

Rainbow Ramsey Theorems for Colorings Establishing Negative Partition Relations

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Abstract

Given a function f a subset of its domain is a rainbow subset for f if f is one to one on it. We start with an old Erdős Problem: Assume f is a coloring of the pairs of ω_1 with three colors such that every subset A of ω_1 of size ω_1 contains a pair of each color. Does there exist a rainbow triangle? We investigate rainbow problems and results of this style for colorings of pairs establishing negative “square bracket” relations.

1 Introduction and History

Anti-Ramsey theorems appeared, probably, for the first time in a paper of Richard Rado in [9], in 1973, claiming the existence of subsets with elements of different colors of the domain of a given coloring. Later in the game, the more expressive name of a rainbow subset was coined. In this paper we will mostly consider 2-partitions i.e. colorings f of unordered pairs of a set. A subset of pairs will be called a *rainbow subset (for f)* if f is one-to-one on it. Our starting point will be a problem of Paul Erdős that was stated long before any of this names were coined:

Assume $f : [\omega_1]^2 \rightarrow 3$ is a 2-partition of ω_1 with three colors such that each subset $A \subseteq \omega_1$ of size ω_1 contains a pair of each color. Does there exist a

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rainbow triangle for f ?

This is Problem 68 of [3] written in 1967. We restate it in the jargon of partition relations developed in [5]:

Problem 1.1. *Assume $f : [\omega_1]^2 \rightarrow 3$ establishes $\omega_1 \not\rightarrow [\omega_1]_3^2$. Does there exist a rainbow triangle for f ?*

We knew that the answer is affirmative under some stronger conditions e.g.

Fact 1.2. *Assume $f : [\omega_1]^2 \rightarrow 3$ establishes $\omega_1 \not\rightarrow [(\omega, \omega_1)]_3^2$. (i.e. For $A \in [\omega_1]^\omega, B \in [\omega_1]^{\omega_1}$ f takes all three values on $[A, B]^{1,1}$). Then there exists a rainbow triangle for f .*

However, in those early days, we could only construct an f satisfying the condition of 1.2 using CH .

Definition 1.3. *For a coloring $d : [k]^2 \rightarrow \omega_1$, $k \leq \omega$ we write $d \Rightarrow f$ if there is a one-to-one map $\Phi : k \rightarrow \omega_1$ such that*

$$d(\{n, m\}) = f(\{\Phi(n), \Phi(m)\})$$

for $n, m \in k$.

We could generalize 1.2 to

Fact 1.4. *Assume $f : [\omega_1]^2 \rightarrow \omega_1$ establishes $\omega_1 \not\rightarrow [(\omega, \omega_1)]_{\omega_1}^2$. Then $d \Rightarrow f$ for an arbitrary $d : [\omega]^2 \rightarrow \omega_1$*

As we already mentioned we could not verify in *ZFC* that this does not hold vacuously and it bothered us that we could not lift it e.g. replacing ω, ω_1 by ω_1, ω_2 respectively. The next steps were taken in a paper of Shelah [10] written in 1975. He proved

Theorem 1.5. *Shelah [10]*

1. *CH implies that 1.1 fails for some f with ω colors*
2. *\diamond implies that 1.1 fails for an f with ω_1 colors*

Shelah also showed in [10] that a possible “lifting” of Fact 1.4 is consistently false say adding one Cohen-real to a model of GCH . In more detail, he constructed a graph of size ω_1 from the Cohen real which does not embed isomorphically into any graph of the ground model. Then any graph of the ground model establishing the partition relation $\omega_2 \not\rightarrow [(\omega_2, \omega_1)]_{\omega_1}^2$ satisfies the same relation in the new model, and we have a graph of size ω_1 that does not embed to it.

Knowing all this, in our 1978 paper [2] we stated implicitly a generalization of 1.4.

Theorem 1.6. [2] *Assume that f establishes $\omega_1 \not\rightarrow [(\omega_1; \omega_1)]_{\omega}^2$. Then $d \Rightarrow f$ for an arbitrary $d : [\omega]^2 \rightarrow \omega$*

The symbol with the semi-colon “;” means that all ω_1 by ω_1 “half-graph”s are totally multicolored i.e. for every $A, B \subseteq \omega_1, |A| = |B| = \omega_1$ and $n < \omega$ there are $\alpha \in A, \beta \in B, \alpha < \beta$ such that $f(\{\alpha, \beta\}) = n$. I want to mention that [2] seems to be the first paper in print where this important concept was used. I think it was invented (discovered) by Fred Galvin. It was proved 37 years later by Justin Moore that

Theorem 1.7. [7] *Moore (ZFC) There is an f establishing $\omega_1 \not\rightarrow [(\omega_1; \omega_1)]_{\omega_1}^2$.*

This is a byproduct of Moore’s result [7] showing the existence of L -spaces in ZFC. All that said above justifies revisiting the old Problem 1.1.

2 $\not\Rightarrow$ relations

First we remark that we still do not know if the hypotheses of either clauses of Theorem 1.5 can be proved under weaker conditions. Next we want to show that a Theorem 1.7 type generalization can not hold if we only assume that each $[A]^2$ with $|A| = \omega_1$ is totally multicolored.

Theorem 2.1. *There exist a rainbow $d : [4]^2 \rightarrow 6$ and an $f : [\omega_1]^2 \rightarrow 6$ establishing $\omega_1 \not\rightarrow [\omega_1]_6^2$ such that*

$$d \not\Rightarrow f.$$

Proof: First we define $e : [4]^2 \rightarrow W$ and $g : [\omega_1]^2 \rightarrow W$ where

$$W = \{(+, +), (+, -), (-, +), (-, -)\}.$$

Let $e(\{0, 1\}) = (+, -)$, $e(\{1, 2\}) = (-, +)$, $e(\{2, 3\}) = (+, -)$,
 $e(\{0, 3\}) = (-, +)$, $e(\{0, 2\}) = (+, +)$, $e(\{1, 3\}) = (-, -)$.

Let $<_R$ and $<_A$ be real and Aronszajn type orderings of ω_1 . For $\alpha < \beta < \omega_1$ let $g(\alpha, \beta) = (u, v)$ with $u, v \in \{+, -\}$, where $u = +$ iff $\alpha <_A \beta$ and $v = +$ iff $\alpha <_R \beta$.

It is a well known property of these orderings that for all $B \in [\omega_1]^{\omega_1}$ there are $C, D, E, F \in [B]^{\omega_1}$ such that $C <_A D$, $C <_R D$, $E <_A F$ and $F <_R E$. This implies that each $B \in [\omega_1]^{\omega_1}$ contains a complete ω_1 by ω_1 halfgraph for g in each of the colors in W .

It is an easy exercise to see that $e \not\Rightarrow g$ holds. Let now h establish Moore's Theorem 1.7. Then $k = (g, h)$ establishes $\omega_1 \not\rightarrow [\omega_1]_{\omega_1}^2$. Using k and e it is a matter of easy calculation to get f and d satisfying the theorem. \square

Next we are going to investigate the cases when f establishes

$$\omega_1 \not\rightarrow [(\omega_1, \omega_1)]_{\gamma}^2,$$

i.e. all ω_1 by ω_1 subgraphs are totally multicolored for some γ .

Fact 2.2. *Assume f establishes $\omega_1 \not\rightarrow [(\omega_1, \omega_1)]_{\gamma}^2$. Let $d : [3]^2 \rightarrow 3$ be one-to-one. Then $d \Rightarrow f$, i.e. all possible rainbow triangles exist.*

Proof: The assumption implies that for some $\alpha \in \omega_1$ both sets

$$\{\beta \in \omega_1 : f(\alpha, \beta) = d(0, 1)\}, \{\gamma \in \omega_1 : f(\alpha, \gamma) = d(0, 2)\}$$

are of cardinality ω_1 . \square

Fact 2.3. *There exist a rainbow $d : [5]^2 \rightarrow 10$ and an $f : [\omega_1]^2 \rightarrow 10$ establishing $\omega_1 \not\rightarrow [(\omega_1, \omega_1)]_{10}^2$ such that*

$$d \not\Rightarrow f$$

Proof (in outline): Define $e : [5]^2 \rightarrow 2$ with the stipulation

$$e(\{i, j\}) = 0 \text{ for } i < 5 \text{ and } j \cong i + 1 \pmod{5}.$$

That is, e is a ‘‘pentagon without a diagonal’’. Let $d : [5]^2 \rightarrow 10$ be one-to-one such that $d(\{i, i + 1\}) < 5$ iff $e(\{i, i + 1\}) = 0$. Let $<_R$ be a real type ordering of ω_1 . Let $g(\alpha, \beta) : [\omega_1]^2 \rightarrow 2$ be the ‘‘Sierpinski’’ partition. That is $g(\alpha, \beta) = 0$ iff $\alpha <_R \beta$ for $\alpha < \beta < \omega_1$. It is well known, that every complete bipartite ω_1 by ω_1 contains a complete bipartite ω_1 by ω_1 halfgraph in both colors for g . Again by Moore's theorem, we can take an h establishing $\omega_1 \not\rightarrow [(\omega_1; \omega_1)]_5^2$. Set $f \cong g \cdot 5 + h$. Then f establishes $\omega_1 \not\rightarrow [(\omega_1, \omega_1)]_{10}^2$ and $d \Rightarrow f$ would imply $e \Rightarrow g$, that is known to be false. \square

Problem 2.4. Can we improve 2.3 to have in it a $d : [4]^2 \rightarrow 6$ and an f establishing $\omega_1 \not\rightarrow [(\omega_1, \omega_1)]_6^2$?

3 Rainbow Theorems

Theorem 3.1. Assume $f : [\omega_1]^2 \rightarrow \omega$ establishes $\omega_1 \not\rightarrow [(\omega_1, \omega_1)]_\omega^2$. Then there exists an infinite rainbow set.

Proof: Assume f establishes $\omega_1 \not\rightarrow [(\omega_1, \omega_1)]_\omega^2$. We use A, B, C, \dots to denote subsets of ω_1 of size ω_1 , N, M, \dots to denote infinite subsets of ω , and we set

$$f_j(x) = \{y \in \omega_1 : f(x, y) = f(\{x, y\}) = j\}$$

for $j < \omega$ and $x \in \omega_1$.

3.1.1. Assume $B \cap C = \emptyset$ and

$$\forall n \in M \forall x \in B (|f_n(x) \cap C| \leq \omega).$$

Then $\forall n \in M \forall C' \subseteq C \exists y \in C' (|f_n(y) \cap B| = \omega_1)$.

Otherwise we could pick, by transfinite induction, a pair (B', C'') omitting the color n . □

Let $(*)(A, N)$ be the following property of A and N :

There are $B, C \subseteq A$ and $M \subseteq N$ such that

$$\forall B' \subseteq B \forall C' \subseteq C \forall m \in M \exists x \in B' (|f_m(x) \cap C'| = \omega_1).$$

When $(*)(A, N)$ holds we denote by

$$B(A, N), C(A, N), M(A, N)$$

suitable sets B, C, M respectively, with $B \cap C = \emptyset$.

3.1.2. Assume that for some A_0, N_0 , $(*)(A, N)$ holds for all $A \subseteq A_0$ and $N \subseteq N_0$. Then there is an infinite rainbow subset.

Proof: Define A_k, B_k, N_k by induction on $k < \omega$. Assume A_k, N_k are defined. Let $B_k = B(A_k, N_k)$, $A_{k+1} = C(A_k, N_k)$, $N_{k+1} = N(A_k, N_k)$. Let $\{N'_k : k < \omega\}$ be a disjoint refinement of $\{N_k : k < \omega\}$ and let

$$N'_k = \{n_i^k : i < \omega\}$$

be a one-to-one enumeration of N_k' for $k < \omega$. It is now easy to pick $x_i \in A_i$ for $i < \omega$ in such a way that $c(x_i, x_j) = n_j^i$ for $i < j < \omega$. This proves 3.1.2, as $\{x_i : i < \omega\}$ is an infinite rainbow set. \square

Hence to finish the proof of Theorem 3.1 it is sufficient to prove

3.1.3. *Assume $(*)(A, N)$ is false for some A and N . Then A has an infinite rainbow subset.*

Let $N = \bigcup_{k < \omega} N_k$, $A = \bigcup_{k < \omega} A_k$ be disjoint partitions. To prove 3.1.3 we first prove

3.1.4. *There are $x \in A_0$ and $\{n_i \in N_0 : 1 \leq i < \omega\}$ one to one such that*

$$|f_{n_i}(x) \cap A_i| = \omega_1$$

for $1 \leq i < \omega$.

For an $x \in A_0$ we try to choose $n_i : 1 \leq i < \omega$ by induction on i . Assume we chose $n_k : 1 \leq k \leq i$ with $|f_{n_k}(x) \cap A_k| = \omega_1$. If there is always an n such that

$$(+) \quad n \in N_0 \setminus \{n_k : 1 \leq k \leq i\} \text{ and } |f_n(x) \cap A_{i+1}| = \omega_1$$

we can choose n_{i+1} the smallest of these and 3.1.3 is true. If not, let $i(x)$ be the smallest i for which (+) fails. If (+) fails for all $x \in A_0$ then for some $1 \leq i < \omega$ and $M = N_0 \setminus \{n_k : 1 \leq k \leq i\}$

$$C = \{x \in A_0 : i(x) = i\}$$

has cardinality ω_1 .

Choosing $B = A_{i+1}$ we get that

$$|f_n(x) \cap B| \leq \omega$$

for $n \in M$ and $x \in C$. But then, by 3.1.2 for all $n \in M$ there is $x \in B$ with $|f_n(x) \cap C| = \omega_1$, a contradiction to the assumption that $(*)(A, N)$ is false. This shows 3.1.4. To finish the proof of 3.1.3 and Theorem 3.1, we can use 3.1.4 inductively. \square

Here is a problem that has not been looked at very thoroughly:

Problem 3.2. *Under the conditions of 3.1, is there a rainbow set containing all the colors?*

Theorem 3.3. For every $1 < k < \omega$ there is an $n \in \omega$, $\binom{k}{2} \leq n$ such that every f satisfying $\omega_1 \not\rightarrow [(\omega_1, \omega_1)]_n^2$ has a rainbow set of size k .

Proof: We prove the following statement by induction on $2 \leq k < \omega$. There is an $n < \omega$ such that if $Dom(f) \subseteq [\omega_1]^2$ satisfies $\omega_1 \not\rightarrow [(\omega_1, \omega_1)]_n^2$ (note that this means that for all $A, B \subseteq \omega_1$ with $|A| = |B| = \omega_1$ and for all $i < n$ there are $\alpha \in A$, $\beta \in B$, $\{\alpha, \beta\} \in Dom(f)$ such that $f(\{\alpha, \beta\}) = i$) and $\{A_i : i < n\}$ are pairwise disjoint subsets of ω_1 of size ω_1 , then there is a rainbow partial transversal P , ($[P]^2 \subseteq Dom(f)$) of size k for these sets. Just like in the proof of 3.1 put

$$f_j(x) = \{y \in \omega_1 : f(x, y) = f(\{x, y\}) = j\}$$

for $j < \omega$ and $x \in \omega_1$. Assume n is good for k and $A_0 \cdots A_{2n-1}$ are pairwise disjoint subsets of ω_1 of size ω_1 with union A .

Let (*) denote the following statement:

There are x, i_x, N_x, φ_x such that $x \in A_{i_x}$, $N_x \subseteq 2n \setminus \{i_x\}$, $\varphi_x : N_x \rightarrow 2n$ is one-to-one,

$$|f_{\varphi_x(j)}(x) \cap A_j| = \omega_1$$

for $j \in N_x$, and $|N_x| = n$. If (*) holds for an x then applying the Induction Hypothesis for the sets:

$$f_{\varphi_x(j)}(x) \cap A_j : j \in N_x$$

and for the color set $2n \setminus \varphi[N_x]$ we get a rainbow partial transversal of size k for these sets, and adding x to it we get a rainbow transversal of size $k + 1$ for the sets $A_0 \cdots A_{2n-1}$.

If (*) is false, choosing an N_x of maximal size for $x \in A$ we will have $|N_x| \leq n - 1$ for $x \in A$. By thinning out, we get sets $B_i \subseteq A_i : i < 2n$ of size ω_1 and $N_i, M_i \subseteq 2n : i < 2n$ such that $N_x = N_i$ and $M_i = \varphi_x[N_x]$ for $x \in B_i$ for $i < 2n$.

Then $i \mapsto N_i$ is a set mapping of order at most $n - 1$ on $2n$. By a theorem of deBruijn and Erdős, from 1951, there are $i \neq j$ such that $i \notin N_j$ and $j \notin N_i$. As $|M_i \cup M_j| < 2n$ we can choose $l \notin M_i \cup M_j$. By the maximality of N_i we know that $|f_l(x) \cap B_j| \leq \omega$ for $x \in B_i$ and likewise $|f_l(x) \cap B_i| \leq \omega$ for $x \in B_j$. We then could pick, by an easy transfinite induction, sets $C_i \subseteq B_i$ and $C_j \subseteq B_j$ both of size ω_1 such that the color l is missing from the bipartite (ω_1, ω_1) determined by C_i and C_j . This contradicts the assumption. \square

Corollary 3.4. *In Theorem 3.3, n can be chosen to be 2^{k-2} for $2 \leq k < \omega$.*

Problem 3.5. *Can n be chosen to $\binom{k}{2}$ in Theorem 3.3 ?*

4 Resurrecting the problem for larger cardinals

We explained in section 1 how Shelah's example described in 1.5 forced us to consider problems only for underlying sets of size at most ω_1 . In [2] written in 1978 we tried to ask if we can get every graph of size ω_1 as an induced subgraph provided the graph shows $\omega_2 \not\rightarrow [(\omega_1, \omega)]_{\omega_1}^2$, a stronger assumption that one can only make consistent. Recently Soukup showed that the simple method of adding one Cohen real gives a negative answer as well. Working through the material of this paper I realized that this trick only kills questions of \Rightarrow type. The following is probable the simplest problem I can not solve:

Problem 4.1. *Assume GCH and let f establish*

$$\omega_2 \not\rightarrow [(\omega_1, \omega_2)]_{\omega_1}^2.$$

Does there exist a rainbow subset of size ω_1 for f ?

In fact we do not know a single case, where for some $\kappa > \lambda > \omega$ for some $f : [\kappa]^2 \rightarrow \lambda$ establishes $\kappa \not\rightarrow [(\kappa, \kappa)]_{\lambda}^2$ and for all such f there is an uncountable rainbow set.

5 Finitary Problems

In our paper [4] we considered finitary Ramsey problems and proved in 1989

Theorem 5.1. *([4], Theorem 1.3, Erdős - Hajnal) Assume $2 \leq k, s < \omega$ and $d : [k]^2 \rightarrow s$. Then there are n_0 and a real number $r > 0$ such that for all $f : [n]^2 \rightarrow s$ establishing*

$$n \not\rightarrow [e^{r\sqrt{\log n}}]_s^2$$

$d \Rightarrow f$ holds.

In fact we only wrote down the proof of this result for $s = 2$. Janos Pach kindly communicated to us that he can prove a much stronger result for a great many cases. Most relevant to this paper he can prove:

Theorem 5.2. [6](J.Fox, J.Pach) *There are n_0 and $\varepsilon > 0$ such that for any $n > n_0$ and f establishing*

$$n \not\rightarrow [n^\varepsilon]_3^2$$

there is a rainbow triangle for f .

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