A Spectral Strong Approximation Theorem for Measure Preserving Actions

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Abstract

Let Γ be a finitely generated group acting by probability measure preserving maps on the standard Borel space (X, μ). We show that if $H \leq \Gamma$ is a subgroup with relative spectral radius greater than the global spectral radius of the action, then H acts with finitely many ergodic components and spectral gap on (X, μ). This answers a question of Shalom who proved this for normal subgroups.

1 Introduction

Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let $L^2(X) = L^2(X, \mathcal{B}, \mu)$ denote the space of square integrable measurable real functions on X and let $L_0^2(X) \subseteq L^2(X)$ be the subspace of functions with zero integral. Let $\mathrm{Aut}(X, \mu)$ denote the Polish group of μ-preserving Borel isomorphisms of $(X, \mathcal{B}, \mu)$.

Let λ be a Borel probability measure on $\mathrm{Aut}(X, \mu)$. One can associate the averaging operator $M_\lambda : L^2(X) \to L^2(X)$ defined by

$$(fM_\lambda)(x) = \int\limits_{a \in \mathrm{Aut}(X,\mu)} f(xa)d\lambda(a) \quad (f \in L^2(X), x \in X)$$

Let the top of the spectrum of λ be

$$\rho^+(\lambda) = \rho^+(X, \mathcal{B}, \mu, \lambda) = \sup_{0 \neq f \in L_0^2(X)} \frac{\langle fM_\lambda, f\rangle}{\langle f, f\rangle}$$

and let the norm of λ be

$$\rho(\lambda) = \|M_\lambda\| = \sup_{0 \neq f \in L_0^2(X)} \sqrt{\frac{\langle fM_\lambda, fM_\lambda\rangle}{\langle f, f\rangle}}.$$

Then we have

$$0 \leq \rho^+(\lambda) \leq \rho(\lambda) \leq 1.$$

The operator M_λ is self-adjoint when λ is *symmetric*, that is, $\lambda = \lambda^{-1}$ where λ^{-1} is obtained by composing λ with the inverse map. In general, we have $\rho(\lambda)^2 = \rho^+(\lambda\lambda^{-1})$. The inequality $\rho^+(\lambda) \geq 0$ is proved in Proposition 6.

Let Γ be a countable group and let λ be a symmetric probability measure on Γ such that the support of λ generates Γ. The averaging operator M_λ now naturally acts on $l^2(\Gamma)$ and is self-adjoint. Let the *spectral radius of* λ be

$$\rho(\lambda) = \sup_{0 \neq f \in l^2(\Gamma)} \frac{\langle f M_\lambda, f \rangle}{\langle f, f \rangle} = \| M_\lambda \| = \lim_{n \to \infty} \sqrt[2n]{p_{e,e,2n}}$$

where $p_{e,e,n}$ is the probability of return for the λ-random walk on Γ in n steps.

A *p.m.p. action* φ *of* Γ *on* (X, B, μ) is a homomorphism from Γ to $\mathrm{Aut}(X, \mu)$. The push-forward $\varphi(\lambda)$ is a Borel probability measure on $\mathrm{Aut}(X, \mu)$ and the action φ has spectral gap if and only if $\rho^+(\varphi(\lambda)) < 1$. In Proposition 11 we show that

$$\rho(\lambda) \leq \rho^+(\varphi(\lambda)).$$

Let H be an arbitrary subgroup of Γ and let λ be a symmetric probability measure on Γ. Then M_λ also acts on $l^2(H \backslash \Gamma)$ where $H \backslash \Gamma$ is the set of right cosets of H in Γ. Let the *relative spectral radius*

$$\rho(\Gamma, H, \lambda) = \sup_{0 \neq f \in l^2(H \backslash \Gamma)} \frac{\langle f M_\lambda, f \rangle}{\langle f, f \rangle} = \lim_{n \to \infty} \sqrt[2n]{p_{e,H,2n}} = \| M_\lambda \|$$

where $p_{e,H,n}$ is the probability that the λ-random walk of length n starting at e ends in H. We have

$$\rho(\lambda) \leq \rho(\Gamma, H, \lambda) \leq 1$$

The quantity $\rho(\Gamma, H, \lambda)$ can be thought of as the *dimension of* H *in* Γ.

Our first theorem says that when $H \leq \Gamma$ is too big compared to a p.m.p. action φ of Γ, it can not effectively 'hide in the action'. For a finite symmetric set S of Γ let λ_S denote the uniform probability measure on S.

Theorem 1 *Let* Γ *be generated by the finite symmetric set* S *and let* φ *be a p.m.p. action of* Γ*. Then for every subgroup* H *of* Γ *with*

$$\rho(\Gamma, H, \lambda_S) > \rho^+(\varphi(\lambda_S))$$

there exists a finitely generated subgroup H' *of* H *such that* H' *has finitely many ergodic components and* H' *acts on each component with spectral gap.*

Theorem 1 answers a question of Shalom, who proved it in the case when H is normal in Γ [Sha]. We use a different approach from Shalom, which also provides an explicit generating set for H' and effective bounds on the expansion properties of this generating set.

The proof of Theorem 1 uses geometric expansion. Let λ be a Borel probability measure on $\mathrm{Aut}(X, \mu)$. For a Borel subset $Y \in \mathcal{B}$ let

$$e_\lambda(Y) = \int_{a \in \mathrm{Aut}(X,\mu)} \mu(Ya \diagdown Y) d\lambda(a) = \langle \chi_Y M_\lambda, \chi_{Y^c} \rangle .$$

be the probability that a λ-random edge starting at Y leaves Y. Let the *expansion constant of* λ be

$$\mathrm{h}(\lambda) = \mathrm{h}(X, \mathcal{B}, \mu, \lambda) = \inf \left\{ \frac{e_\lambda(Y)}{\mu(Y)(1 - \mu(Y))} \mid Y \in \mathcal{B}, 0 < \mu(Y) < 1 \right\}.$$

We call the system λ an *expander* if $\mathrm{h}(\lambda) > 0$. Adapting Cheeger's inequalities for group actions in [LyN] gives the estimates

$$1 - \mathrm{h}(\lambda) \leq \rho^+(\lambda) \leq 1 - \mathrm{h}(\lambda)^2/8.$$

In particular, λ is an expander if and only if $\rho^+(\lambda) < 1$. See Proposition 5 for details.

The core of Theorem 1 is a general result saying that every large enough convex part of an expander measure keeps expanding, at least on small enough subsets. A Borel subset $Y \in \mathcal{B}$ is λ-*invariant*, if $e_\lambda(Y) = 0$. When Y is λ-invariant and $\mu(Y) > 0$, we can naturally restrict λ to $\mathrm{Aut}(Y, \frac{1}{\mu(Y)}\mu)$. We say that Y is λ-*ergodic*, if Y is λ-invariant and every λ-invariant Borel subset $Z \subseteq Y$ satisfies $\mu(Z) = 0$ or $\mu(Y \setminus Z) = 0$.

In the following we do not assume λ and λ_i to be symmetric.

Lemma 2 *Let λ be a Borel probability measure on $\mathrm{Aut}(X, \mu)$, let $\kappa > \rho^+(\lambda)$ and let us decompose*

$$\lambda = \kappa\lambda_1 + (1 - \kappa)\lambda_2$$

where the λ_i are Borel probability measures on $\mathrm{Aut}(X, \mu)$. Then every λ_1-invariant Borel subset $Y \in \mathcal{B}$ of positive measure satisfies

$$\mu(Y) \geq \frac{\kappa - \rho}{1 - \rho}.$$

Also, for every Borel subset $Y \in \mathcal{B}$ with

$$0 < \mu(Y) \leq \frac{1}{2}\frac{\kappa - \rho}{1 - \rho}$$

we have

$$\frac{e_{\lambda_1}(Y)}{\mu(Y)} \geq \frac{1}{2}\frac{\kappa - \rho}{\kappa}.$$

In particular, one can decompose

$$X = \bigcup_{i=1}^{n} X_i \quad (n \leq \frac{1 - \rho}{\kappa - \rho})$$

where the X_i are λ_1-ergodic Borel subsets of positive measure, and λ_1 on X_i has spectral gap $(1 \leq i \leq n)$.

It is natural to ask whether the explicit bounds on e_{λ_1} lead to explicit bounds on $\rho^+(X_i, \mathcal{B}, \mu, \lambda_1)$ in Theorem 2. Even in the case when λ is finitely supported and X stays λ_1-ergodic, the answer is negative in general as follows from the work of the author and Elek [AbE]. However, when X is homogeneous enough, one can indeed get such explicit bounds. The nicest case is when $X = G$ is a compact topological group and μ is the normalized Haar measure. In this case, G acts on X by p.m.p. maps from both the left and the right, and the two actions commute. The following theorem concentrates on the case when G is connected and $\kappa \geq 2\rho^+(\lambda)$.

Corollary 3 *Let G be a compact, connected topological group with normalized Haar measure μ. Let λ be a Borel probability measure on G and let $\kappa \geq 2\rho^+(\lambda) > 0$. Let us decompose*

$$\lambda = \kappa\lambda_1 + (1 - \kappa)\lambda_2$$

where the λ_i are Borel probability measures on G. Then G is λ_1-ergodic and we have

$$\rho^+(\lambda_1) < 1 - \frac{1}{512}\left(\frac{\rho^+(\lambda)}{\log_2(2/\rho^+(\lambda))}\right)^2.$$

In Corollary 13 we also state a version that estimates the spectral radius $\rho(\lambda_1)$ in terms of $\rho(\lambda)$. These estimates have been used in the recent paper of Lindenstrauss and Varju [LiV]. Note that when G is not connected, in particular, when it is a profinite group, one can still get an explicit, but significantly weaker estimate on the spectral gap on the ergodic components of λ_1, using the machinery introduced by the author and Elek in [AbE].

Theorem 1 translates to the following result in terms of profinite actions and property (τ). For a finite d-regular graph G let $\rho^+(G)$ denote the largest nontrivial eigenvalue of the Markov (random walk) operator on G.

Theorem 4 *Let Γ be a group generated by the finite symmetric set S. Let (Γ_n) be a chain of finite index subgroups in Γ and let*

$$\rho^+ = \sup \rho^+(\mathrm{Sch}(\Gamma/\Gamma_n, S)).$$

where $\mathrm{Sch}(\Gamma/\Gamma_n, S)$ is the Schreier graph of the coset action on Γ/Γ_n. If $H \leq \Gamma$ is a subgroup such that

$$\rho(\Gamma, H, \lambda_S) > \rho^+$$

then there exists a finitely generated subgroup $H' \leq H$ such that $(H' \cap \Gamma_n)$ has property (τ) in H'. If the Γ_n are normal in Γ, then there exists $M > 0$ such that $|\Gamma : H'\Gamma_n| < M$.

A special case of Theorem 4 is Shalom's theorem [Sha]: he assumes $\rho^+ = \rho(T_S)$ and that H is a nontrivial normal subgroup of Γ. Here T_S is the $|S|$-regular tree. Indeed, in this case, by Kesten's theorem [Kes], Γ must be a free product of cyclic groups and so H is nonamenable, which, again using Kesten's theorem, implies $\rho(\Gamma, H, \lambda_S) > \rho^+$.

A major motivation of Shalom's result was to find infinite index subgroups of arithmetic groups that have property (τ) with respect to its congruence subgroups. An early provocative question in this direction was the so-called 1-2-3 problem of Lubotzky. In the last decade, this arithmetic direction experienced an enormous activity, starting in the breakthrough works of Helfgott [Hel] and Bourgain-Gamburd [BoG] and continued in a series of deep papers. See [BoG], [BoV], [GoV], [PySz], [BGT] for the latest developments. In particular, proprty (τ) is now known to hold for a large class of arithmetic lattices in semisimple Lie groups acting on their congruence completion, assuming H is Zariski dense. We do not expect that Theorems 1 and 4 will say much new in this direction, because it seems rather nontrivial to effectively estimate the spectral gap of a profinite action. An advantage of our result is that it substitutes the arithmetic language, namely Zariski density and congruence subgroups with a simple spectral condition, in the spirit of Gamburd [Gam], but in the discrete setting.

Ramanujan actions. Let λ be a symmetric probability measure on the countable group Γ and let φ be a p.m.p. action of Γ. We call the triple $(\Gamma, \lambda, \varphi)$ *Ramanujan*, if $\rho(\lambda) = \rho^+(\varphi(\lambda))$. Theorem 1 implies that for a Ramanujan action, every subgroup H where $\rho(\Gamma, H, \lambda) > \rho(\Gamma, 1, \lambda)$ acts with finitely many ergodic components and spectral gap on each components. These actions seem to be tight in many other senses. For instance, a recent theorem of the author, Glasner and Virag [AGV] plus an even more recent result of Bader, Duchesne and Lecureux [BDL] implies that for such actions, the stabilizer of a μ-random $x \in X$ in Γ lies in the amenable radical of Γ a.s. Indeed, as we show in Proposition 11, $\rho^+(\varphi(\lambda)) \geq \rho(\Gamma, H, \lambda)$ where H is a typical stabilizer. Being Ramanujan then implies that $\rho(\Gamma, H, \lambda) = \rho(\lambda)$, which, by [AGV] implies that H is amenable a.s. Now [BDL] implies that every stabilizer of a p.m.p. action that is amenable a.s. lies in the amenable radical.

Straightforward examples for Ramanujan actions are nontrivial Bernoulli actions (or, in another language, i.i.d. processes) of Γ [KeT]. Note that for a Bernoulli action of Γ, every infinite subgroup H acts ergodically and the restricted action is also a Bernoulli, in particular, it has spectral gap if and only if H is nonamenable. So for Bernoulli actions Theorem 1 does not give much new. Another, quite non-trivial examples come from the Lubotzky-Philips-Sarnak construction [LPS] that produces Ramanujan profinite actions for free groups, for suitable ranks and the standard generating set. Recently, Backhausz, Szegedy and Virag [BSV] analyzed the behavior of local algorithms using the notion of a Ramanujan graphing. The connection is that when $\lambda = \lambda_S$ for a symmetric generating set, the triple $(\Gamma, \lambda, \varphi)$ is encoded in a graphing.

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The paper is organized as follows. In Section 2 we define the basic notions and prove some general lemmas on expansion and spectral gap. In particular, we prove Lemma 8, a weaker substitute for Schmidt's lemma [Sch] for probability measures on $\mathrm{Aut}(X, \mu)$. Section 3 contains the proofs of Lemma 2 and Theorem 1. In Section 4 we prove Corollary 3 and Corollary 13 and discuss profinite and Ramanujan actions.

2 Preliminaries

In this section we define the basic notions and state some general lemmas.

We start with a proposition that is well-known in Riemannian geometry and finite graph theory under the name Cheeger inequalities. For the nontrivial part we use the exposition by Lyons and Nazarov [LyN].

Proposition 5 *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space and let λ be a Borel probability measure on* $\mathrm{Aut}(X, \mu)$*. Then we have*

$$1 - \mathrm{h}(\lambda) \leq \rho^+(\lambda) \leq 1 - \frac{\mathrm{h}(\lambda)^2}{8}.$$

Proof. For $Y \in \mathcal{B}$ let $f_Y = \chi_Y - y\chi_X$ and let $y = \mu(Y)$. Then $f \in L_0^2(X, \mu)$ and

$$e_\lambda(Y) = \langle \chi_Y M_\lambda, \chi_X - \chi_Y \rangle = y(1-y) - \langle f_Y M_\lambda, f_Y \rangle .$$

This gives

$$\mathrm{h}(\lambda) \geq \frac{e_\lambda(Y)}{y(1-y)} = 1 - \frac{\langle f_Y M_\lambda, f_Y \rangle}{\langle f_Y, f_Y \rangle} \geq 1 - \rho^+(\lambda).$$

For the other inequality, see e.g. [LyN, Theorem 3.1], where this is proved in the case when λ is the uniform measure on a finite subset S of $\mathrm{Aut}(X, \mu)$. The proof therein goes through without difficulty by formally changing

$$\frac{1}{|S|}\sum_{s\in S}(.) \quad \text{to} \quad \int\limits_{\mathrm{Aut}(X,\mu)} (.)\, d\lambda$$

everywhere. □

Curiously, we did not find a non-probabilistic proof for the following.

Proposition 6 *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space and let λ be a Borel probability measure on $\mathrm{Aut}(X, \mu)$. Then $\rho^+(\lambda) \geq 0$.*

Proof. We can assume that X is the unit circle and μ is the normalized Lebesque measure. Let d be the normalized distance on X, so that the length of X is 1. Let $B(x, r)$ be the ball of radius r around x.

For a parameter $r > 0$ let us define the random vector

$$f_r = \chi_{B(x,r)} - \chi_{B(y,r)}$$

where x, y are independent μ-random elements of X. Then $f_r \in L_0^2(X, \mu)$, $\langle f_r, f_r\rangle \leq 4r$ and

$$\mathcal{P}(\langle f_r, f_r\rangle < 4r) \leq 4r.$$

Let $\gamma \in \mathrm{Aut}(X, \mu)$ be fixed. The expected measure of the intersection $B(x, r) \cap B(y, r)g$ can be computed by fixing y and only using that $B(y, r)g$ has μ-measure $2r$. This gives

$$\mathbb{E}(\mu(B(x, r) \cap B(y, r)\gamma)) = 4r^2.$$

This implies

$$\langle f_r\gamma, f_r\rangle \geq -\left(\mu(B(x, r) \cap B(y, r)\gamma + \mu(B(x, r)\gamma \cap B(y, r)\right)$$

which yields

$$\mathbb{E}(\langle f_r\gamma, f_r\rangle) \geq -8r^2.$$

Let $g_r = f_r$ conditioned on $d(x, y) \geq 2r$. This implies $\langle g_r, g_r\rangle = 4r$. The probability of this event is $(1 - 4r)$, so using $|\langle f_r\gamma, f_r\rangle| \leq 4r$, we have

$$\mathbb{E}\left(\langle g_r\gamma, g_r\rangle\right) \geq -\frac{1}{1-4r}(8r^2 + 16r^2) = -\frac{24r^2}{1-4r}.$$

Let $\gamma \in \mathrm{Aut}(X, \mu)$ be λ-random, independently of g_r. We get

$$\mathbb{E}_{g_r}\left(\langle g_r M_\lambda, g_r\rangle\right) = \mathbb{E}_{g_r}\mathbb{E}_\gamma\left(\langle g_r\gamma, g_r\rangle\right) = \mathbb{E}_\gamma\mathbb{E}_{g_r}\left(\langle g_r\gamma, g_r\rangle\right) \geq -\frac{24r^2}{1-4r}.$$

In particular, for every $r > 0$ there exists $g \in L_0^2(X, \mu)$ such that

$$\frac{\langle gM_\lambda, g\rangle}{\langle g, g\rangle} \geq \frac{-24r}{1-4r}.$$

Letting r tend to zero implies $\rho^+(\lambda) \geq 0$. □

We will use the following easy lemma multiple times. We omit the proof.

Lemma 7 *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space and let λ be a Borel probability measure on* $\mathrm{Aut}(X, \mu)$*. Then for all $Y, Z \in \mathcal{B}$ we have*

$$e_\lambda(Y \cap Z) \leq e_\lambda(Y) + e_\lambda(Z).$$

Similarly, $e_\lambda(Y \backslash Z) \leq e_\lambda(Y) + e_\lambda(Z)$.

In the following lemma we prove that for an ergodic measure, if all small sets expand, then the measure is an expander. For p.m.p. actions of countable groups, this is proved e.g. in [AbE], using Schmidt's lemma [Sch]. Since we could not find a version of this lemma for Borel measures on $\mathrm{Aut}(X, \mu)$, we give a direct proof here that is of combinatorial nature.

Lemma 8 *Let $(X, \mathcal{B}, \mu)$ be a standard Borel probability space. Let λ be an ergodic Borel probability measure on* $\mathrm{Aut}(X, \mu)$*. Assume that there exists $c, c' > 0$ such that for every $Y \in \mathcal{B}$ with $0 < \mu(Y) < c$ we have*

$$\frac{e_\lambda(Y)}{\mu(Y)(1 - \mu(Y))} > c'.$$

Then $\rho^+(\lambda) < 1$.

Proof. Let us define the function $F : [0, 1] \to [0, 1]$ as

$$F(y) = \inf \{e_\lambda(Y) \mid Y \in \mathcal{B},\, \mu(Y) = y\}.$$

Then $F(0) = F(1) = 0$ and F is symmetric to $1/2$. For all $Y, Z \in \mathcal{B}$ with $Y \cap Z = \varnothing$ we have

$$|e_\lambda(Y \cup Z) - e_\lambda(Y)| \leq e_\lambda(Z) \leq \mu(Z) \tag{Sub}$$

which implies that

$$|F(y + z) - F(y)| \leq z \ (y \in [0, 1], z \in [0, 1 - y]).$$

In particular, F is continuous on $[0, 1]$. By the assumption of the lemma, we have

$$F(y) > c'y > 0 \quad (0 < y < c).$$

Let

$$r = \min \{y \in (0, 1] \mid F(y) = 0\} \text{ and } \varepsilon_0 = \max \{F(y) \mid 0 \leq y \leq r\}.$$

For $0 \leq \varepsilon < \varepsilon_0$ let

$$g(\varepsilon) = \min \{z \mid F(y) \geq \varepsilon \text{ for all } y \in [z, r - z]\}.$$

Since $F(0) = F(1) = 0$, $\lim_{\varepsilon \to 0} g(\varepsilon) = 0$. Let $\varepsilon_1 > 0$ such that for all $0 < \varepsilon \leq \varepsilon_1$ we have $g(\varepsilon) \leq r/3$.

Let $Y, Z \in \mathcal{B}$ with $\mu(Y) = \mu(Z) = r$ such that

$$e_\lambda(Y), e_\lambda(Z) \leq \varepsilon/2 \leq \varepsilon_1/2.$$

Using Lemma 7, we have

$$\varepsilon \geq e_\lambda(Y) + e_\lambda(Z) \geq e_\lambda(Y \cap Z) \geq F(\mu(Y \cap Z))$$

which, by the definition of g, implies that

$$\mu(Y \cap Z) \leq g(\varepsilon) \text{ or } \mu(Y \cap Z) \geq r - g(\varepsilon).$$

The assumption $g(\varepsilon) \leq r/3$ ensures that the above two possibilities are mutually exclusive.

Let $\varepsilon_1 > \varepsilon_2 > \ldots$ be a positive sequence converging to 0. Since $F(r) = 0$, for all $n \geq 1$ there exists $Y_n \in \mathcal{B}$ such that $\mu(Y_n) = r$ and $e_\lambda(Y_n) \leq \varepsilon_n/2$. By the above, for every pair of integers $n < m$ exactly one of the following holds:

$$\mu(Y_n \cap Y_m) \leq g(\varepsilon_n) \text{ or } \mu(Y_n \cap Y_m) \geq r - g(\varepsilon_n).$$

We call Y_n and Y_m *overlapping*, if $\mu(Y_n \cap Y_m) \geq r - g(\varepsilon_n)$.

We claim that there exists an infinite subsequence of (Y_n) such that all pairs in the subsequence are overlapping. Assume not. Then the graph defined on $\{Y_n\}$ by the overlapping relation does not admit an infinite complete subgraph, so by the infinite Ramsey theorem, it admits an infinite empty subgraph. Let $m > 2/r$ and choose $Y_{n_1}, \ldots, Y_{n_m}$ from this subgraph such that $g(\varepsilon_{n_i}) < r/(2(m-1))$ $(1 \leq i \leq m)$. Then the sets

$$Y_{n_i} \backslash \bigcup_{j \neq i} Y_{n_j}$$

are mutually disjoint of measure at least $r/2$, thus the sum of their measures is at least $mr/2 > 1$, a contradiction. The claim holds. By passing to this subsequence we can assume that (Y_n) is totally overlapping.

Let us endow the set of measurable subsets of $(X, \mathcal{B}, \mu)$ modulo nullsets with the usual metric

$$d(Y, Z) = \mu(Y \backslash Z \cup Z \backslash Y).$$

By the above, for all $n < m$ we have

$$d(Y_n, Y_m) \leq 2g(\varepsilon_n).$$

That is, in the metric d, Y_n forms a Cauchy sequence. Since d defines a complete metric space, there exists $Y \in \mathcal{B}$ such that $d(Y, Y_n) \to 0$. However, by (Sub), e_λ continuous with respect to d, so we have

$$\mu(Y) = r \text{ and } e_\lambda(Y) = \lim_{n \to \infty} e_\lambda(Y_n) \leq \lim_{n \to \infty} \varepsilon_n/2 = 0.$$

By the ergodicity of λ, this implies that $r = 1$.

In particular, $F(y) > 0$ for $c \leq y \leq 1 - c$ and so $\mathrm{h}(\lambda) > 0$ and by Lemma 5, we have $\rho^+(\lambda) < 1$. $\square$

In the proof of Theorem 1 we use the known trick of making λ (and the associated random walk) lazy, by averaging it with λ_e, the Dirac measure on the identity element. This gives us the following advantages.

Lemma 9 *Let λ be a symmetric Borel probability measure on* $\mathrm{Aut}(X, \mu)$ *and let*

$$\lambda' = \frac{1}{2}\lambda + \frac{1}{2}\lambda_e.$$

Then

$$\frac{1}{2}\rho^+(\lambda)+\frac{1}{2}=\rho^+(\lambda')=\|M_{\lambda'}\|=\|M_{\lambda'}^n\|^{1/n}\quad (n\geq 1)$$

and

$$\mathrm{h}(\lambda')=\frac{1}{2}\mathrm{h}(\lambda).$$

Proof. The spectrum of $M_{\lambda'}$ lies in $[0,1]$ and so $\rho^+(\lambda')=\|M_{\lambda'}\|=\|M_{\lambda'}^n\|^{1/n}$. By definition, we have

$$e_{\lambda'}(Y)=\frac{1}{2}e_\lambda(Y)$$

for all $Y\in\mathcal{B}$ which implies $\mathrm{h}(\lambda')=\frac{1}{2}\mathrm{h}(\lambda)$. □

The first part of the following lemma can be found e.g. in [AbN]. We thank P. Varju for pointing out the second part.

Lemma 10 *Let G be a compact topological group with normalized Haar measure μ and let $A,B\subseteq G$ be measurable subsets of positive measure. Let g be a μ-random element of G. Then the expected value*

$$\mathrm{E}(\mu(Ag\cap B))=\mu(A)\mu(B).$$

When G is connected, for every $k\geq 2$, there exists $g_1,\dots,g_k\in G$ such that

$$\mu(\bigcap_{i=1}^k Ag_i)=\mu(A)^k.$$

Proof. Let

$$U=\{(a,g)\in G\times G\mid a\in A,\, ag\in B\}\,.$$

Then U is measurable in $G\times G$ and using Fubini's theorem, we get

$$\mu(A)\mu(B)=\int_{a\in A}\mu(a^{-1}B)=\mu^2(U)=\int_{g\in G}\mu(Ag\cap B)=E(\mu(Ag\cap B))\,.$$

Let $g_1,\dots,g_k\in G$ be independent μ-random elements and let $Y=\cap_{i=1}^k Ag_i$. By induction on k, we have

$$\mathrm{E}(\mu(Y))=\mu(A)^k.$$

By the connectedness of G, the set of possible values for $\mu(Y)$ is a connected subset of $\mathbb{R}$. Hence, it must contain $\mu(A)^k$. □

3 Measure preserving actions

This section contains the results on p.m.p. actions, in particular, we prove Lemma 2 and Theorem 1.

We start by showing that for a p.m.p. action, the local spectral radius is less than or equal to the global one.

Proposition 11 *Let Γ be a countable group and let λ be a symmetric probability measure on Γ. Let φ be a p.m.p. action of Γ. Then*

$$\rho^+(\varphi(\lambda)) \geq \rho(\lambda).$$

Proof. Both ρ and $\rho^+(\varphi(.))$ are continuous for the l_1 distance on the space of probability measures on Γ, so we can assume that λ is supported on the finite symmetric set S. For $x \in X$ let $H_x = \mathrm{Stab}_\Gamma(x)$ be the stabilizer of x in Γ and let $x\Gamma$ be the orbit of x under Γ. We have

$$\rho(\lambda) = \lim_{n\to\infty} \sqrt[2n]{p_{e,e,2n}} \leq \lim_{n\to\infty} \sqrt[2n]{p_{e,H_x,2n}} = \rho(\Gamma, H_x, \lambda)$$

In particular, we get that for μ-almost every $x \in X$, the norm of M_λ acting on $l^2(x\Gamma)$ is at least $\rho(\lambda)$. (We will not use this, but by the ergodicity of $\varphi(\lambda)$, this norm is independent of x).

For $B \subseteq \Gamma$ let $xB = \{xb \mid b \in B\}$. Let $\varepsilon > 0$. Then for every $x \in X$ there exists a minimal $n(x) \in \mathbb{N}$ and a function $f_x : xS^{n(x)} \to \mathbb{Q}$ such that $\langle f_x, f_x \rangle = 1$,

$$\sum_{y\in S^{n(x)}} f_x(y) = 0 \quad \text{and} \quad \langle f_x M_\lambda, f_x \rangle > \rho(\lambda) - \varepsilon.$$

Since $n(x)$ is a measurable function of x, we can choose f_x to be a measurable function of x, say, by listing all the f_x-es with zero sum and norm 1 and taking the first one satisfying the inequality. We get that there exists a Borel subset $Y \in \mathcal{B}$ with $\mu(Y) > 0$ and $n > 0$ and such that for all $x \in Y$ we have $n(x) = n$. By a standard argument (see e.g. [KeM]), by passing to a subset of positive measure, we can also assume that

$$y_1 S^{n+1} \cap y_2 S^{n+1} = \varnothing \quad (y_1, y_2 \in Y,\ y_1 \neq y_2). \tag{Dis}$$

Let $F : X \to \mathbb{Q}$ be defined by

$$F(z) = \begin{cases} f_x(z) & z \in yS^n,\ y \in Y \\ 0 & \text{otherwise} \end{cases}.$$

By the above, F is well-defined, $F \in L^2_0(X, \mu)$ and

$$\langle F, F \rangle = \int_{y\in Y} \langle f_y, f_y \rangle \, d\mu = \mu(Y).$$

Also, by (Dis), M_λ acts separately on the yS^{n+1} $(y \in Y)$, thus

$$\langle F M_\lambda, F \rangle = \int_{y\in Y} \langle f_y M_\lambda, f_y \rangle \, d\mu > (\rho(\lambda) - \varepsilon)\mu(Y).$$

This implies $\rho^+(\varphi(\lambda)) \geq \rho(\lambda) - \varepsilon$. $\square$

We now prove Lemma 2 saying that a big enough convex part of an expander measure keeps being expander on its ergodic components.

Proof of Lemma 2. Let $\rho = \rho^+(\lambda)$. Fix $Y \in \mathcal{B}$, let $y = \mu(Y)$ and let $e_{\lambda_1} = e_{\lambda_1}(Y)$.

Let $f : X \to \mathbb{R}$ be defined as

$$f(x) = \begin{cases} 1-y & x \in Y \\ -y & x \notin Y \end{cases}.$$

Then $f \in L_0^2(X, \mu)$ and $\langle f, f \rangle = y(1-y)$.

For $a \in \mathrm{Aut}(X, \mu)$ let

$$\delta_a = \delta_a(Y) = \mu(Ya \setminus Y).$$

Then we have

$$\langle fa, f \rangle = (y - \delta_a)(1-y)^2 - 2\delta_a y(1-y) + (1 - y - \delta_a)y^2 = y(1-y) - \delta_a.$$

and using $\delta_a \leq y$ this gives

$$y(1-y) = \langle f, f \rangle \geq \langle fa, f \rangle = y(1-y) - \delta_a \geq -y^2.$$

Using the decomposition of λ, we have

$$\begin{aligned}
\int\limits_{a \in \mathrm{Aut}(X,\mu)} \langle fa, f \rangle \, d\lambda(a) &= \kappa \int\limits_{a \in \mathrm{Aut}(X,\mu)} \langle fa, f \rangle \, d\lambda_1(a) + (1-\kappa) \int\limits_{a \in \mathrm{Aut}(X,\mu)} \langle fa, f \rangle \, d\lambda_2(a) \\
&\geq \kappa \int\limits_{a \in \mathrm{Aut}(X,\mu)} \langle fa, f \rangle \, d\lambda_1(a) - (1-\kappa)y^2 \\
&= -(1-\kappa)y^2 + \kappa y(1-y) - \kappa \int\limits_{a \in \mathrm{Aut}(X,\mu)} \delta_a d\lambda_1(a) \\
&= y(\kappa - y) - \kappa e_{\lambda_1}.
\end{aligned}$$

By the definition of ρ we have

$$\int\limits_{a \in \mathrm{Aut}(X,\mu)} \langle fa, f \rangle \, d\lambda(a) \leq \rho \langle f, f \rangle = \rho y(1-y).$$

Putting together the two inequalities we get

$$y(\kappa - y) - \kappa e_{\lambda_1} \leq \rho y(1-y)$$

which yields

$$\frac{e_{\lambda_1}}{y} \geq \frac{\kappa - \rho}{\kappa} - y\left(\frac{1-\rho}{\kappa}\right). \qquad \text{(Exp)}$$

In particular, if

$$0 < y < \frac{\kappa - \rho}{1 - \rho}$$

then $e_{\lambda_1} > 0$, so Y can not be λ-invariant. This implies that we can decompose

$$X = \bigcup_{i=1}^{n} X_i \quad \left(n \leq \frac{1-\rho}{\kappa - \rho}\right)$$

where the X_i are λ_1-ergodic Borel subsets of positive measure.

Using (Exp) again, for

$$0 < y \leq \frac{1}{2}\frac{\kappa - \rho}{1 - \rho}$$

we get

$$\frac{e_{\lambda_1}}{y} \geq \frac{1}{2}\frac{\kappa - \rho}{\kappa}.$$

In particular, there exists $c, c' > 0$ such that for each X_i $(1 \leq i \leq n)$ and every $Y \subseteq X_i$ with $0 < \mu(Y) \leq c$ we have $e_{\lambda_1}/y \geq c\prime$. By Lemma 8 this implies that λ_1 has spectral gap on the X_i. The lemma holds. □

We are ready to prove the main theorem of the paper.

Proof of Theorem 1. Let

$$\lambda' = \frac{1}{2}\lambda_S + \frac{1}{2}\lambda_e.$$

Then $\varphi(\lambda') = \frac{1}{2}\varphi(\lambda_S) + \frac{1}{2}\lambda_e$, by Lemma 9 we have

$$\rho(\Gamma, H, \lambda') - \rho^+(\varphi(\lambda')) = \frac{1}{2}\left(\rho(\Gamma, H, \lambda') - \rho^+(\varphi(\lambda'))\right) > 0 \tag{A}$$

Recall that

$$\rho(\Gamma, H, \lambda') = \lim_{n\to\infty} \sqrt[2n]{p_{e,H,2n}}$$

where

$$p_{e,H,n} = \sum_{h\in H} p_{e,h,n} = \sum_{h\in H} \langle \chi_e M_{\lambda'}^n, \chi_h\rangle$$

is the probability that the λ'-random walk of length n starting at e ends at H. By (A) there exists an even integer n such that

$$p_{e,H,n} > \rho^+(\varphi(\lambda'))^n.$$

Fix this n. Let λ be the n-fold convolution of λ'. That is, $\lambda(\{g\}) = p_{e,g,n}$ $(g \in \Gamma)$. Let $\kappa = \lambda(H)$. By Lemma 9 we have

$$\kappa = p_{e,H,n} > \rho^+(\varphi(\lambda'))^n = \rho^+(\varphi(\lambda)).$$

Since M_λ is self-adjoint, we have

$$\rho^+(\lambda) = \rho^+(\lambda')^n = \|M_{\lambda'}\|^n.$$

Let the measures λ_1 and λ_2 on Γ be defined by

$$\lambda_1(g) = \begin{cases} \frac{1}{\kappa}\lambda(g) & g \in H \\ 0 & g \notin H \end{cases} \text{ and } \lambda_2(g) = \begin{cases} 0 & g \in H \\ \frac{1}{1-\kappa}\lambda(g) & g \notin H \end{cases}.$$

That is, we decompose λ according to $\Gamma = H \cup (\Gamma\backslash H)$. Then the λ_i are symmetric probability measures on Γ and we have

$$\lambda = \kappa\lambda_1 + (1 - \kappa)\lambda_2$$

which implies

$$\varphi(\lambda) = \kappa\varphi(\lambda_1) + (1-\kappa)\varphi(\lambda_2).$$

Let $\rho = \rho^+(\varphi(\lambda))$. Applying Lemma 2 on this convex sum, we get the decomposition

$$X = \bigcup_{i=1}^{n} X_i \quad (n \leq \frac{1-\rho}{\kappa-\rho})$$

where the X_i are $\varphi(\lambda_1)$-ergodic Borel subsets of positive measure, and the restriction of $\varphi(\lambda_1)$ on X_i has spectral gap $(1 \leq i \leq n)$.

Since λ' is supported on $S \cup \{e\}$, the support T of λ_1 is contained in $H \cap (S \cup \{e\})^n$, in particular, it is finite. Let $H' \leq H$ be the subgroup generated by T. Then H' acts on X_i with spectral gap $(1 \leq i \leq n)$. □

Remark. It is easy to see that finitely many bad eigenvalues will not disturb Theorem 1. More precisely, we can define the *essential top of the spectrum of* λ as the infimum of $\rho^+(\lambda)$ acting on the orto-complements of finite dimensional M_λ-invariant subspaces of $L_2(X)$. Theorem 1 then works with the essential top of the spectrum as input.

4 Homogeneous and profinite actions

In this section we prove Corollary 3 and some versions of it using norm instead of ρ^+. Then we translate Theorem 1 to the profinite setting to obtain Theorem 4.

Proof of Corollary 3. Let $X = G$ with the standard Borel sets and let μ be the normalized Haar measure on G. Then G acts on itself from the left by μ-preserving maps, so λ gives a Borel measure on $\mathrm{Aut}(X, \mu)$. Let $\rho = \rho^+(\lambda) > 0$.

Applying Lemma 2 we can decompose

$$G = \bigcup_{i=1}^{n} X_i \quad (n \leq \frac{1-\rho}{\kappa-\rho})$$

where the X_i are λ_1-ergodic Borel subsets of positive measure, and λ_1 on X_i has spectral gap $(1 \leq i \leq n)$. Since the right and left G-actions commute, for all $g \in G$ and $1 \leq i \leq n$, $X_i g$ is also λ_1-ergodic. By Lemma 10 there exists $g \in G$ such that $\mu(X_1 \cap X_1 g) = \mu(X_1)^2$. Since $\mu(X_1 \cap X_1 g)$ is also λ_1-ergodic, we have $\mu(X_1) = 1$ and so G is λ_1-ergodic.

Again using Lemma 2 and $\kappa \geq 2\rho > 0$ we get that for every Borel subset $Y \in \mathcal{B}$ with

$$0 < \mu(Y) \leq \frac{1}{2}\rho \leq \frac{1}{2}\frac{\kappa-\rho}{1-\rho} \tag{Small}$$

we have

$$e_{\lambda_1}(Y) \geq \frac{1}{2}(1-\frac{\rho}{\kappa})\mu(Y) \geq \frac{1}{4}\mu(Y). \tag{Exp}$$

Let $Z \in \mathcal{B}$ with $0 < \mu(Z) \leq 1/2$. Then there exists k such that

$$\frac{1}{2}\mu(Z)\rho \leq \mu(Z)^k \leq \frac{1}{2}\rho.$$

By Lemma 10 there exists $g_1, \ldots, g_k \in G$ such that for $Y = \cap_{i=1}^k Z g_i$ we have $\mu(Y) = \mu(Z)^k$.

Using Lemma 7 and (Exp) we get

$$k e_{\lambda_1}(Z) \geq e_{\lambda_1}(Y) \geq \frac{1}{4}\mu(Y) = \frac{1}{4}\mu(Z)^k \geq \frac{1}{8}\rho\mu(Z)$$

which gives

$$\frac{e_{\lambda_1}(Z)}{\mu(Z)(1-\mu(Z))} \geq \frac{\rho}{8k(1-\mu(Z))} > \frac{\rho}{8k} \geq \frac{\rho}{8\log_2(2/\rho)}$$

Since Z was arbitrary and $e_{\lambda_1}(Z) = e_{\lambda_1}(Z^c)$, this yields

$$h(\lambda_1) \geq \frac{\rho}{8\log_2(2/\rho)}$$

which, by Lemma 5, implies

$$\rho^+(\lambda_1) \leq 1 - \frac{\mathrm{h}(\lambda_1)^2}{8} \leq 1 - \frac{1}{512}\left(\frac{\rho^+(\lambda)}{\log_2(2/\rho^+(\lambda))}\right)^2.$$

The Corollary holds. □

With a more careful analysis one can certainly shave off the constant 512. It is not clear whether the log term can be omitted here. The square comes from playing back and forth between ρ^+ and h, and thus using both Cheeger inequalities.

Now we prove a version that uses κ in the estimate but in turn allows $\rho^+(\lambda) = 0$.

Lemma 12 *Let G be a compact, connected topological group with normalized Haar measure μ. Let λ be a Borel probability measure on G and let $\kappa > 2\rho^+(\lambda)$. Let us decompose*

$$\lambda = \kappa\lambda_1 + (1-\kappa)\lambda_2$$

where the λ_i are Borel probability measures on G. Then G is λ_1-ergodic and we have

$$\rho^+(\lambda_1) < 1 - \frac{1}{2^{11}}\left(\frac{\kappa}{\log_2(4/\kappa)}\right)^2.$$

The proof is identical to the proof of Corollary 3 above, except that in (Small) we change $\mu(Y) \leq \rho/2$ to $\mu(Y) \leq \kappa/4$ and then throughout the whole proof, we use $\kappa/2$ instead of ρ everywhere.

This leads to the following corollary, that uses ρ instead of ρ^+ both as input and output. Note that the paper [LiV] uses this form.

Corollary 13 *Let G be a compact, connected topological group with normalized Haar measure μ. Let λ be a Borel probability measure on G and let $\kappa > 2\rho(\lambda)$. Let us decompose*

$$\lambda = \kappa\lambda_1 + (1-\kappa)\lambda_2$$

where the λ_i are Borel probability measures on G. Then G is λ_1-ergodic and we have

$$\rho(\lambda_1) < 1 - \frac{1}{2^{11}}\left(\frac{\kappa}{\log_2(4/\kappa)}\right)^2.$$

Proof. For $g \in G$ let λg denote the g-translate of λ. Then we have

$$\rho^+(\lambda g) \le \rho(\lambda) \quad (g \in G)$$

Using Lemma 12 on $\lambda g, \lambda_1 g$ and $\lambda_2 g$ we get

$$\rho^+(\lambda_1 g) < r = 1 - \frac{1}{2^{11}} \left(\frac{\kappa}{\log_2(4/\kappa)} \right)^2 .$$

Let $f \in L_0^2(X, \mu)$ with $\langle f, f \rangle = 1$. Then

$$\left\langle f M_{\lambda_1}, f g^{-1} \right\rangle = \langle f M_{\lambda_1 g}, f \rangle < r.$$

This yields

$$\langle f M_{\lambda_1}, f M_{\lambda_1} \rangle = \int \left\langle f M_{\lambda_1}, f g^{-1} \right\rangle d\lambda_1(g) < r$$

impying

$$\rho(M_{\lambda_1}) < r.$$

The corollary holds. □

Remark. Let Γ be a countable group and let λ be a symmetric probability measure on Γ with averaging operator M_λ acting on $l^2(\Gamma)$. In the paper we implicitely use that $\rho^+(\lambda) = \|M_\lambda\|$. The fact itself is folkore, but we found a couple of incomplete proofs in the literature, so we include a sketch here. Let μ be the spectral measure of M_λ. Then for all $k \ge 0$, the k-th moment of μ equals $p_{e,e,k}$, the probability of return in k steps for the λ-random walk on Γ. In particular, all the moments of μ are non-negative. This easily implies that the top of the support of μ is at least the absolute value of the bottom of the support, which then implies $\rho^+(\lambda) = \|M_\lambda\|$.

We are ready to prove Theorem 4.

Proof of Theorem 4. Let X denote the boundary of the coset tree of Γ with respect to (Γ_n). This is the inverse limit of the coset spaces Γ/Γ_n (see [AbN] for an exposition). Note that when the Γ_n are normal, X equals the profinite completion of Γ with respect to (Γ_n). Then Γ acts on X by a p.m.p. action. This action is always ergodic and has spectral gap if and only if (Γ_n) has property (τ) in Γ.

Let λ be the uniform measure on the symmetric set S. Since $L^2(X)$ as a Γ-space is the union of $L^2(\Gamma/\Gamma_n)$, we have

$$\rho^+(\lambda) = \sup \rho^+(\Gamma/\Gamma_n, \lambda)$$

where $\rho^+(\Gamma/\Gamma_n, \lambda)$ is the top of the spectrum of λ acting on $L_0^2(\Gamma/\Gamma_n)$. Similarly, we have $\rho(\lambda) = \sup \rho(\Gamma/\Gamma_n, \lambda)$.

Since $\rho(\Gamma, H, \lambda) > \rho^+$, we can apply Theorem 1 and get that there exists a finitely generated subgroup $H' \le H$ such that X has finitely many ergodic H'-components and H' acts on each component with spectral gap. This is equivalent to saying that there exists $M > 0$ such that the action of H' on Γ/Γ_n has at most M orbits and uniform spectral gap ($n \ge 0$). In particular, $(H' \cap \Gamma_n)$ has property (τ) in H'. When the Γ_n are normal in Γ, the coset action of H' on Γ/Γ_n is fixedpoint-free and the number of its orbits equals the index $|\Gamma : H'\Gamma_n|$. □

When the Γ_n are not normal and the finite actions of Γ are far from regular, we do not expect that the index stays bounded in general. Random actions on rooted trees, in the spirit of [Gla] may give counterexamples.

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