Obligatory subsystems of triple systems

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Abstract

We determine a class of triple systems such that each must occur in a triple system with uncountable chromatic number that omits $\mathcal{T}_0$ (the system consisting of two triples on four vertices). This class contains all odd circuit of length ≥ 7. We also show that consistently there are two finite triple systems such that they can separately be omitted by uncountably chromatic triple systems but not both.

1 Introduction

The first result on finite subgraphs of uncountably chromatic graphs was obtained by P. Erdős and R. Rado, who proved that there exist arbitrarily large chromatic triangle-free graphs ([6]). For years, Erdős wanted to generalize this to show that there are arbitrarily large chromatic graphs which omit short circuits. This would have been the transfinite common generalization of the above mentioned theorem and one of his theorems on finite graphs ([1]). To his great amazement, it turned out to be false; he and A. Hajnal proved ([3]), that every uncountably chromatic graph contains a circuit of length 4, in fact, every finite bipartite graph. They also proved that for every uncountable cardinal κ and natural number r there is a κ-chromatic graph of cardinality κ that omits all circuits $C_3, C_5, \dots, C_{2r+1}$, that is, all circuits of odd length up to and including $2r+1$. Notice that this describes which

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finite graphs must occur in every κ-chromatic graph for any $\kappa > \aleph_0$: these are precisely the bipartite graphs.

Erdős and Hajnal started to generalize these results in [3] by investigating the corresponding problem for r-hypergraphs, that is, systems of r-element sets, for $3 \leq r < \omega$. The case $r = 3$ already gives rise to surprising new phenomena. For example, there is an uncountable chromatic triple system of cardinality c^+ which has no two members intersecting in two elements, yet there is no such system of cardinality $\aleph_1$ ([3], [4]). The long paper [2] was devoted to these problems and here we comment on some problems of that paper.

Problem 10. Give a characterization of the finite triple systems that occur in every triple system with chromatic number $> \aleph_0$.

That paper considers this problem rather hopeless and so do we. Following the suggestions of [2] we will consider two instances stated there. $\mathcal{T}_0$ is the triple system $\{\{x, y, z\}, \{x, y, u\}\}$, that is, the only triple system with two edges on four vertices, C_5 is the pentagon ($\mathcal{T}_8$ in [2]). Here, and in what follows C_n is the triple system with the vertices $\{x_i, y_i : i < n\}$ and the triples

$$\{x_0, x_1, y_0\}, \{x_1, x_2, y_1\}, \ldots, \{x_{n-1}, x_0, y_{n-1}\}.$$

The vertices $x_0, \ldots, x_{n-1}$ are the *inner vertices* of the circuit.

Problem 10 A. Is it true that either $\mathcal{T}_0$ or C_5 occurs in every triple system of chromatic number $> \aleph_0$?

Problem 10 B. Let $\mathcal{S}_0$, $\mathcal{S}_1$ be finite triple systems. Assume that there is an uncountably chromatic triple system that omits $\mathcal{S}_0$ and another one omitting $\mathcal{S}_1$. Does it follow that there is one that omits both?

Notice that Theorem 11.6 of [2] implies, assuming CH and using absoluteness, the existence of an uncountably chromatic triple system omitting C_5, making the first problem a special case of the second. Further, the above mentioned results of Erdős and Hajnal easily imply that the graph analogue of the second problem is true.

If $1 \leq n < \omega$ then $\mathcal{M}(n)$ is the following triple system. The vertices are the distinct points $\{x(i, j), y(i, j), z(i, j) : i, j < n\}$, the edges are those of the form $\{x(i,j), y(i,k), z(j,k)\}$ for some $i,j,k < n$. $\mathcal{M}^+(n)$ is the following system. The vertex set is

$$\{0,1,2\} \times \{x(i,j), y(i,j), z(i,j) : i,j < n\}.$$

The edges are the sets of the form $\{(\varepsilon_0, x(i,j)), (\varepsilon_1, y(i,k)), (\varepsilon_2, z(j,k))\}$ where $i < j < k$ and $\{\varepsilon_0, \varepsilon_1, \varepsilon_2\} = \{0,1,2\}$.

Our Theorems 1 and 2 work toward a characterization of the obligatory finite subsystems $\mathcal{H}$ of those $> \aleph_0$ chromatic triple systems that do not contain $\mathcal{T}_0$. Each $\mathcal{M}(n)$ is such an $\mathcal{H}$, and each such $\mathcal{H}$ occurs as a subsystem of some $\mathcal{M}^+(n)$.

We will prove that each C_{2k+1} is a subset of some $\mathcal{M}(n)$ for $k \geq 3$. This, interesting as it is, barely misses **Problem 10A** and does not solve **Problem 10B** either, since we do not know how to construct large chromatic C_7-less triple systems. To achieve this goal we need Theorem 3 that gives consistently an $\mathcal{M}(n)$ omitted by an uncountably chromatic triple system. This $\mathcal{M}(n)$ and $\mathcal{T}_0$ give a negative answer to **Problem 10B** using Theorem 1.

Though these are all the results we have, before passing on to proofs we want to comment on **Problem 10C.** of [2]. We say that a finite triple system $\mathcal{H}$ is *obligatory* for $\kappa(\geq \omega)$ if it is contained in every triple system of chromatic number greater than κ. The problem asks if every $\mathcal{H}$ obligatory for some $\kappa \geq \omega$ is obligatory for ω. This is a special case of a "Taylor-type" problem for triple systems which in spite of recent spectacular advances for graph problems is totally unsolved. For a tale on these results see eg. [7].

2 Theorems 1 and 2

Theorem 1. *If $\mathcal{H}$ is a triple system with* $\mathrm{Chr}(\mathcal{H}) > \omega$ *and $\mathcal{T}_0 \not\leq \mathcal{H}$ then $\mathcal{M}(n) \leq \mathcal{H}$ holds for every $n < \omega$.*

Proof. Let $\mathcal{H}$ be a triple system with $\mathrm{Chr}(\mathcal{H}) > \omega$, $\mathcal{T}_0 \not\leq \mathcal{H}$. Let V be the set of vertices and $\kappa = |V|$. We can assume that $\mathrm{Chr}(\mathcal{H}') \leq \omega$ holds for every $\mathcal{H}' \leq \mathcal{H}$ with $|\mathcal{H}'| < \kappa$. (Otherwise we replace $\mathcal{H}$ with a uncountably chromatic subsystem of least cardinality.)

Let $N_1 \prec \cdots \prec N_\alpha \prec \cdots$ be a continuous chain of elementary submodels of $H((2^\kappa)^+; \in, \mathcal{H})$ with $|N_\alpha| < \kappa$ for $1 \leq \alpha < \kappa$. Set $V_0 = \emptyset$, and $V_\alpha = N_\alpha \cap V$

for $1 \leq \alpha < \kappa$. Notice that $|V_\alpha| < \kappa$ and V is the continuous, increasing union of the V_α's. Set $W_\alpha = V_{\alpha+1} - V_\alpha$. Then, $|W_\alpha| < \kappa$ holds for every $\alpha < \kappa$, and V is the disjoint union of the W_α's. Define $\mathcal{H}_\alpha = \mathcal{H} \cap [W_\alpha]^3$, then $\mathrm{Chr}(\mathcal{H}_\alpha) \leq \omega$ holds for $\alpha < \kappa$ by our choice of $\mathcal{H}$. The vertex disjoint union of countably chromatic systems, $\bigcup\{\mathcal{H}_\alpha : \alpha < \kappa\}$ is countably chromatic. As $\mathcal{H}$ omits $\mathcal{T}_0$ and the N_α's are elementary submodels, there are no elements $H \in \mathcal{H}$ with $|H \cap V_\alpha| = 2$ for some $\alpha < \kappa$.

Set

$$\mathcal{G}_\alpha = \{\{x, y\} \in [W_\alpha]^2 : \exists z \in V_\alpha, \{x, y, z\} \in \mathcal{H}\}.$$

Again, as $\mathcal{H}$ omits $\mathcal{T}_0$, for every $\{x, y\} \in \mathcal{G}_\alpha$ there is a unique $z(x, y) \in V_\alpha$ with $\{x, y, z(x, y)\} \in \mathcal{H}$.

By our previous remarks the system

$$\bigcup_{\alpha<\kappa} \{\{x, y, z(x, y)\} : \{x, y\} \in \mathcal{G}_\alpha\}$$

is uncountably chromatic and this necessitates that for some $\alpha < \kappa$ the graph $\mathcal{G}_\alpha$ is uncountably chromatic. Fix such an α. By a theorem of Erdős-Hajnal (cf. [3]), $\mathcal{G}_\alpha$ contains a complete bipartite graph $K_{n,n}$ for every $n < \omega$.

We therefore get the vertices $\{x(i), y(i), z(i, j) : i, j < n\}$ such that $\{x(i), y(j), z(i, j)\} \in \mathcal{H}$. Nothing excludes the possibility that some of the verices $\{z(i, j) : i, j < n\}$ coincide. In order to handle this problem we color the edge $\langle i, j\rangle \in n \times n$ by color $z(i, j)$ and observe that this is a coloring of the edges of $K_{n,n}$ in such a way that any two edges with a common endpoint have different colors.

We need therefore the following lemma.

Lemma 1. *For every natural number r there is a natural number N with the following property. If $|S| = |T| = N$ and $f : S \times T \to \omega$ is a function with $f(x, y) \neq f(x, y')$ $(y \neq y')$ and $f(x, y) \neq f(x', y)$ $(x \neq x')$ then there are $A \in [S]^r$, $B \in [T]^r$ such that f assumes different values on $A \times B$.*

Proof. In order to show that $N = r^3$ suffices assume that $|S| = r^3$, $|T| = r$, and $f : S \times T \to \omega$ is as described. For $x \in S$ set

$$P(x) = \{x' \in S : \exists y, y' \in T f(x, y) = f(x', y')\}$$

and notice that $|P(x)| \leq r^2$ (there can be only one element for every pair $\langle y, y'\rangle$). Then, we can select $A \in [S]^r$ with $x' \notin P(x)$ for $x \neq x' \in A$ and then A is as required (with $B = T$). □

Using this lemma we get that for every $n < \omega$ there exist distinct points $\{x(i), y(j), z(i,j) : i, j < n\}$ such that $z(i,j) \in V_\alpha$, $x(i), y(j) \in W_\alpha$ and $\{x(i), y(j), z(i,j)\} \in \mathcal{H}$ $(i, j < n)$. If we set $Z = \{z(i,j) : i, j < n\}$, $X = \{x(i), y(i) : i < n\}$ then the above statement can be written out as a formula $\Phi(Z, X)$. If we show that there exist sets $X_0, \ldots, X_{n-1}$ disjoint from Z and each other such that $\Phi(Z, X_k)$ holds for $k < n$ then we get that $\mathcal{H}$ contains an $\mathcal{M}(n)$.

To show that we can inductively select $X_0, \ldots, X_{n-1}$ it suffices to show that if $|S| \leq 2n^2$ then there is a set U with $\Phi(Z, U)$ and $S \cap U = \emptyset$. If this fails, then, as N_α is an elementary submodel, there is such an $S \in N_\alpha$, therefore $S \subseteq V_\alpha$, so specifically $S \cap X \neq \emptyset$, which is impossible. This contradiction concludes the argument. □

Corollary. *If* $\mathrm{Chr}(\mathcal{H}) > \omega$ *and* $\mathcal{T}_0 \not\leq \mathcal{H}$ *then* $\mathcal{H}$ *contains every circuit of length 4 or* ≥ 6*.*

Proof. It suffices to show that these circuits occur in $\mathcal{M}(n)$ for n sufficiently large. Indeed, for $n \geq 2$ the circuit C_{2n} can be found in $\mathcal{M}(n)$ with the inner vertices

$$x(0,0), y(0,0), x(0,1), y(0,1), \ldots, x(0,n-1), y(0,n-1)$$

and edges

$$\{x(0,0), y(0,0), z(0,0)\}, \ldots, \{x(0,0), y(0,n-1), z(0,n-1)\}$$

Finally, we show that for $2n+1 \geq 7$, the circuit C_{2n+1} can be embedded into $\mathcal{M}(n+1)$. By the above argument C_{2n-2} embeds into $\mathcal{M}(n-1)$ Now replace the first triple $\{x(0,0), y(0,0), z(0,0)\}$ by the following path of length 3, consisting of vertices $x(0,0), y(0,n-1), x(0,n), y(0,0)$ and edges

$$\{x(0,0), y(0,n-1), z(0,n-1)\}, \{x(0,n), y(0,n-1), z(n,n-1)\},$$

$$\{x(0,n), y(0,0), z(n,0)\}.$$

□

Theorem 2. *Assume that the finite triple system* $\mathcal{F}$ *has the property that for some* $\kappa \geq \omega$*, every triple system* $\mathcal{H}$ *with* $\mathrm{Chr}(\mathcal{H}) > \kappa$ *either has* $\mathcal{T}_0 \leq \mathcal{H}$ *or* $\mathcal{F} \leq \mathcal{H}$*. Then* $\mathcal{F} \leq \mathcal{M}^+(n)$ *for some* $n < \omega$*.*

Proof. We first prove that $\mathcal{F}$ is tripartite, i.e., there are disjoint sets A, B, C such that for every $H \in \mathcal{F}$

$$|H \cap A| = |H \cap B| = |H \cap C| = 1$$

holds. We can assume that $\mathcal{F}$ is connected. Let r be the number of edges in $\mathcal{F}$. We define the following triple system $\mathcal{H}$. The vertex set is

$$V = [\exp_{4r+4}(\kappa)^+]^{4r+5}.$$

A triple $\{A, B, C\} \in \mathcal{H}$ for $A = \{a_1, \ldots, a_{4r+5}\}_<$, $B = \{b_1, \ldots, b_{4r+5}\}_<$, $C = \{c_1, \ldots, c_{4r+5}\}_<$, if

$$b_i = a_{i+1} \quad (1 \leq i \leq 4r+3), \quad a_{4r+5} < b_{4r+4} < b_{4r+5},$$

$$c_1 = a_1, \quad c_i = a_{i+2} \ (2 \leq i \leq 4r+2), \quad c_{4r+3} = a_{4r+5}$$

$$c_{4r+4} = b_{4r+4}, \quad c_{4r+5} = b_{4r+5}.$$

We first argue that $\mathcal{H}$ does not contain a $\mathcal{T}_0$. For this, notice that if $\{A, B, C\} \in \mathcal{H}$ then each of A, B, C is covered by the other two. Further, if $\{X, Y\} \subseteq \{A, B, C\}$ then from X, Y we can recover which of A, B, C they are equal to. Namely,

if $\min(X) = \min(Y)$, and $\max(X) < \max(Y)$, then $X = A$ and $Y = C$;

if $\max(X) = \max(Y)$, and $\min(X) > \min(Y)$, then $X = B$ and $Y = C$;

and finally,

if $\min(X) < \min(Y)$ and $\max(X) < \max(Y)$, then $X = A$ and $Y = B$.

These observations show that $\mathcal{T}_0 \not\leq \mathcal{H}$.

By our assumptions $\mathcal{F}$ can be embedded into $\mathcal{H}$. Fix an embedding of $\mathcal{F}$ into $\mathcal{H}$. Let the vertices of that embedding be $\{A_0, \ldots, A_{r-1}\}$. As $\mathcal{F}$ is connected, we can assume that every A_i $(i > 0)$ is joined with some A_j with $j < i$, that is, A_i and A_j are contained in some edge of $\mathcal{H}$.

Let ξ be the middle element of A_0. Let x_i be the index of ξ in A_i $(i < r)$. Clearly, $x_0 = 2r + 3$, and if A_i is joined to A_j with $j < i$ then $x_i = x_j \pm 1$ or $x_i = x_j \pm 2$. This implies that $5 \leq x_i \leq 4r + 1$ for every $i < r$. Finally, if $\{A_i, A_j, A_k\} \in \mathcal{H}$ then $x_j = x_i \pm 1$ and $x_k = x_i \pm 2$ (or for some permutation of i, j, k). This implies that the mapping $A_i \mapsto x_i \bmod 3$ shows that the subhypergraph on $\{A_0, \ldots, A_{r-1}\}$ is tripartite.

We therefore obtained that $\mathcal{F}$ is tripartite. (Note that this part was already proved in [8].)

Consider finally the following triple system. The vertex set is $V = \left[(2^\kappa)^+\right]^2$, and the set of edges is

$$\mathcal{H} = \left\{\{\{x,y\},\{x,z\},\{y,z\}\} : x < y < z < (2^\kappa)^+\right\}.$$

By the Erdős-Rado theorem $\mathrm{Chr}(\mathcal{H}) > \kappa$ and $\mathcal{T}_0 \not\leq \mathcal{H}$. We have, therefore that $\mathcal{F} \leq \mathcal{H}$, in other words, $\mathcal{F}$ is isomorphic to a finite, tripartite subsystem of $\mathcal{H}$.

The latter object can be given as follows. There is a natural number n, a coloring $f : [n]^2 \to 3$ and $\mathcal{F}$ can be embedded into

$$\left\{\{\{i,j\},\{i,k\},\{j,k\}\} : \{i,j,k\} \in [n]^3, f(i,j) = 0, f(i,k) = 1, f(j,k) = 2\right\}$$

which is $\mathcal{M}^+(n)$. □

3 Theorem 3

Theorem 3. *There is a number $n < \omega$ such that it is consistent that there is a triple system $\mathcal{H}$ with $\mathrm{Chr}(\mathcal{H}) = |\mathcal{H}| = \aleph_1$ and $\mathcal{M}(n) \not\leq \mathcal{H}$.*

Proof. We introduce two ordered triple systems.

Let $\mathcal{T}_0'$ denote the following system: the vertex set is $\{x,y,z,u\}_<$, and the system consists of $\{x,z,u\}$ and $\{y,z,u\}$.

Let $\mathcal{K}$ contain the different vertices $\{x(i,j), y(i), z(j) : i,j < 2\}$ and the ordered edges $\{x(i,j), y(i), z(j)\}_<$. Notice that the order of the vertices is not totally determined, for example $x(0,0) < x(0,1)$ and $x(0,1) < x(0,0)$ are equally possible.

A condition is of the form $p = (s,t,h,\mathcal{H})$ where $s \in [\omega_1]^{<\omega}$, $t \subseteq s$, $h : t \to s$ is a regressive function, $\mathcal{H} \subseteq [t]^3$. We further stipulate the following: $\mathcal{T}_0' \not\leq \mathcal{H}$, $\mathcal{K} \not\leq \mathcal{H}$, and if $\{x,y,z\} \in \mathcal{H}$ then $y,z > h(x)$.

We define $p' = (s',t',h',\mathcal{H}') \leq p = (s,t,h,\mathcal{H})$ if the following hold: $s' \supseteq s$, $t = s \cap t'$, $h' \supseteq h$, $\mathcal{H} = \mathcal{H}' \cap [s]^3$.

Lemma 1. $(P,\leq)$ *is ccc.*

Proof. With the usual arguments it suffices to show that $p' = (s',t',h',\mathcal{H}')$ and $p'' = (s'',t'',h'',\mathcal{H}'')$ are compatible if they are isomorphic and

$$s' \cap s'' < s' - s'' < s'' - s'.$$

Set $\Delta = s' \cap s''$, $A = s' - s''$, $B = s'' - s'$. We only have to show that

$$p = (s' \cup s'', t' \cup t'', h' \cup h'', \mathcal{H}' \cup \mathcal{H}'')$$

is a condition.

What is not obvious here is that $\mathcal{T}_0', \mathcal{K} \not\leq \mathcal{H}' \cup \mathcal{H}''$. Assume that $H = \{x, z, u\}$ and $H' = \{y, z, u\}$ form a $\mathcal{T}_0'$ in $\mathcal{H}' \cup \mathcal{H}''$. Then $H \in \mathcal{H}'$ and $H' \in \mathcal{H}''$ (or vice versa). But then, $H' \subseteq \Delta \cup A$ and $H'' \subseteq \Delta \cup B$, so $u \in (\Delta \cup A) \cap (\Delta \cup B) = \Delta$ and so H, H' are both in $\mathcal{H}'$, a contradiction.

Assume finally that the vertices

$$\{x(0,0), x(0,1), x(1,0), x(1,1), y(0), y(1), z(0), z(1)\}$$

form a $\mathcal{K}$ in $\mathcal{H}' \cup \mathcal{H}''$. If $z(0), z(1) \in B$ then all the remaining points must be in $\Delta \cup B$, so $\mathcal{K} \leq \mathcal{H}''$, an impossibility. A similar contradiction is obtained if $z(0), z(1) \in A$. The only possibility left is that $z(0) \in A$, $z(1) \in B$, and then the remaining elements (the x's and the y's) must be in Δ. We now use the fact that p' and p'' are isomorphic and let $\pi : \Delta \cup A \to \Delta \cup B$ be an isomorphism between them. If $\pi(z(0)) \neq z(1)$, then $\{x(i,j), y(i) : i, j < 2\} \cup \{\pi(z(0)), z(1)\}$ form a $\mathcal{K}$ in $\mathcal{H}''$, a contradiction. If, however, $\pi(z(0)) = z(1)$, then $\{x(0,0), x(0,1), y(0), z(0)\}$ form a $\mathcal{T}_0'$ in $\mathcal{H}'$, a contradiction again. □

If $G \subseteq P$ is generic, set $\mathcal{G} = \bigcup\{\mathcal{H} : (s, t, h, \mathcal{H}) \in G\}$, a triple system on ω_1. Obviously, $\mathcal{G}$ does not contain a $\mathcal{K}$, neither a $\mathcal{T}_0'$.

Lemma 2. *Some condition forces that* $\mathrm{Chr}(\mathcal{G}) = \aleph_1$.

Proof. Assume indirectly that 1 forces that $f : \omega_1 \to \omega$ is a good coloring for $\mathcal{G}$. Let

$$f \in N_0 \prec N_1 \prec \left(H(2^{\aleph_1})^+, \in, \Vdash\right)$$

be countable elementary submodels with $\delta_0 < \delta_1 < \omega_1$ where $\delta_i = N_i \cap \omega_1$ ($i < 2$). Set

$$p = (\{\delta_0, \delta_1\}, \{\delta_1\}, \{(\delta_1, \delta_0)\}, \emptyset)$$

and select $q \leq p$ in such a way that $q \Vdash f(\delta_1) = i$ for some $i < \omega$. If $q = (s, t, h, \mathcal{H})$, $\Delta = s \cap \delta_0$, $A = s \cap [\delta_0, \delta_1)$, and $B = s \cap [\delta_1, \omega_1)$ then there is a formula Φ such that $\Phi(\Delta, A, B)$ states this and also describes the isomorphism type of q. As N_0, N_1 are elementary submodels, there exist A', A'', B', B'' such $\Phi(\Delta, A', B')$ and $\Phi(\Delta, A'', B'')$ also hold, and

$$\Delta < A < A' < A'' < B < B' < B''.$$

Accordingly, there are conditions

$$q' = (\Delta \cup A' \cup B', t', h', \mathcal{H}')$$

and

$$q'' = (\Delta \cup A'' \cup B'', t'', h'', \mathcal{H}'')$$

such that if $\delta_1' = \min(B')$ and $\delta_1'' = \min(B'')$ then $q' \Vdash f(\delta_1') = i$ and $q'' \Vdash f(\delta_1') = i$. Now set

$$q^* = (s^*, t^*, h^*, \mathcal{H}^*)$$

where

$$s^* = \Delta \cup A \cup A' \cup A'' \cup B \cup B' \cup B'',$$
$$t^* = t \cup t' \cup t'', h^* = h \cup h' \cup h'',$$
$$\mathcal{H}^{**} = \mathcal{H} \cup \mathcal{H}' \cup \mathcal{H}'', \quad \mathcal{H}^* = \mathcal{H}^{**} \cup \{H\}, \quad H = \{\delta_1, \delta_1', \delta_1''\}.$$

If we show that q^* is a condition, then we get the desired contradiction as q^* forces on the one hand that $H = \{\delta_1, \delta_1', \delta_1''\} \in \mathcal{G}$ on the other hand that $f(\delta_1) = f(\delta_1') = f(\delta_1'') = i$.

To check the condition on h^* we only have to show that if $x, y \in H$ then $h^*(y) < x$. This holds, as

$$\delta_0 < \delta_0' < \delta_0'' < \delta_1 < \delta_1' < \delta_1''$$

and $h^*(\delta_1) = \delta_0$, $h^*(\delta_1') = \delta_0'$, $h^*(\delta_1'') = \delta_0''$.

We still have to show that $\mathcal{T}_0', \mathcal{K} \not\leq \mathcal{H}^*$. A purported $\mathcal{T}_0'$ cannot contain H as this latter set meets every element in $\mathcal{H}^{**}$ in one element, at the most. In the remaining case the last two points of $\mathcal{T}_0'$ are in $\Delta \cup A \cup B$ or in $\Delta \cup A' \cup B'$ or in $\Delta \cup A'' \cup B''$, that is one of q, q', q'' is not a condition, a contradiction.

Assume finally that $\mathcal{H}^*$ contains the system $\mathcal{K}$ with the vertices

$$\{x(i,j), y(i), z(j) : i, j < 2\}.$$

The edge set of this $\mathcal{K}$ does not contain H as there is no path of length 3 between any two of δ_1, δ_1', and δ_1''. The vertices $z(0)$, $z(1)$ cannot be contained in one of the sets $\Delta \cup A \cup B$, $\Delta \cup A' \cup B'$, $\Delta \cup A'' \cup B''$, as that would imply a copy of $\mathcal{K}$ in $\mathcal{H}$, $\mathcal{H}'$, or $\mathcal{H}''$. We obtain, therefore, that $z(0)$ and $z(1)$ are in two of the sets $A \cup B$, $A' \cup B'$, $A'' \cup B''$. We can now conclude as in Lemma 1. $\square$

To finish the proof of the theorem we need the following lemma.

Lemma 3. *There is a natural number n such that for every ordering of the vertices of $\mathcal{M}(n)$ it contains a $\mathcal{K}$.*

Proof. Let $n < \omega$ be so large that

$$\begin{pmatrix} n \\ n \\ n \end{pmatrix} \to \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}^{1,1,1}_{6}$$

holds. The statement that such an n exists is a standard fact of Ramsey-theory. It can be proved by repeated applications of the pidgeon hole principle. For example, $n = 1,008,127$ suffices.

We show that $\mathcal{M}(n)$ has the required property. Assume that the vertices of $\mathcal{M}(n)$, $\{x(i,j), y(i,j), z(i,j) : i,j < n\}$ are ordered in some way. For every triplet $\langle i,j,k\rangle$ the vertices of the edge $\{x(i,j), y(i,k), z(j,k)\}$ can have six different ordering. If we color $\langle i,j,k\rangle$ with colors $0,1,\ldots,5$ describing which of the possibilities occurs, then we get a coloring $n\times n\times n \to 6$. By the above partition relation property there is a homogeneous subset with 2 elements in each class.

This homogeneous set gives a triple system isomorphic to the following. The vertices are $\{x(i,j), y(i,j), z(i,j) : i,j < 2\}$ and the edges are $\{x(i,j), y(i,j), z(j,k)\}_<$. But then,

$$\{x(0,0), x(0,1), x(1,0), x(1,1), y(0,0), y(1,0), z(0,0), z(1,0)\}$$

spans a $\mathcal{K}$. □

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