its domain in $\mathbb{A}$ and codomain in $\mathbb{B}$. These morphisms will be referred to as the* ***diagonal morphisms*** *or*

2000 *Mathematics Subject Classification.* Primary 03G30; secondary 18A15, 18A40.
Key words and phrases. algebraic logic, category, profunctor, bimodule, Morita equivalence, bridge, situation.
This work was based on the papers of H. Andréka - I. Németi - I. Sain, and was supported by Ildikó Sain.

diagonal arrows *of the bridge* $\mathbb{H}$.
We introduce the notation $\mathbb{H}\colon \mathbb{A} \Rightarrow \mathbb{B}$ *to mean that* $\mathbb{H}$ *is a directed bridge from* $\mathbb{A}$ *to* $\mathbb{B}$.
$\mathtt{Dg}^{\mathbb{H}}$ *denotes the class of all diagonals of* $\mathbb{H}$, *i.e.*

$$\mathtt{Dg}^{\mathbb{H}} := Mor\mathbb{H} \setminus (Mor\mathbb{A} \cup Mor\mathbb{B}) = \{\vartheta \in Mor\mathbb{H} \mid {}_{dom}\,\vartheta \in \mathbb{A}, {}_{cod}\,\vartheta \in \mathbb{B}\}.$$

Most often we define a bridge $\mathbb{H}$ via giving

$$\mathbb{A}, \quad \mathbb{B} \text{ and } \mathtt{Dg}^{\mathbb{H}} = \cup\{(a \shortmid b)_{\mathbb{H}} \mid a \in Ob\mathbb{A}, b \in Ob\mathbb{B}\}$$

plus all the identities of the form

$$\alpha{\cdot}v{\cdot}\beta = u \text{ with } \alpha \in Mor\mathbb{A},\ u, v \in \mathtt{Dg}^{\mathbb{H}} \text{ and } \beta \in Mor\mathbb{B}.$$

Remark. Let $\mathbb{H}\colon \mathbb{A} \Rightarrow \mathbb{B}$ be a bridge, and $\mathtt{Dg} := \mathtt{Dg}^{\mathbb{H}}$. Note that $\mathbb{A}$ acts from the left on $\mathtt{Dg}$ (by composition), while $\mathbb{B}$ acts from the right, and these actions determine $\mathbb{H}$ uniquely. So, one could identify this **bimodule** ${}_{\mathbb{A}}\mathtt{Dg}_{\mathbb{B}}$ with the bridge itself.
In the literature these categorial bimodules are also called **profunctors** (in e.g. [10]) or **distributors** (in e.g. [11]).

Examples:

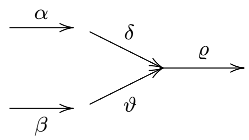
(Ex1) Let $\mathbb{F}\colon \mathbb{S}\mathsf{et} \Rightarrow \mathbb{G}\mathsf{rp}$ be the bridge (from the category of sets to that of groups), whose diagonal morphisms are the set theoretical functions, i.e., for a set S and a group $\mathcal{G}$, the hom-set $(S \shortmid \mathcal{G})_{\mathbb{F}}$ is defined by

$$(S \shortmid \mathcal{G})_{\mathbb{F}} := \{\alpha \mid \alpha\colon S \to \mathcal{G} \text{ is a function}\}.$$

The left and right compositions in $\mathbb{F}$ are just the ordinary compositions of functions.

(Ex2) Let $\mathbb{A}$ be an arbitrary category. We define a bridge $\mathbb{C}\colon \mathbb{A} \times \mathbb{A} \Rightarrow \mathbb{A}$ by setting $\mathtt{Dg}^{\mathbb{C}} := \{\langle\delta, \vartheta\rangle \in Mor\mathbb{A} \times Mor\mathbb{A} \mid {}_{cod}\,\delta = {}_{cod}\,\vartheta\}$, with the following composition rules:
For $\langle\alpha, \beta\rangle \in Mor(\mathbb{A} \times \mathbb{A})$, $\langle\delta, \vartheta\rangle \in \mathtt{Dg}^{\mathbb{C}}$ and $\varrho \in Mor\mathbb{A}$, set

$$\langle\alpha, \beta\rangle{\cdot}\langle\delta, \vartheta\rangle{\cdot}\varrho := \langle\alpha\delta\varrho, \beta\vartheta\varrho\rangle$$



whenever the right side of the equation is defined.

(Ex3) Let $\mathbb{A}\mathsf{b}$ denote the category of Abelian groups. There is a bridge $\mathbb{T}\colon \mathbb{A}\mathsf{b} \times \mathbb{A}\mathsf{b} \Rightarrow \mathbb{A}\mathsf{b}$, in which the diagonals $\langle A, B\rangle \to C$ are just the bilinear functions $A \times B \to C$. Note that, in $\mathbb{T}$, each pair $\langle A, B\rangle \in Ob(\mathbb{A}\mathsf{b} \times \mathbb{A}\mathsf{b})$ is reflected by the tensor product $A \otimes B$ in $\mathbb{A}\mathsf{b}$.

(Ex4) We can define a bridge $\mathbb{P}\colon \mathbb{S}\mathsf{et} \Rightarrow \mathbb{S}\mathsf{et}^{op}$ with diagonal morphisms all the binary relations between sets. If $\mathrm{f}\colon A \to B$ and $\mathrm{h}\colon V \leftarrow U$ are functions and $\mathtt{r} \subseteq B \times V$ is a diagonal of $\mathbb{P}$ from B to V, then the composition $\mathrm{f}\cdot\mathtt{r}\cdot\mathrm{h}$ is defined by $a(\mathrm{f}\cdot\mathtt{r}\cdot\mathrm{h})u \overset{def}{\Leftrightarrow} \mathrm{f}(a)\ \mathtt{r}\ \mathrm{h}(u)$ for $a \in A,\ u \in U$.

$$\begin{array}{ccccc} A & \xrightarrow{\mathrm{f}} & B & & \\ & & \vdots\ \mathtt{r} & & \\ & & V & \xleftarrow{\mathrm{h}} & U \end{array}$$

Definition 1.2. *Let* $\mathrm{F}\colon \mathbb{A} \to \mathbb{B}$ *be a functor. We construct the* ***bridge of*** F $(\mathbb{B}\mathsf{r}(\mathrm{F})\colon \mathbb{A} \Rightarrow \mathbb{B})$*, and the* ***dual bridge of*** F $(\mathbb{B}\mathsf{r}^{op}(\mathrm{F})\colon \mathbb{B} \Rightarrow \mathbb{A})$*, by setting*

$$\mathtt{Dg}^{\mathbb{B}\mathsf{r}(\mathrm{F})} := \{\langle a, \beta\rangle \ \mid \ a \in Ob\mathbb{A}, \beta \in Mor\mathbb{B},\ \mathit{dom}\,\beta = \mathrm{F}a\}$$

$$\alpha\cdot\langle a,\beta\rangle\cdot\beta' := \langle \mathit{dom}\,\alpha, \mathrm{F}\alpha\cdot\beta\cdot\beta'\rangle,\ \ \textit{whenever}\ \mathit{cod}\,\alpha = a\ \textit{and}\ \mathit{cod}\,\beta = \mathit{dom}\,\beta'$$

$$\mathtt{Dg}^{\mathbb{B}\mathsf{r}^{op}(\mathrm{F})} := \{\langle \beta, a\rangle \ \mid \ \beta \in Mor\mathbb{B}, a \in Ob\mathbb{A},\ \mathit{cod}\,\beta = \mathrm{F}a\}$$

$$\beta'\cdot\langle \beta, a\rangle\cdot\alpha := \langle \beta'\cdot\beta\cdot\mathrm{F}\alpha, \mathit{cod}\,\alpha\rangle,\ \ \textit{whenever}\ \mathit{dom}\,\alpha = a\ \textit{and}\ \mathit{cod}\,\beta' = \mathit{dom}\,\beta\ .$$

These constructions are analogous to the situation when a homomorphism $\mathrm{f}\colon A \to B$ between rings determines a left A-, right B-module, as well as a left B-, right A-module structure (both on the additive reduct of B).

Let $\mathbb{F}\colon \mathbb{A} \Rightarrow \mathbb{B}$ and $\mathbb{G}\colon \mathbb{A} \Rightarrow \mathbb{B}$ be bridges. We call a functor $\mathrm{H}\colon \mathbb{F} \to \mathbb{G}$ a **bridge morphism**, iff H is identical on both $\mathbb{A}$ and $\mathbb{B}$. Such a bridge morphism maps the hom-set $(a \shortmid b)_{\mathbb{F}}$ to the hom-set $(a \shortmid b)_{\mathbb{G}}$ for each pair of objects $a \in Ob\mathbb{A},\ b \in Ob\mathbb{B}$. We say that the bridges $\mathbb{F}$ and $\mathbb{G}$ are **isomorphic** (in notation $\mathbb{F} \cong \mathbb{G}$), iff there is an invertible bridge morphism between them.

We call a bridge $\mathbb{F}\colon \mathbb{A} \Rightarrow \mathbb{B}$ **functorial**, iff there is a functor $\mathrm{F}\colon \mathbb{A} \to \mathbb{B}$ such that $\mathbb{B}\mathsf{r}(\mathrm{F}) \cong \mathbb{F}$. We call $\mathbb{F}$ **opfunctorial**, iff there is a functor G such that $\mathbb{B}\mathsf{r}^{op}(\mathrm{G}) \cong \mathbb{F}$.
We remark at this point that the concept of a functorial bridge is essentially the same as the concept of a ***saturated anafunctor*** defined by M. Makkai in [12].

The next proposition gives a characterisation for functorial bridges. Similar characterisation for opfunctorial bridges is obtained by duality.

Proposition 1.3. *For a bridge* $\mathbb{U}\colon \mathbb{A} \Rightarrow \mathbb{B}$*, the following assertions are equivalent:*
1) $\mathbb{U}$ *is functorial.*
2) There is a map $\mathrm{U}_0\colon Ob\mathbb{A} \to Ob\mathbb{B}$ *such that for all* $a \in Ob\mathbb{A}$*,*

$$(a \shortmid b)_{\mathbb{U}} \simeq (\mathrm{U}_0 a \shortmid b)_{\mathbb{B}}\ \ \textit{naturally in}\ b \in Ob\mathbb{B}.$$

3) $\mathbb{B}$ *is a* ***reflective*** *subcategory of* $\mathbb{U}$*.*

Note that the proof of *3)⇒1)* uses the ***axiom of choice***.

In other words, a bridge $\mathbb{A} \Rightarrow \mathbb{B}$ is **functorial** iff all objects of $\mathbb{A}$ can be reflected in $\mathbb{B}$. Dually, a bridge $\mathbb{A} \Rightarrow \mathbb{B}$ is **opfunctorial** iff all objects of $\mathbb{B}$ can be coreflected in $\mathbb{A}$.

Note that examples (Ex1), (Ex3) and (Ex4) above are functorial bridges, and that (Ex1), (Ex2) and (Ex4) are opfunctorial. (Cf. Theorem 1.5.)

In the literature of category theory one can find several functors defined on objects only, because it is understood that there is a unique natural way to extend them to morphisms. This is the case, for example, when we know a left [or a right] adjoint of our current functor (e.g. product or limit functors). As a generalization of this, in Corollary 1.4, we give a necessary and sufficient condition, which, in practice, verifies easily that a given mapping on objects extends naturally to morphisms.

Corollary 1.4. *Let* $\mathbb{A}$ *and* $\mathbb{B}$ *be categories, and let* $\mathrm{U}_0 : Ob\mathbb{A} \to Ob\mathbb{B}$ *be a map defined on objects only. Then the following statements are equivalent:*
1) U_0 *extends to a functor* $\mathrm{U} : \mathbb{A} \to \mathbb{B}$ *(so that* $\mathrm{U}\restriction_{Ob\mathbb{A}} = \mathrm{U}_0$*).*
2) There is a bridge $\mathbb{A} \Rightarrow \mathbb{B}$ *in which each* $a \in Ob\mathbb{A}$ *is reflected by a diagonal arrow whose codomain is* $\mathrm{U}_0 a$.
3) There is a bridge $\mathbb{B} \Rightarrow \mathbb{A}$ *in which each* $a \in Ob\mathbb{A}$ *is coreflected by a diagonal arrow whose domain is* $\mathrm{U}_0 a$.

The next result asserts that a bridge expresses an **adjoint situation** iff it is both functorial and opfunctorial:

Theorem 1.5 (Adjoint bridge theorem)**.** *Let* $\mathrm{L}\colon \mathbb{A} \to \mathbb{B}$ *and* $\mathrm{R}\colon \mathbb{B} \to \mathbb{A}$ *be functors. Then we have*

$$\mathrm{L} \textit{ is left adjoint to } \mathrm{R} \qquad \Longleftrightarrow \qquad \mathbb{B}\mathsf{r}(\mathrm{L}) \cong \mathbb{B}\mathsf{r}^{op}(\mathrm{R})$$

Proof. If $\mathbb{B}\mathsf{r}^{op}(\mathrm{R}) \cong \mathbb{B}\mathsf{r}(\mathrm{L})$, then

$$(\mathrm{L}a \mid b)_{\mathbb{B}} \;\simeq\; (a \mid b)_{\mathbb{B}\mathsf{r}(\mathrm{L})} \;\simeq (a \mid b)_{\mathbb{B}\mathsf{r}^{op}(\mathrm{R})} \;\simeq\; (a \mid \mathrm{R}b)_{\mathbb{A}}$$

naturally in both $a \in Ob\mathbb{A}$ and $b \in Ob\mathbb{B}$.
Conversely, $(a \mid \mathrm{R}b)_{\mathbb{A}} \simeq (\mathrm{L}a \mid b)_{\mathbb{B}}$ implies $(a \mid \mathrm{R}b)_{\mathbb{A}} \simeq (a \mid b)_{\mathbb{B}\mathsf{r}(\mathrm{L})}$, hence $\mathbb{B}\mathsf{r}^{op}(\mathrm{R}) \cong \mathbb{B}\mathsf{r}(\mathrm{L})$. □

Let $\mathbb{A}$ and $\mathbb{B}$ be two categories. Let $\mathbb{F}\mathrm{UN}(\mathbb{A},\mathbb{B})$ denote the functor category, the morphisms of which are the natural transformations between functors $\mathbb{A} \to \mathbb{B}$, and let $\mathbb{B}\mathrm{R}(\mathbb{A},\mathbb{B})$ denote the ***bridge category***, the morphisms of which are the bridge morphisms between bridges $\mathbb{A} \Rightarrow \mathbb{B}$.

Proposition 1.6. *The mapping* $\mathbb{B}\mathsf{r}\colon Ob\mathbb{F}\textsc{un}(\mathbb{A},\mathbb{B}) \to Ob\mathbb{B}\textsc{r}(\mathbb{A},\mathbb{B})$ *extends to a contravariant full and faithful functor (i.e. one which is bijective between hom-sets), and the mapping* $\mathbb{B}\mathsf{r}^{op}\colon Ob\mathbb{F}\textsc{un}(\mathbb{A},\mathbb{B}) \to Ob\mathbb{B}\textsc{r}(\mathbb{B},\mathbb{A})$ *extends to a covariant full and faithful functor* $\mathbb{F}\textsc{un}(\mathbb{A},\mathbb{B}) \to \mathbb{B}\textsc{r}(\mathbb{B},\mathbb{A})$ *in a natural way.*

Proof. Consider the bridge $\mathbb{B}\textsc{r}(\mathbb{A},\mathbb{B}) \Rightarrow \mathbb{F}\textsc{un}(\mathbb{A},\mathbb{B})^{op}$, in which the diagonals $\underset{\mathbb{A}\Rightarrow\mathbb{B}}{\mathbb{U}} \to \underset{\mathbb{A}\to\mathbb{B}}{\mathrm{F}}$ are those functors $\mathrm{T}\colon\mathbb{U}\to\mathbb{B}$ which satisfy $\mathrm{T}\restriction_{\mathbb{A}}=\mathrm{F}$ and $\mathrm{T}\restriction_{\mathbb{B}}= id_{\mathbb{B}}$. We compose such a $\mathrm{T}\colon\mathbb{U}\to\mathrm{F}$ with a natural transformation $\tau\colon\mathrm{F}\leftarrow\mathrm{G}$ by defining $\mathrm{T}\circ\tau\colon\mathbb{U}\to\mathrm{G}$ as follows:
for $\underset{a\to b}{u} \in \mathtt{Dg}^{\mathbb{U}}$, set $(\mathrm{T}\circ\tau)u := \tau_a\cdot\mathrm{T}u\colon\mathrm{G}a\to b$.

$$\begin{array}{ccc}
\mathbb{U} & & \\
\mathrm{T}\,\vdots & & \\
\mathrm{F} & \xleftarrow[\tau]{} & \mathrm{G}
\end{array}
\quad : \quad
\begin{array}{c} a \\ u\big\downarrow \\ b \end{array}
\;\overset{\mathrm{T}\circ\tau}{\longmapsto}\;
\begin{array}{ccc}
\mathrm{G}a & \xrightarrow{\tau_a} & \mathrm{F}a \\
 & & \big\downarrow \mathrm{T}u \\
 & & b
\end{array}$$

Left compositions are just the functor compositions.

In this bridge each functor F is coreflected by $\mathbb{B}\mathsf{r}(\mathrm{F})$. So, by Corollary 1.4, $\mathbb{B}\mathsf{r}$ extends naturally to a functor $\mathbb{F}\textsc{un}(\mathbb{A},\mathbb{B})^{op}\to\mathbb{B}\textsc{r}(\mathbb{A},\mathbb{B})$. Along this functor each bridge morphism $\mathrm{K}\colon\mathbb{B}\mathsf{r}(\mathrm{F})\to\mathbb{B}\mathsf{r}(\mathrm{G})$ has a unique preimage $\tau\colon\mathrm{G}\to\mathrm{F}$, defined by the equalities $\mathrm{K}\langle a, 1_{\mathrm{F}a}\rangle = \langle a,\tau_a\rangle$ in $\mathbb{B}\mathsf{r}(\mathrm{G})$ [for $a\in Ob\mathbb{A}$] (such a τ_a is indeed from $\mathrm{G}a$ to $\mathrm{F}a$).

We can similarly define another opfunctorial bridge $\mathbb{B}\textsc{r}(\mathbb{A},\mathbb{B})\Rightarrow\mathbb{F}\textsc{un}(\mathbb{B},\mathbb{A})$ attesting that $\mathbb{B}\mathsf{r}^{op}$ extends naturally to a functor, and we can verify that it is bijective on the hom-sets, too. ☐

Remark. We can define the **composition** of bridges $\mathbb{F}\colon\mathbb{A}\Rightarrow\mathbb{B}$ and $\mathbb{G}\colon\mathbb{B}\Rightarrow\mathbb{C}$, in an analogous way as tensor product of bimodules is defined: The diagonals of the composition bridge $\mathbb{F}\cdot\mathbb{G}$ will be presented by the pairs $\langle f,g\rangle$ of *composable* arrows $f\in\mathtt{Dg}^{\mathbb{F}}$, $g\in\mathtt{Dg}^{\mathbb{G}}$ (that is, $cod\,f = dom\,g$), so that $\langle f\beta, g\rangle$ and $\langle f,\beta g\rangle$ represents the same diagonal (whenever $\beta\in Mor\mathbb{B}$, $cod\,f = dom\,\beta$ and $cod\,\beta = dom\,g$). {See also [11, 1].}
Then we obtain a ***bicategory*** $\mathbb{C}\mathsf{at}^{br}$ whose objects are the small categories, whose arrows are the bridges between them, and whose 2-cells are the bridge morphisms.

Note that, by Proposition 1.6, the ordinary 2-category $\mathbb{C}\mathsf{at}$ of categories, functors and natural transformations, is fully embedded in $\mathbb{C}\mathsf{at}^{br}$ in two ways, *both being contravariant at one level.*

Corollary 1.7. *Functors* $\mathrm{F},\mathrm{G}\colon\mathbb{A}\to\mathbb{B}$ *are naturally isomorphic if and only if their bridges* $\mathbb{B}\mathsf{r}(\mathrm{F})$ *and* $\mathbb{B}\mathsf{r}(\mathrm{G})$ *are isomorphic [resp. iff their opfunctorial bridges,* $\mathbb{B}\mathsf{r}^{op}(\mathrm{F})$ *and* $\mathbb{B}\mathsf{r}^{op}(\mathrm{G})$ *are isomorphic].*

In [2] we introduced the notion of an *undirected* (or *symmetric*) *bridge*, and proved Theorem 1.9.

Definition 1.8. *A category $\mathbb{H}$ is an* ***undirected bridge*** *between categories $\mathbb{A}$ and $\mathbb{B}$ iff $\mathbb{A}$ and $\mathbb{B}$ are disjoint full subcategories of $\mathbb{H}$ and $Ob\mathbb{H} = Ob\mathbb{A} \cup Ob\mathbb{B}$. (That is, we allow diagonal morphisms in both directions.) We denote it by $\mathbb{H}\colon \mathbb{A} \rightleftharpoons \mathbb{B}$.*

Clearly, any directed bridge is automatically undirected, and any undirected bridge $\mathbb{H}\colon \mathbb{A} \rightleftharpoons \mathbb{B}$ determines two directed bridges (the *parts* of $\mathbb{H}$), one from $\mathbb{A}$ to $\mathbb{B}$ and one from $\mathbb{B}$ to $\mathbb{A}$, by forgetting the arrows in the other direction.

Theorem 1.9. *Categories $\mathbb{A}$ and $\mathbb{B}$ are equivalent if and only if there is an undirected bridge $\mathbb{H}\colon \mathbb{A} \rightleftharpoons \mathbb{B}$, in which every $a \in Ob\mathbb{A}$ is isomorphic to some $b \in Ob\mathbb{B}$ and every $b \in Ob\mathbb{B}$ is isomorphic to some $a \in Ob\mathbb{A}$.*

Note that constructing an equivalence functor from such an undirected bridge required the *axiom of choice*.

Categories $\mathbb{A}$ and $\mathbb{B}$ are called **Morita equivalent** iff there is an invertible bridge between them.
If $\mathbb{H}\colon \mathbb{A} \rightleftharpoons \mathbb{B}$ is an undirected bridge of a category equivalence, then its parts are inverses of each other with respect to bridge composition. So, equivalent categories are also Morita equivalent.

However, there are invertible bridges which does not determine a category equivalence. But, it is shown in [11] that categories $\mathbb{A}$ and $\mathbb{B}$ are Morita equivalent if and only if their *Cauchy completions* are equivalent.

2. Logic

In this section we introduce an abstract language which is a common generalization of most of the familiar logics with semantics. When a mathematician works on a theorem, he usually does not formulate the statements explicitly in a regular first order language, but writes and draws a 'situation' that he is given and a 'situation' that he is to prove.

Definition 2.1. *According to papers* [7, 8]*, by an* ***abstract language (with semantics)*** *we mean a triple $\langle \mathcal{M}, \models, \mathcal{F} \rangle$ where $\mathcal{M}$ and $\mathcal{F}$ are classes, and $\models \subseteq \mathcal{M} \times \mathcal{F}$ is a binary relation. Elements of $\mathcal{M}$ are called* ***models****, those of $\mathcal{F}$ are called* ***formulas****, and $\models$ is the relation of* ***validity****.*

Abstractly, any bridge $\mathbb{L}\colon \mathbb{S} \Rightarrow \mathbb{M}$ determines such an abstract language, $\mathtt{Lang}(\mathbb{L}) := \langle \mathcal{M}, \models, \mathcal{F} \rangle$, in the following way.

The **models** will be just the objects of the target category $\mathbb{M}$. That is,

$$\mathcal{M} := Ob\mathbb{M} \ .$$

The source category $\mathbb{S}$ is considered as the category of **situations**, and a diagonal morphism $S \to m$ is considered as an **interpretation** of the situation S in the model m. Note that we require compositions of interpretations with morphisms of $\mathbb{M}$ (from the right). By Proposition 2.3, we can ***think about*** $\mathbb{S}$ as a partial ordering, in which, the existence of a morphism $A \to B$ expresses that the situation B 'contains' the situation A (i.e. we are given more data in B). Then, we can think about the composition of an interpretation $u\colon B \to m$ with $A \to B$ (from the left) as a ***restriction*** of u to the smaller situation A.

The **formulas** will be the trees in $\mathbb{S}$ ***without infinite paths***.

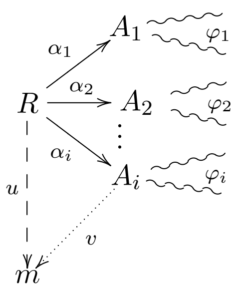
Precisely, by an **abstract tree** we mean a simple directed graph which has a vertex r (called the **root**), such that for each vertex x there is a ***unique*** directed path from r to x. (Note that cycles are excluded.)
Then, a **tree** in a category $\mathbb{A}$ is just a graph morphism $F \to \mathbb{A}$ from an abstract tree F to the underlying directed graph of $\mathbb{A}$.

$$\mathcal{F} := \mathsf{Fa}(\mathbb{S}) := \{\varphi\colon F \to \mathbb{S} \ \mid \ F \text{ is an abstract tree without an inifinite path}\}$$

We consequently use small letters $r, a, b, c \dots$ for the vertices of an abstract tree F (usually r for the root), and the corresponding capital letters $R, A, B, C \dots$ for their realizations in $\mathbb{S}$ (So that $R = \varphi(r),\ A = \varphi(a)$, etc.).

Validity is defined via game theory.
For any tree $\varphi \in \mathsf{Fa}(\mathbb{S})$ and any model $m \in Ob\mathbb{M}$ we define a game played by two players: ADAM and EVE. (Some authors call them *Abelard* and *Eloise*.)

ADAM ***starts*** the game: he has to choose a diagonal $\underset{R\to m}{u}$ from the root of φ to m.
Then EVE ***continues*** by picking an edge $r \to a_i$ of F (which is mapped, say, to $\alpha_i \in Mor\mathbb{S}$ by φ), and a diagonal $\underset{A_i\to m}{v}$ which makes $u = \alpha_i \cdot v$.



Then the match is going on in the full subtree φ_i (whose root is A_i). It is ADAM's turn to ***continue*** by choosing an edge $a_i \to b_{ij}$ of F (mapped, say, to β_{ij} by φ) and a diagonal $\underset{B_{ij}\to m}{w}$ satisfying $v = \beta_{ij}\cdot w$, and so on...

The player who cannot make any more moves, ***loses*** the game (and then the other one ***wins***). We define

$$m \models \varphi \overset{def}{\Leftrightarrow} \text{EVE has a } \textit{winning strategy}.$$

Note that a ***simple tree*** $\varphi: A \to B \in \mathsf{Fa}(\mathbb{S})$ (i.e. one which is based on the abstract tree with two vertices) is ***valid*** in a model m if and only if m is ***injective*** with respect to φ in $\mathbb{L}$, that is, for each arrow $u: A \to m$ there is a $v: B \to m$ such that $\varphi\cdot v = u$.

Examples.

In the language of *categories*, a situation is just an ***abstract diagram***, that is a directed graph endowed with a set of parallel paths (called ***commutativity conditions***). If $\mathbb{S}$ is the partial ordering of abstract diagrams ordered by inclusion, then the corresponding language is invariant under category equivalence (cf. [9]).

A *first order* situation [of similarity type t] is just a ***presentation***, that is a pair $\langle X, \Gamma\rangle$, where X is a set (of ***variables***) and Γ is a set of ***atomic formulas*** over X. An interpretation of $\langle X, \Gamma\rangle$ in a first order model $\mathcal{M}$ of similarity type t is such an ***evaluation*** $v: X \to \mathcal{M}$ which makes Γ true. It is important that Γ contains only *atomic* formulas, because we need to compose interpretations with *homomorphisms*, and homomorphisms are exactly those mappings which preserve all atomic formulas at a fixed evaluation.
Note that the finite trees of finite first order situations give back the original first order language. Cf. [2, 1].

These situations can also be used for the language of *partial algebras.* Note that, in this case, an interpretation of an equation '$\tau = \sigma$' in a partial algebra $\mathcal{A}$ requires the existence of the evaluated terms τ and σ in $\mathcal{A}$.

Let $\mathtt{L} = \langle \mathcal{M}, \models, \mathcal{F}\rangle$ and $\mathtt{L}' = \langle \mathcal{M}, \hat{\models}, \mathcal{F}'\rangle$ be languages with the same class of models. For formulas $f \in \mathcal{F}$ and $f' \in \mathcal{F}'$ we introduce the relation $\approx \subseteq \mathcal{F}\times\mathcal{F}'$ of ***equivalence***:

$$f \approx f' \overset{def}{\Leftrightarrow} (\forall m \in \mathcal{M}) \Big(m \models f \Leftrightarrow m \hat{\models} f'\Big)$$

Definition 2.2. *Languages* $\mathtt{L}$ *and* $\mathtt{L}'$ *are said to be* ***equivalent*** *(in notation* $\mathtt{L} \cong \mathtt{L}'$*), iff for each* $f \in \mathcal{F}$ *there is an* $f' \in \mathcal{F}'$ *such that* $f \approx f'$ *and vice-versa: each* $f' \in \mathcal{F}'$ *is equivalent to some* $f \in \mathcal{F}$.

The next theorems state that the category of *situations* can be specialized.

Theorem 2.3. *Let* $\mathbb{L}:\mathbb{S}\Rightarrow\mathbb{M}$ *be a bridge. Then there exists a partial ordered class* $\mathbb{S}'$ *and a bridge* $\mathbb{L}':\mathbb{S}'\Rightarrow\mathbb{M}$ *such that*

$$\mathtt{Lang}(\mathbb{L})\cong\mathtt{Lang}(\mathbb{L}')\ .$$

Moreover, if $\mathbb{L}$ *was functorial, then* $\mathbb{L}'$ *can be chosen to be functorial, too.*

Proof. Let $Ob\mathbb{S}'$ be the class of all finite chains of composable arrows of $\mathbb{S}$. We define a path to precede another one iff the former is an initial segment of the latter. I.e.,

$$Ob\mathbb{S}':=\{\langle\alpha_1,\alpha_2,\dots,\alpha_n\rangle\mid n\in\omega,\ \mathit{cod}\,\alpha_i=\mathit{dom}\,\alpha_{i+1}\}\quad\text{and}$$

$$\langle\alpha_1,\dots,\alpha_n\rangle\leqslant\langle\beta_1,\dots,\beta_m\rangle\ \overset{def}{\Leftrightarrow}\ n\leqslant m\text{ and for }i\leqslant n:\ \alpha_i=\beta_i.$$

We set $(\langle\alpha_1,\dots,\alpha_n\rangle\ \shortmid\ m)_{\mathbb{L}'}:=(\mathit{cod}\,\alpha_n\shortmid m)_{\mathbb{L}}$, that is, the $\mathbb{L}'$-diagonals from a chain will be exactly the $\mathbb{L}$-diagonals from the ending of the chain. This $\mathbb{L}'$ is as required. □

Theorem 2.4. *For any bridge* $\mathbb{L}:\mathbb{S}\Rightarrow\mathbb{M}$, *there exists a subcategory* $\mathbb{S}'\leqslant\left(\mathbb{S}\mathsf{et}^{\mathbb{M}}\right)^{op}$ *and a bridge* $\mathbb{L}':\mathbb{S}'\Rightarrow\mathbb{M}$ *such that*

$$\mathtt{Lang}(\mathbb{L})\cong\mathtt{Lang}(\mathbb{L}')\ .$$

Proof. The bridge $\mathbb{L}$ determines (and is determined by) a functor $\mathbb{S}^{op}\to\mathbb{S}\mathsf{et}^{\mathbb{M}}$. Let $\mathbb{S}'$ be its image, and let $\mathbb{L}'$ be the restriction of the bridge $\left(\mathbb{S}\mathsf{et}^{\mathbb{M}}\right)^{op}\Rightarrow\mathbb{M}$, in which a diagonal morphism $\mathrm{F}\to m$ is just an element of the set $\mathrm{F}m$. □

Recall that a tree is **simple** iff it is based on the abstract tree with two vertices. The following theorem is about that 'any mathematical sentence' can be written in the form "$\forall A\ \exists B$" for some situations A and B.

Theorem 2.5. *Let* $\mathbb{L}:\mathbb{D}\Rightarrow\mathbb{M}$ *be a bridge. Then there exists another bridge* $\mathbb{L}':\mathbb{D}'\Rightarrow Ob\mathbb{M}$ *to the discrete category of objects of* $\mathbb{M}$, *for which every tree of* $\mathbb{L}$ *is equivalent to a* ***simple tree*** *of* $\mathbb{L}'$, *yet* $\mathtt{Lang}(\mathbb{L})\cong\mathtt{Lang}(\mathbb{L}')$.

Proof. The idea of the proof is to introduce new, `derived situations` by the original formulas of $\mathbb{L}$. For details, see [1]. □

3. Preservation theorems

In [3] a general Birkhoff type theorem is proved. The operations **H** (of taking `homomorphic images`), **S** (of taking `subalgebras`) and **P** (of taking `products`) are replaced there by more general operations on the class of models, depending on certain varying classes of morphisms. Let $\mathbb{M}$ denote an arbitrary category. For instance, instead of **H** and **S** we consider the operations

$$\overleftarrow{\mathsf{Q}}K := \{a \in Ob\mathbb{M} \mid \exists \underset{a\to k}{\sigma} \in \mathsf{Q}: \; k \in K\} \quad (K \subseteq Ob\mathbb{M})$$

$$\overrightarrow{\mathsf{Q}}K := \{b \in Ob\mathbb{M} \mid \exists \underset{k\to b}{\sigma} \in \mathsf{Q}: \; k \in K\} \quad (K \subseteq Ob\mathbb{M})$$

for any class $\mathsf{Q} \subseteq Mor\mathbb{M}$ of morphisms. In particular, if $\mathbb{M}$ is a variety, and Q is the class of all surjective homomorphisms, then $\overrightarrow{\mathsf{Q}}$ is just **H**, while, if Q is the class of all embeddings, then $\overleftarrow{\mathsf{Q}}$ is just **S**.

Now, let us fix an arbitrary bridge $\mathbb{L}: \mathbb{S} \Rightarrow \mathbb{M}$ with an initial object $O \in Ob\mathbb{S}$, for the rest of the paper.
We define a binary relation $\mathtt{T} \subseteq Mor\mathbb{S} \times Mor\mathbb{M}$ as follows:
$\alpha \;\mathtt{T}\; \beta$, iff all commutative squares of the form $\alpha v = u\beta \quad (u, v \in \mathtt{Dg}^{\mathbb{L}})$ has the "***diagonal fill-in property***", i.e. there exists a ***diagonal fill-in*** $d \in \mathtt{Dg}^{\mathbb{L}}$ such that $\alpha d = u$ and $d\beta = v$.

$$\alpha \;\mathtt{T}\; \beta \;\overset{def}{\Leftrightarrow}\; \forall \quad \begin{array}{ccc} \cdot & \xrightarrow{\alpha} & \cdot \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \cdot & \xrightarrow[\beta]{} & \cdot \end{array} \quad \exists \quad \begin{array}{ccc} \cdot & \xrightarrow{\alpha} & \cdot \\ {\scriptstyle u}\downarrow & \swarrow{\scriptstyle d} & \downarrow{\scriptstyle v} \\ \cdot & \xrightarrow[\beta]{} & \cdot \end{array}$$

Let $\curlyvee$ and $\curlywedge$ denote the operations belonging to the Galois connection induced by $\mathtt{T}$, that is, for a subclass Q of morphisms of $\mathbb{M}$,
$\curlyvee(\mathsf{Q}) := \{\alpha \in \mathbb{S} \mid \forall \sigma \in \mathsf{Q} : \alpha \;\mathtt{T}\; \sigma\}$, and similarly,
$\curlywedge(\mathsf{R}) := \{\beta \in \mathbb{M} \mid \forall \rho \in \mathsf{R} : \rho \;\mathtt{T}\; \beta\}$ for a subclass $\mathsf{R} \subseteq Mor\mathbb{S}$.

An edge of an abstract tree $\mathcal{F}$ is due to Adam or Eve according to which player may choose it once. That is, edges on odd level (e.g. those that start at the root) are due to Eve and the rest to Adam. By a ***branch of*** Eve in a tree $\varphi: \mathcal{F} \to \mathbb{S}$ we mean the image $\varphi(e)$ of an edge e of Eve in $\mathcal{F}$.
Similarly, a vertex z of $\mathcal{F}$ belongs to Adam [resp. to Eve], iff the unique path $r \rightsquigarrow z$ contains an odd [resp. even] number of vertices.

For a tree $\varphi: \mathcal{F} \to \mathbb{S}$, let $O \to \varphi$ denote the tree obtained by placing one more vertex and one more edge in front of $\mathcal{F}$, having the vertex mapped to the initial object O in $\mathbb{S}$. We note that $O \to \varphi$ is just the ***negation*** of φ, since in the corresponding games the roles of the players are simply exchanged.

Definition 3.1. *Let $\mathsf{Q} \subseteq Mod\mathbb{M}$ be any class of morphisms in the model category. We will call a tree φ ($\in \mathsf{Fa}(\mathbb{S})$) a* $\overleftarrow{\mathsf{Q}}$***-tree*** *or* $\overleftarrow{\mathsf{Q}}$***-formula****, iff each branch of* Eve *is in $\curlyvee(\mathsf{Q})$. We will call φ a* $\overrightarrow{\mathsf{Q}}$***-tree****, iff $O \to \varphi$ is a $\overleftarrow{\mathsf{Q}}$-tree [i.e.* Adam*'s branches are all in $\curlyvee(\mathsf{Q})$, including the 0^{th} branch $O \to R$].*

We say that a class $K \subseteq Ob\mathbb{M}$ of models is **axiomatized** by a class $\mathcal{F} \subseteq \mathsf{Fa}(\mathbb{D})$ of trees iff

$$K = \{m \in Ob\mathbb{M} \mid \forall\varphi \in \mathcal{F} : m \models \varphi\} \ .$$

Theorem 3.2. *Let $\mathsf{Q} \subseteq Mor\mathbb{M}$ be an arbitrary class of morphisms. If a class $K \subseteq Ob\mathbb{M}$ is axiomatizable by $\overleftarrow{\mathsf{Q}}$-formulas, then $\overleftarrow{\mathsf{Q}}K \subseteq K$.*
Similarly, if K is axiomatizable by $\overrightarrow{\mathsf{Q}}$-formulas, then $\overrightarrow{\mathsf{Q}}K \subseteq K$.
If, in addition, all identities belong to Q then the conclusions hold with equality ($\overleftarrow{\mathsf{Q}}K = K$ and $\overrightarrow{\mathsf{Q}}K = K$, resp.).

The proof can be found in [1].

Applying this theorem to an algebraic first order language (i.e. which has only operation symbols), yields to the familiar propositions that a class of algebras is closed under **S** [resp. **H**], if it is definable by universal [resp. positive] formulas.

Definition 3.3. *We call $\varphi\colon F \to \mathbb{S}$ a **P-tree**, iff there is no branching at* ADAM*'s vertices in F, i.e. each vertex of* ADAM *has* `at most one` *successor.*

Note that branching at a vertex of ADAM corresponds to disjunction [for EVE has a choice there], so **P**-trees represent essentially the *formulas without disjunction.*

Let $\mathrm{H}\colon \mathbb{D} \to \mathbb{M}$ be a functor, which has a limit $\overleftarrow{\mathsf{lim}}\mathrm{H}$ in $\mathbb{M}$. We say that the limit of H **works** in the bridge $\mathbb{L}\colon \mathbb{S} \Rightarrow \mathbb{M}$, iff $\mathrm{J}(\overleftarrow{\mathsf{lim}}\mathrm{H}) = \overleftarrow{\mathsf{lim}}(\mathrm{J}\circ\mathrm{H})$, where $\mathrm{J}\colon \mathbb{M} \to \mathbb{L}$ is the embedding functor.

Lemma 3.4. *If the bridge $\mathbb{L}\colon \mathbb{S} \Rightarrow \mathbb{M}$ is functorial, then every limit in $\mathbb{M}$ works in $\mathbb{L}$.*

Theorem 3.5. *Suppose that $\mathbb{M}$ has products, and every product of $\mathbb{M}$* `works` *in $\mathbb{L}$. Then, if a class $K \subseteq Ob\mathbb{M}$ of models is axiomatizable by **P**-trees then $\mathbf{P}K = K$.*

Note that instead of the operation **P** of taking products, papers [3, 4, 1] deal with reduced products with respect to $\mathcal{F} \in \mathcal{H}$ for an arbitrary but fixed class $\mathcal{H}$ of filters. Then, choosing $\mathcal{H}$ to be the class of all one elemented filters give back **P**, while $\mathcal{H} :=$ {ultrafilters} gives the operation **Up** of taking ultraproducts. In particular, as a consequence of a suitable more general version of Theorem 3.5, we can obtain a general *Łoś lemma* (cf. [1, 6]).

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(B. Pécsi) Department of Algebra and Number Theory, ELTE University, Budapest

E-mail address: aladar@renyi.hu

URL: http://renyi.hu/~aladar