MATCHINGS IN BENJAMINI–SCHRAMM CONVERGENT GRAPH SEQUENCES

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Abstract. We introduce the *matching measure* of a finite graph as the uniform distribution on the roots of the matching polynomial of the graph. We analyze the asymptotic behavior of the matching measure for graph sequences with bounded degree.

A graph parameter is said to be estimable if it converges along every Benjamini–Schramm convergent sparse graph sequence. We prove that the normalized logarithm of the number of matchings is estimable. We also show that the analogous statement for *perfect* matchings already fails for d–regular bipartite graphs for any fixed $d \geq 3$. The latter result relies on analyzing the probability that a randomly chosen perfect matching contains a particular edge.

However, for any sequence of d–regular bipartite graphs converging to the d–regular tree, we prove that the normalized logarithm of the number of perfect matchings converges. This applies to random d–regular bipartite graphs. We show that the limit equals to the exponent in Schrijver's lower bound on the number of perfect matchings.

Our analytic approach also yields a short proof for the Nguyen–Onak (also Elek–Lippner) theorem saying that the matching ratio is estimable. In fact, we prove the slightly stronger result that the independence ratio is estimable for claw-free graphs.

1. Introduction

In this paper, we study the asymptotic behavior of the number of matchings and perfect matchings for Benjamini–Schramm convergent sequences of finite graphs. Benjamini-Schramm convergence was introduced in [3] and has been under intense investigation since then.

For a finite graph G, a finite rooted graph α and a positive integer r, let $\mathbb{P}(G, \alpha, r)$ be the probability that the r-ball centered at a uniform random vertex of G is isomorphic to α (as a rooted graph). A sequence of finite graphs (G_n) is *sparse* if the set of degrees of vertices in G_n $(n \geq 1)$ is bounded. A sparse graph sequence (G_n) is *Benjamini–Schramm convergent* if for all finite rooted graphs α and $r > 0$, the probabilities $\mathbb{P}(G_n, \alpha, r)$ converge. This means that one cannot distinguish G_n and $G_{n'}$ for large n and n' by sampling them at a random vertex with a fixed radius of sight.

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The graph parameter $p(G)$ is *estimable* in a class $\mathcal{C}$ of finite graphs if the sequence $p(G_n)$ converges for all Benjamini–Schramm convergent sparse graph sequences (G_n) in $\mathcal{C}$. When $\mathcal{C}$ is the class of all finite graphs, we simply say that $p(G)$ is estimable.

Let $v(G)$, $e(G)$, $\mathbb{M}(G)$, and $\mathrm{pm}(G)$ stand for the number of vertices, edges, matchings, and perfect matchings in the graph G, respectively. We write $\nu(G)$ and $\alpha(G)$ for the maximal size of a matching, respectively an independent vertex set in G.

1.1. **Estimable matching parameters.** There are several examples of seemingly 'global' graph parameters that turn out to be estimable. A striking example is the following theorem of R. Lyons [19].

Theorem 1.1 (R. Lyons). *Let $\tau(G)$ denote the number of spanning trees in the graph G. Then the* tree entropy per site

$$\frac{\ln \tau(G)}{v(G)}$$

is estimable in the class of connected graphs.

Our first result shows that a similar statement is true for the number of matchings.

Theorem 1.2. *The* matching entropy per site

$$\frac{\ln \mathbb{M}(G)}{v(G)}$$

is estimable.

This will be proved as a part of Theorem 3.5. For the proof, we apply the machinery developed by the first three authors and T. Hubai in the papers [1, 8]. In particular, results in [8] show that if $f(G,x)$ is a graph polynomial satisfying certain conditions and ρ_G is the uniform distribution on the roots of $f(G,x)$, then for every fixed k, the graph parameter

$$\int z^k \mathrm{d}\rho_G$$

is estimable.

When considering the matching polynomial as $f(G,x)$, we get the definition of the *matching measure* and that the matching measure weakly converges for Benjamini–Schramm convergent sequences of graphs. This leads to Theorem 1.2. Note that a modification of the algorithm 'CountMATCHINGS' in [7] yields an alternative proof of Theorem 1.2.

Considering the *independence polynomial* as $f(G,x)$, however, also yields an extension of the following theorem of H. Nguyen and K. Onak [22] (independently proved by G. Elek and G. Lippner [9]).

Theorem 1.3 (Nguyen–Onak and Elek–Lippner). *The* matching ratio $\nu(G)/v(G)$ *is estimable.*

A graph is *claw-free* if it does not contain the complete bipartite graph $K_{1,3}$ as an induced subgraph. Our extension is the following.

Theorem 1.4. *The* independence ratio $\alpha(G)/v(G)$ *is estimable in the class of claw-free graphs.*

This will be proved as a part of Theorem 2.5. By the following argument, Theorem 1.4 indeed extends Theorem 1.3. The *line graph* $L(G)$ of the graph G has vertex set $V(L(G)) = E(G)$, and $e, f \in E(G)$ are adjacent in $L(G)$ if they share an endpoint in G. Trivially $\nu(G) = \alpha(L(G))$, so we have

$$\frac{\nu(G)}{v(G)} = \frac{\nu(G)}{e(G)} \cdot \frac{e(G)}{v(G)} = \frac{\alpha(L(G))}{v(L(G))} \cdot \frac{e(G)}{v(G)}$$

that is, the matching ratio of G equals the independence ratio of $L(G)$ times the edge density of G. The edge density is clearly estimable. Using that line graphs are claw-free, and that $(L(G_n))$ is Benjamini–Schramm convergent if (G_n) is, Theorem 1.4 implies Theorem 1.3.

Note that the independence ratio is not estimable in general. Indeed, random d-regular graphs and random d-regular *bipartite* graphs converge to the same object, the d-regular tree, but by a result of B. Bollobás [4], the independence ratio of a sequence of random d-regular graphs is bounded away from $1/2$ a.s.

1.2. **Matchings and perfect matchings in graphs with essentially large girth.** The girth $g(G)$ of the graph G is defined to be the length of the shortest cycle in G. If (G_n) is a sequence of d–regular graphs with $g(G_n) \to \infty$, then (G_n) Benjamini–Schramm converges, since every r-ball of G_n will be isomorphic to the r-ball of the d-regular tree for large enough n. More generally, we say that (G_n) is of *essentially large girth* (or converges to the d-regular tree), if for any fixed k, the number of k–cycles in G_n is $o(v(G_n))$ as $n \to \infty$. Important examples are sequences of random d–regular graphs and bipartite graphs.

The following theorem is the main result of this paper.

Theorem 1.5. *Let (G_n) be a sequence of d–regular graphs with essentially large girth. Then the following hold.*

(a) *We have*

$$\lim_{n\to\infty} \frac{\ln \mathbb{M}(G_n)}{v(G_n)} = \frac{1}{2} \ln S_d,$$

where

$$S_d = \frac{1}{\xi^2}\left(\frac{d-1}{d-\xi}\right)^{d-2}, \qquad \xi = \frac{2}{1+\sqrt{4d-3}}.$$

In particular, $S_3 = 16/5$.

(b) *For the number of perfect matchings* $\mathrm{pm}(G_n)$*, we have*

$$\limsup_{n\to\infty} \frac{\ln \mathrm{pm}(G_n)}{v(G_n)} \le \frac{1}{2} \ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right).$$

(c) *If, in addition, the graphs (G_n) are bipartite, then*

$$\lim_{n\to\infty} \frac{\ln \mathrm{pm}(G_n)}{v(G_n)} = \frac{1}{2} \ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right).$$

Theorem 1.5 is related to the following famous result of A. Schrijver [23].

Theorem 1.6 (Schrijver)**.** *For any d–regular bipartite graph G on $v(G) = 2 \cdot n$ vertices, we have*

$$\mathrm{pm}(G) \ge \left(\frac{(d-1)^{d-1}}{d^{d-2}}\right)^n.$$

In other words, for a d–regular bipartite graph G we have

$$\frac{\ln \mathrm{pm}(G)}{v(G)} \geq \frac{1}{2} \ln \left(\frac{(d-1)^{d-1}}{d^{d-2}} \right).$$

For $d = 3$, this theorem was proved by M. Voorhoeve [25]. Then A. Schrijver [23] proved it for every d. A very elegant new proof was given by L. Gurvits [14]. For a simplified version of Gurvits's proof see [17].

A. Schrijver and W. G. Valiant proved in [24] that the exponent

$$\frac{(d-1)^{d-1}}{d^{d-2}}$$

cannot be improved by showing that for a random d–regular bipartite multigraph the statement is asymptotically sharp. I. Wanless noticed in [26] that the same holds if we do not allow multiple edges. More refined results are proved by B. Bollobás and B. D. McKay [5]. Theorem 1.5 shows that the only thing that is relevant here about random d–regular bipartite graphs is that they are of essentially large girth. In particular, this makes the sharpness statement of A. Schrijver and W. G. Valiant constructive, as Part (c) of Theorem 1.5 now allows us to *construct* bipartite graphs with an asymptotically minimal number of perfect matchings. Indeed, we simply have to construct d–regular bipartite graphs with large girth, which is known to be possible in various ways. See for instance [10] for constructing d–regular graphs with large girth and note that if G is d–regular, then the (weak) direct product $G \times K_2$ is d–regular bipartite and satisfies $g(G \times K_2) \geq g(G)$.

1.3. **Perfect matchings with no assumption on girth.** It is natural to ask whether Theorem 1.2 holds for the number $\mathrm{pm}(G)$ of perfect matchings rather than the number $\mathbb{M}(G)$ of all matchings. It is easy to see that in the class of all graphs, the perfect matching entropy per site

$$\frac{\ln \mathrm{pm}(G)}{v(G)}$$

is not estimable. Indeed, one can consider a large graph with many perfect matchings and then add an isolated vertex to it. Then the two graphs are very close in local statistics, but the latter graph has no perfect matching. This is of course a quite cheap example. On the other hand, it turns out that the situation does not get much better even for the class of d–regular bipartite graphs.

Notation. Given a finite graph admitting at least one perfect matching, and an edge e, let $p(e)$ denote the probability that e is contained in a uniformly chosen random perfect matching of the graph.

We shall prove that perfect matchings can get quite unevenly distributed.

Theorem 1.7. *For any integer $d \geq 3$, there exists a constant $0 < c < 1$ such that for any positive integer $n \geq d$ there exists a d–regular bipartite simple graph on $2 \cdot n$ points with an edge e such that*

$$p(e) > 1 - c^n.$$

This leads to the following.

Theorem 1.8. *Fix $d \geq 3$. The perfect matching entropy per site*

$$\frac{\ln \mathrm{pm}(G)}{v(G)}$$

is not estimable in the class of d–regular bipartite simple graphs.

The phenomenon in Theorem 1.8 does not occur for *expander* bipartite graphs. Indeed, it can be deduced from Corollary 1 of D. Gamarnik and D. Katz [12] that for any $\delta > 0$, the perfect matching entropy per site is estimable for d–regular bipartite δ-expander graphs. We thank D. Gamarnik for pointing this out for us. The phenomenon in Theorem 1.7 cannot occur either for expander bipartite graphs: an edge probability cannot be exponentially close to 1. In fact, we shall prove the following stronger statement about edge probabilities.

Theorem 1.9. *Let $n \geq 2$, $\delta > 0$, let G be a δ-expander bipartite graph of maximum degree d on $2 \cdot n$ vertices, and e an edge of G. Assume that G admits a perfect matching. Then*

$$p(e) \geq \frac{1}{d} n^{-2 \cdot \ln(d-1)/\ln(1+\delta)}.$$

1.4. **Structure of the paper.** The paper is organized as follows. In Section 2, we gather a few known results about the independence polynomial and prove Theorem 2.5 about independent vertex sets in claw-free graphs. The reader only interested in matchings can skip this section without harm.

In Section 3, we gather some known results about the matching polynomial and prove Theorem 3.5 about matchings. In Section 4, we prove Theorem 1.5 about matchings in essentially large girth graphs. In Section 5, we prove the negative results: Theorem 1.7 and Theorem 1.8. In Section 6 we give the proof of Theorem 1.9 about expanders. Finally, in Section 7 we pose some open problems.

2. Independent sets in claw-free graphs

2.1. The independence polynomial.

Definition 2.1. Let G be a graph on $v(G)$ vertices. Let $\alpha(G)$ be the maximal size of an independent vertex set, and let $i_k(G)$ denote the number of independent sets of size k. Then the *independence polynomial* $I(G, x)$ is defined as follows:

$$I(G, x) = \sum_{k=0}^{\alpha(G)} i_k(G) x^k.$$

Note that $i_0(G) = 1$. The *independence measure* σ_G is defined as

$$\sigma_G = \frac{1}{v(G)} \sum_{\lambda} r(G, \lambda) \delta_\lambda$$

where λ runs through the roots of $I(G, x)$, $r(G, \lambda)$ is the multiplicity of λ as a root of $I(G, x)$ and δ_λ denotes the Dirac measure at λ.

Note that unless G is the empty graph, the independence measure is not a probability measure.

Many graph parameters related to independent sets can be read off from the independence measure.

Definition 2.2. For a finite graph G let κ_G denote the size of a uniform random independent subset of G.

So κ_G is a random variable depending on G.

Besides the number of all independent sets

$$\mathbb{I}(G) = I(G,1) = \sum_{k=0}^{\alpha(G)} i_k(G)$$

we shall also be interested in the expected size

$$\mathbb{E}\kappa_G = \frac{\sum k i_k(G)}{\sum i_k(G)}$$

and the variance

$$\mathbb{D}^2\kappa_G = \frac{\sum (k - \mathbb{E}\kappa_G)^2 i_k(G)}{\sum i_k(G)}.$$

Proposition 2.3. (a) *For the* independent set entropy per vertex, *we have*

$$\frac{\ln \mathbb{I}(G)}{v(G)} = \int \ln|1-x| \mathrm{d}\sigma_G(x).$$

(b) *The normalized expected value*

$$\frac{\mathbb{E}\kappa_G}{v(G)} = \int \frac{1}{1-x} d\sigma_G(x).$$

(c) *The normalized variance*

$$\frac{\mathbb{D}^2\kappa_G}{v(G)} = \int \frac{-x}{(1-x)^2} \mathrm{d}\sigma_G(x).$$

(d) *The independence ratio of G equals*

$$\alpha(G)/v(G) = \sigma_G(\mathbb{C}).$$

Proof. (a) We have

$$\mathbb{I}(G) = \sum_{k=0}^{\alpha(G)} i_k(G) = I(G,1).$$

Thus,

$$\frac{\ln \mathbb{I}(G)}{v(G)} = \frac{\ln |I(G,1)|}{v(G)} = \int \ln|1-x| \mathrm{d}\sigma_G(x).$$

(b) We have

$$\frac{\mathbb{E}\kappa_G}{v(G)} = \frac{1}{v(G)} \frac{\sum k i_k(G)}{\sum i_k(G)} = \frac{1}{v(G)} \frac{I'(G,1)}{I(G,1)} = \int \frac{\mathrm{d}\sigma_G(x)}{1-x}.$$

(c) Let $\lambda_1, \dots, \lambda_{\alpha(G)}$ be the roots of the polynomial $I(G,x)$. We have

$$\left(\sum \frac{1}{1-\lambda_i}\right)^2 = \sum \frac{1}{(1-\lambda_i)^2} + \sum_{i\neq j} \frac{1}{(1-\lambda_i)(1-\lambda_j)},$$

i.e.,

$$\left(v(G) \int \frac{\mathrm{d}\sigma_G(x)}{1-x}\right)^2 = v(G) \int \frac{\mathrm{d}\sigma_G(x)}{(1-x)^2} + \frac{I''(G,1)}{I(G,1)},$$

in other words,

$$(\mathbb{E}\kappa_G)^2 = v(G)\int \frac{\mathrm{d}\sigma_G(x)}{(1-x)^2} + \mathbb{E}(\kappa_G(\kappa_G - 1)),$$

so

$$\mathbb{D}^2\kappa_G = \mathbb{E}\kappa_G^2 - (\mathbb{E}\kappa_G)^2 = \mathbb{E}\kappa_G - v(G)\int \frac{\mathrm{d}\sigma_G(x)}{(1-x)^2},$$

and the claim follows using statement (b).

(d) Obvious from the definition.

□

To study the behaviour of the independence measure in a convergent graph sequence, we need to have some control on the location of the roots in terms of the greatest degree in a graph. It follows from Dobrushin's lemma that all roots of $I(G,x)$ have absolute value greater than

$$\beta := \frac{\exp(-1)}{d+1},$$

where d is the greatest degree in G, cf. [8, Corollary 5.10].

The following lemma has its roots in [18], see also [2].

Lemma 2.4. *For all $R > 1$ we have*

$$\sigma_G(|x| \geq R) \leq \frac{\ln(1/\beta)}{\ln R}. \tag{2.1}$$

Proof. The product of the roots of $I(G,x)$ is, in absolute value, equals $1/i_{\alpha(G)}(G) \leq 1$. Thus, for any $R > 1$ we have

$$R^{\sigma_G(|x|\geq R)}\beta \leq 1,$$

which proves the lemma. □

2.2. **Claw-free graphs.** When G is claw-free, all roots of $I(G,x)$ are real by [6].

The following theorem deals with the behaviour of the independence polynomial in Benjamini–Schramm convergent sequences of claw-free graphs.

Theorem 2.5. *Let (G_n) be a Benjamini–Schramm convergent claw-free graph sequence with absolute degree bound d. Set $H = (-\infty, -\beta]$. Then the sequence of independence measures $\sigma_n = \sigma_{G_n}$ converges weakly to a measure σ on H. As $n \to \infty$, we have*

$$\frac{\ln \mathbb{I}(G_n)}{v(G_n)} \to \int_H \ln(1-x)\mathrm{d}\sigma(x),$$

$$\frac{\mathbb{E}\kappa_{G_n}}{v(G_n)} \to \int_H \frac{\mathrm{d}\sigma(x)}{1-x},$$

$$\frac{\mathbb{D}^2\kappa_{G_n}}{v(G_n)} \to \int_H \frac{-x}{(1-x)^2}\mathrm{d}\sigma(x),$$

and

$$\alpha(G_n)/v(G_n) \to \sigma(H).$$

In particular, $(\ln \mathbb{I})/v$, $\mathbb{E}\kappa/v$, $\mathbb{D}^2\kappa/v$ and α/v are estimable graph parameters for claw-free graphs.

Note that this recovers Theorem 1.4.

Proof. We consider the graph polynomial

$$f(G,x)=x^{v(G)}I(G,1/x)=\sum_{k=0}^{\alpha(G)} i_k(G)x^{v(G)-k}.$$

Let $\tau=\tau_G$ be the probability measure of uniform distribution on the roots of $f(G,x)$. For G claw-free with greatest degree d, this measure is supported on $K=[-1/\beta,0]$. The graph polynomial $f(G,x)$ is monic of degree $v(G)$, and it is multiplicative with respect to disjoint union of graphs because

$$i_k(G)=\sum_{k_1+k_2=k} i_{k_1}(G_1)i_{k_2}(G_2)$$

whenever $G=G_1\cup G_2$ is a disjoint union. The coefficient $i_k(G)$ of $x^{v(G)-k}$ is the number of induced subgraphs of G that are isomorphic to the empty graph on k points. By the well-known and easy [8, Fact 3.2], this can be expressed as a finite linear combination

$$i_k(G)=\sum_H c_{H,k}H(G),$$

where $H(G)$ is the number of (not necessarily induced) subgraphs of G that are isomorphic to H.

Note that $\mathbb{C}\setminus K$ is connected and K has empty interior (as a subset of $\mathbb{C}$). By [8, Theorem 4.6(a)], it follows that the sequence $\int_K g\mathrm{d}\tau_{G_n}$ converges for all continuous $g:K\to\mathbb{R}$. Thus, the sequence $\int_H g\mathrm{d}\sigma_n$ converges for any continuous $g:H\to\mathbb{R}$ that tends to zero at $-\infty$. Using (2.1), we see that this last decay assumption may be dropped, so σ_n converges weakly.

Since $1\notin H$, we have

$$\frac{\ln\mathbb{I}(G_n)}{v(G_n)}=\int_H \ln(1-x)\mathrm{d}\sigma_n\to\int_H \ln(1-x)\mathrm{d}\sigma.$$

The other statements follow from Proposition 2.3 the same way. □

3. Matching polynomial and Benjamini–Schramm convergence

Definition 3.1. Let G be a graph on $v(G)=v$ vertices and let $m_k(G)$ denote the number of matchings of size k. Then the *matching polynomial* $\mu(G,x)$ is defined as follows:

$$\mu(G,x)=\sum_{k=0}^{\lfloor v/2\rfloor}(-1)^k m_k(G)x^{v-2k}.$$

Note that $m_0(G)=1$. The *matching measure* ρ_G is defined as

$$\rho_G=\frac{1}{v(G)}\sum_\lambda r(G,\lambda)\delta_\lambda$$

where λ runs through the roots of $\mu(G,x)$, $r(G,\lambda)$ is the multiplicity of λ as a root of $\mu(G,x)$ and δ_λ denotes the Dirac measure at λ.

Let γ_G denote the number of edges in a uniform random matching of G.

Remark 3.2. Let $L(G)$ be the line graph of G. Then $m_k(G)=i_k(L(G))$, so

$$\mu(G,x)=x^{v(G)}I(L(G),-1/x^2).$$

Therefore, if x runs over the nonzero roots of $\mu(G,x)$, then $-1/x^2$ runs over the roots of $I(L(G),x)$ twice. Remember that $\sigma_{L(G)}$ assigns weight $1/v(L(G)) = 1/e(G)$ times the multiplicity to each root of $I(L(G),x)$, while ρ_G assigns weight $1/v(G)$ times the multiplicity to each root of $\mu(G,x)$. Thus, for any function g defined on the roots of $I(L(G),x)$, we have

$$2e(G)\int g(x)\mathrm{d}\sigma_{L(G)}(x) = v(G)\int_{x\neq 0} g(-1/x^2)\mathrm{d}\rho_G(x).$$

Using Remark 3.2, almost all results in this section follow from their counterparts in Section 2. Converting the results requires about the same amount of work as redoing the proofs. In some cases we will do the latter for the convenience of the reader who is only interested in matchings and therefore skipped Section 2.

The fundamental theorem for the matching polynomial is the following.

Theorem 3.3 (Heilmann and Lieb [16]). (a) *The roots of the matching polynomial $\mu(G,x)$ are real.*

(b) *If $d \geq 2$ is an upper bound for all degrees in G, then all roots of $\mu(G,x)$ have absolute value $\leq 2\sqrt{d-1}$.*

Many graph parameters related to matchings can be read off from the matching measure. Besides the number

$$\mathbb{M}(G) = \sum_{k=0}^{\alpha(G)} m_k(G)$$

of all matchings, we shall also be interested in the expectation

$$\mathbb{E}\gamma_G = \frac{\sum k m_k(G)}{\sum m_k(G)}$$

and also in the variance

$$\mathbb{D}^2\gamma_G = \frac{\sum (k - \mathbb{E}_G\gamma_G)^2 m_k(G)}{\sum m_k(G)}.$$

Proposition 3.4. (a) *For the* matching entropy per vertex*, we have*

$$\frac{\ln \mathbb{M}(G)}{v(G)} = \frac{1}{2}\int \ln(1+x^2)\mathrm{d}\rho_G(x).$$

The normalized expected value of γ_G equals

$$\frac{\mathbb{E}_G\gamma_G}{v(G)} = \frac{1}{2}\int \frac{x^2}{1+x^2}\mathrm{d}\rho_G(x).$$

The normalized variance of γ_G equals

$$\frac{\mathbb{D}^2\gamma_G}{v(G)} = \frac{1}{2}\int \frac{x^2}{(1+x^2)^2}\mathrm{d}\rho_G(x).$$

The matching ratio equals

$$\frac{\nu(G)}{v(G)} = \frac{1-\rho_G(\{0\})}{2}.$$

(b) *For the* perfect matching entropy per vertex*, we have*

$$\frac{\ln \mathrm{pm}(G)}{v(G)} = \int \ln|x|\mathrm{d}\rho_G(x).$$

Proof. (a) All statements follow from Proposition 2.3 and Remark 3.2. However, we give direct proofs for the first two statements.

The number of matchings in G is

$$\mathbb{M}(G) = \sum_{k=0}^{\lfloor v/2 \rfloor} m_k(G) = |\mu(G, \sqrt{-1})|.$$

Thus,

$$\frac{\ln \mathbb{M}(G)}{v(G)} = \frac{\ln |\mu(G, \sqrt{-1})|}{v(G)} = \int \ln |\sqrt{-1} - x| \mathrm{d}\rho_G(x) = \frac{1}{2} \int \ln(1 + x^2) \mathrm{d}\rho_G(x).$$

We have

$$\frac{\mathbb{E}\gamma_G}{v(G)} = \frac{1}{v(G)} \frac{\sum k m_k(G)}{\sum m_k(G)} = \frac{1}{2}\left(1 - \frac{1}{v(G)} \frac{|\mu'(G, \sqrt{-1})|}{|\mu(G, \sqrt{-1})|}\right) =$$
$$= \frac{1}{2}\left(1 - \sqrt{-1} \int \frac{\mathrm{d}\rho_G(x)}{\sqrt{-1} - x}\right) = \frac{1}{2} \int \frac{x^2}{1 + x^2} \mathrm{d}\rho_G(x).$$

The statement for the normalized variance is straightforward from Proposition 2.3(c) using Remark 3.2.

(b) The number of perfect matchings in G equals

$$\mathrm{pm}(G) = |\mu(G, 0)|.$$

Thus,

$$\frac{\ln \mathrm{pm}(G)}{v(G)} = \frac{\ln |\mu(G, 0)|}{v(G)} = \int \ln |x| \mathrm{d}\rho_G(x).$$

□

The following theorem deals with the behaviour of the matching measure in a Benjamini–Schramm convergent graph sequence.

Theorem 3.5. *Let (G_n) be a Benjamini–Schramm convergent graph sequence with absolute degree bound $d \geq 2$. Set $\omega = 2\sqrt{d-1}$ and $K = [-\omega, \omega]$. Then the sequence of matching measures $\rho_n = \rho_{G_n}$ converges weakly to a probability measure ρ on K. Moreover, we have*

(a)

$$\frac{\ln \mathbb{M}(G_n)}{v(G_n)} \to \frac{1}{2} \int_K \ln(1 + x^2) \mathrm{d}\rho(x),$$
$$\frac{\mathbb{E}\gamma_{G_n}}{v(G_n)} \to \frac{1}{2} \int_K \frac{x^2}{1 + x^2} \mathrm{d}\rho(x),$$
$$\frac{\mathbb{D}^2\gamma_{G_n}}{v(G_n)} \to \frac{1}{2} \int \frac{x^2}{(1 + x^2)^2} \mathrm{d}\rho(x),$$
$$\frac{\nu(G_n)}{v(G_n)} \to \frac{1 - \rho(\{0\})}{2}.$$

In particular, $(\ln \mathbb{M})/v$, $\mathbb{E}\gamma/v$, $\mathbb{D}^2\gamma/v$ and ν/v are estimable graph parameters.

(b)

$$\limsup_{n \to \infty} \frac{\ln \mathrm{pm}(G_n)}{v(G_n)} \leq \int_K \ln |x| \mathrm{d}\rho(x).$$

Note that part (a) recovers Theorem 1.3.

Proof. The matching polynomial $\mu(G,x)$ is monic of degree $v(G)$ and multiplicative with respect to disjoint union of graphs. The coefficient of $x^{v(G)-k}$ is of the form $c_k H_k(G)$, where c_k is a constant, H_k is a graph, and $H_k(G)$ is the number of subgraphs isomorphic to H_k in G. These are the only properties we need besides the Heilmann–Lieb Theorem.

Note that $\mathbb{C}\setminus K$ is connected and K has empty interior (as a subset of $\mathbb{C}$). By [8, Theorem 4.6(a)], it follows that the sequence $\int_K g \mathrm{d}\rho_n$ converges for all continuous $g: K \to \mathbb{R}$, i.e., ρ_n converges weakly to a measure ρ.

(a) Since $\ln(1+x^2)$ is continuous on K, we have

$$\frac{\ln \mathbb{M}(G_n)}{v(G_n)} = \frac{1}{2}\int_K \ln(1+x^2)\mathrm{d}\rho_n \to \frac{1}{2}\int_K \ln(1+x^2)\mathrm{d}\rho.$$

The statements for the expectation and variance follow from Proposition 3.4 in the same way.

From Remark 3.2 and formula (2.1), we see that

$$\sup_n \rho_n(0<|x|\le \delta) \to 0$$

as $\delta \to 0$. Therefore, $\rho_n(\{0\}) \to \rho(\{0\})$ as $n\to\infty$. Thus, the statement for the matching ratio follows from Proposition 3.4.

(b) Let $u(x) = \ln|x|$ and $u_k(x) = \max(u(x), -k)$ for $k = 1, 2, \dots$. Then

$$\frac{\ln \mathrm{pm}(G)}{v(G)} = \int u \mathrm{d}\rho_G \le \int u_k \mathrm{d}\rho_G \qquad (k=1,2,\dots).$$

Thus,

$$\limsup_{n\to\infty} \frac{\ln \mathrm{pm}(G_n)}{v(G_n)} \le \lim_{n\to\infty}\int_K u_k \mathrm{d}\rho_n = \int_K u_k \mathrm{d}\rho \qquad (k=1,2,\dots),$$

since the measures ρ_n are supported on the compact interval K not depending on n, and u_k is continuous and bounded on K.

Since $u_k \ge u_{k+1}$ and $u_k \to u$ pointwise, the claim follows using the Monotone Convergence Theorem. □

Remark 3.6. An alternative proof for the weak convergence of ρ_n is possible. Indeed, there is a very nice interpretation of the k-th power sum $p_k(G)$ of the roots of the matching polynomial. It counts the number of closed tree-like walks of length k in the graph G: see chapter 6 of [13]. We don't go into the details of 'tree-like walks'; all we need is that these are special type of walks, consequently we can count them by knowing all $(k/2)$-balls centered at the vertices of the graph G. In particular, this implies that for all k, the sequence $p_k(G_n)/v(G_n)$ is convergent, and the weak convergence of ρ_n follows.

Remark 3.7. One can ask whether estimability of a certain graph parameter actually means that one can get *explicit* estimates on the parameter from knowing the R-neighborhood statistics of a finite graph for some large R. In general, this is not clear. However, using Lemma 2.4, one can indeed get such estimates. For instance, when G is d-regular and has girth at least R, its matching measure has the same first R moments as the matching measure of H, where H is a d-regular bipartite graph with girth at least R. Since $\rho_H(\{0\}) = 0$, by Lemma 2.4 we get that the matching ratio of G is at least $1/2 - O(1/\ln R)$. Of course, this is a weak estimate but is obtained by purely analytic means.

4. Graphs with large girth

In this section we study d–regular graphs with large girth, in particular, we prove Theorem 4.5. We start by looking at matching measures of regular graphs with essentially large girth. Recall that (G_n) essentially has large girth if, for all k, the number of k–cycles is $o(v(G_n))$.

Theorem 4.1. *Let (G_n) be a sequence of d–regular graphs with essentially large girth. Then ρ_{G_n} converges weakly to the measure with density function*

$$f_d(x) = \frac{d\sqrt{4(d-1)-x^2}}{2\pi(d^2-x^2)}\chi_{[-2\sqrt{d-1},2\sqrt{d-1}]}.$$

From now on, we follow the notations of B. McKay. Let

$$\omega = 2\sqrt{d-1},$$

and

$$f_d(x) = \frac{d\sqrt{\omega^2-x^2}}{2\pi(d^2-x^2)}\chi_{[-\omega,\omega]}$$

as in Theorem 4.1.

The proof of Theorem 4.1 will be an easy application of the following lemma.

Lemma 4.2. [13] *Let G be a graph and let $\phi(G,x)$ denote the characteristic polynomial of its adjacency matrix. Let $\mathcal{C}$ denote the set of two-regular subgraphs of G, i.e., these subgraphs are disjoint union of cycles. For $C \in \mathcal{C}$, let $k(C)$ denote the number of components of C. Then*

$$\phi(G,x) = \sum_{C\in\mathcal{C}}(-2)^{k(C)}\mu(G\setminus C,x).$$

Proof of Theorem 4.1. First let (G_n) be a graph sequence for which the girth $g(G_n)\to\infty$. If $g(G) > k$, then Lemma 4.2 implies that the first k coefficients of $\phi(G,x)$ and $\mu(G,x)$ coincide. This implies that the first k moments of the uniform distribution arising from the roots of $\phi(G,x)$ and $\mu(G,x)$ coincide too. Since $g(G_n)\to\infty$, this means that for any fixed k, the moments arising from $\phi(G,x)$ and $\mu(G,x)$ converge to the same limit, actually the two sequences are the same for large enough values. Then the statement of the theorem follows from B. McKay's work [20] on the spectral distribution of random graphs.

In the general case, consider an auxiliary graph sequence (H_n) of d–regular graphs such that $g(H_n)\to\infty$. The sequence G_1, H_1, G_2, H_2, ... is Benjamini–Schramm convergent, and the theorem follows using Theorem 3.5. □

We shall need the following lemma, implicit in McKay's work, on the density function $f_d(x)$.

Lemma 4.3. *Let γ be a complex number that does not lie on either of the real half-lines $(-\infty,-1/\omega]$ and $[1/\omega,\infty)$. Set*

$$\eta = \frac{1-\sqrt{1-4(d-1)\gamma^2}}{2(d-1)\gamma^2} = \frac{2}{1+\sqrt{1-\omega^2\gamma^2}}.$$

Then

$$\int_{-\omega}^{\omega} f_d(x)\ln(1-\gamma x)\mathrm{d}x = \frac{d-2}{2}\ln\left(\frac{d-1}{d-\eta}\right) - \ln\eta \tag{4.1}$$

and

$$\int_{-\omega}^{\omega} f_d(x)\frac{\mathrm{d}x}{1-\gamma x} = \frac{d-2-d\sqrt{1-\omega^2\gamma^2}}{2(d^2\gamma^2-1)} = \frac{d-1}{(d/\eta)-1}. \tag{4.2}$$

We also have

$$\int_{-\omega}^{\omega} f_d(x)\frac{x}{(1-\sqrt{-1}x)^2}\mathrm{d}x = \frac{8d(d-1)^2\sqrt{-1}}{2(d-2)(4d-3)+(d^2+1)\sqrt{4d-3}}. \tag{4.3}$$

Note that we use the principal branch of the square root and the logarithm function.

Proof. Since both sides of (4.1) are holomorphic in γ, we may assume that $|\gamma| < 1/\omega$. Then formula (4.1) is a special case of [21, Lemma 3.5].

For (4.2) we may write $1/(1-\gamma x) = \sum_{i=0}^{\infty}\gamma^i x^i$. This reduces the statement to [21, Theorem 2.3(a)]. Alternatively, we could differentiate (4.1) with respect to γ and use the fact that f_d is a density function to get (4.2).

The formula (4.3) is the derivative with respect to γ of (4.2) at the point $\gamma = \sqrt{-1}$. Indeed, at that point

$$\left(\frac{1}{\eta}\right)' = -2\sqrt{-1}\frac{d-1}{\sqrt{4d-3}},$$

so

$$\left(\frac{d}{\eta}-1\right)' = -2\sqrt{-1}\frac{d(d-1)}{\sqrt{4d-3}},$$

while

$$\frac{d}{\eta}-1 = \frac{d-2+\sqrt{4d-3}}{2},$$

so

$$\left(\frac{d}{\eta}-1\right)^2 = \frac{d^2+1+2(d-2)\sqrt{4d-3}}{4},$$

whence

$$\sqrt{4d-3}\left(\frac{d}{\eta}-1\right)^2 = \frac{2(d-2)(4d-3)+(d^2+1)\sqrt{4d-3}}{4},$$

and the claim follows. ☐

Lemma 4.4.

$$\int_{-\omega}^{\omega} f_d(x)\ln|x|\mathrm{d}x = \frac{1}{2}\ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right).$$

Proof. Let γ be purely imaginary. We note that

$$\ln|1-\gamma x| - \ln|\gamma| \to \ln|x|$$

monotonously as $\gamma \to +\infty\sqrt{-1}$. So, in the integral, we may replace $\ln|x|$ by this difference and then take the limit. It is easy to check that $\eta \to 0$ and $|\gamma||\eta| \to 1/\sqrt{d-1}$. This implies the statement of the lemma using the real part of formula (4.1) from the previous lemma. ☐

We are ready to prove our main result.

Theorem 4.5. *Let (G_n) be a sequence of d–regular graphs with essentially large girth.*

(a) *We have*

$$\frac{\ln \mathbb{M}(G_n)}{v(G_n)} \to \frac{1}{2}\ln S_d,$$

where

$$S_d = \frac{1}{\xi^2}\left(\frac{d-1}{d-\xi}\right)^{d-2}, \qquad \xi = \frac{\sqrt{4d-3}-1}{2(d-1)} = \frac{2}{1+\sqrt{4d-3}}.$$

In particular, $S_3 = 16/5$.

For the expected size of a uniform random matching we have

$$\frac{\mathbb{E}\gamma_{G_n}}{v(G_n)} \to \frac{d}{2}\cdot\frac{1-\xi}{d-\xi}.$$

For $d = 3$*, this limit is* $3/10$.

For the variance, we have

$$\frac{\mathbb{D}^2\gamma_{G_n}}{v(G_n)} \to \frac{2d(d-1)^2}{2(d-2)(4d-3)+(d^2+1)\sqrt{4d-3}}.$$

For $d = 3$*, this limit is* $1/2$.

(b) *For the number of perfect matchings* $\mathrm{pm}(G_n)$*, we have*

$$\limsup_{n\to\infty}\frac{\ln \mathrm{pm}(G_n)}{v(G_n)} \le \frac{1}{2}\ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right).$$

(c) *If, in addition, the graphs* (G_n) *are bipartite, then*

$$\lim_{n\to\infty}\frac{\ln \mathrm{pm}(G_n)}{v(G_n)} = \frac{1}{2}\ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right).$$

Proof. (a) By Theorem 4.1 we know that ρ_{G_n} converges weakly to the measure ρ with density function $f_d(x)$. Put $\gamma = \sqrt{-1}$ into Lemma 4.3, then η becomes the ξ of Theorem 4.5 and we are done by Theorem 3.5(a), since

$$\frac{1}{2}\ln(1+x^2) = \Re\ln(1-\sqrt{-1}x),$$

$$\frac{x^2}{1+x^2} = 1-\Re\frac{1}{1-\sqrt{-1}x},$$

and

$$\frac{x^2}{(1+x^2)^2} = \frac{1}{2}\Im\frac{x}{(1-\sqrt{-1}x)^2}.$$

(b) This statement immediately follows from Theorem 3.5(b) and Lemma 4.4.

(c) The claim follows from part (b) and Schrijver's theorem (Theorem 1.6). □

Remark 4.6. Friedland's Lower Matching Conjecture (LMC) [11] asserts that if G is a d–regular bipartite graph on $v(G) = 2\cdot n$ vertices and $m_k(G)$ denotes the number of k-matchings as before, then

$$m_k(G) \ge \binom{n}{k}^2\left(\frac{d-t}{d}\right)^{(d-t)n}(td)^{tn},$$

where $t = k/n$. This is still open, but an asymptotic version, Friedland's Asymptotic Lower Matching Conjecture, was proved by L. Gurvits [15]. Namely, if G is a d–regular bipartite graph on $2 \cdot n$ vertices, then

$$\frac{\ln m_k(G)}{n} \geq t \ln\left(\frac{d}{t}\right) + (d-t)\ln\left(1 - \frac{t}{d}\right) - 2(1-t)\ln(1-t) + o_n(1),$$

where $t = k/n$. Note that Gurvits's result implies that for any d–regular bipartite graph G, we have

$$\frac{\ln \mathbb{M}(G)}{v(G)} \geq \frac{1}{2}\ln S_d.$$

Indeed, the maximum of the function

$$t \ln\left(\frac{d}{t}\right) + (d-t)\ln\left(1 - \frac{t}{d}\right) - 2(1-t)\ln(1-t)$$

is $\ln S_d$, and if we apply the statement to the disjoint union of many copies of the graph G, then the $o_n(1)$ term will disappear. This shows that large girth graphs have an asymptotically minimal number of matchings among bipartite graphs.

5. Negative results: perfect matchings of bipartite graphs

It is easy to show that the *perfect matching entropy per vertex,* defined as

$$\frac{\ln \mathrm{pm}(G)}{v(G)},$$

is not an estimable graph parameter since sampling cannot distinguish a large graph with many perfect matchings from the same large graph with an additional isolated vertex (which has no perfect matchings). We shall show that even if we consider d–regular bipartite graphs, the situation does not change.

Notation. Given a finite graph G admitting at least one perfect matching, and given an edge e of G, let $p(e)$ be the probability that a uniform random perfect matching contains e.

Construction 5.1. Let G be a d–regular bipartite graph. Recall the following well-known construction of an n-fold cover of G.

Consider n disjoint copies of G, erase all n copies of the edge $e = \{x, y\}$, and restore d–regularity by adding n new edges: connect each copy of x to the copy of y in the (cyclically) next copy of G. This gives us a graph G'.

We now calculate how edge probabilities are transformed. If we had $p(e) = 1/(x+1)$ in G, then, for each new edge e', we have

$$p(e') = \frac{p(e)^n}{p(e)^n + (1-p(e))^n} = \frac{1}{x^n + 1}$$

in G'. This is because any perfect matching of G' contains either all new edges or none, and the perfect matchings of these two types are in obvious bijections with n-tuples of perfect matchings of G containing, respectively not containing e.

Let $f \in E(G)$ be an edge adjacent to e. Let $f' \in E(G')$ be the corresponding edge adjacent to e'. Then

$$p(f') = p(f)\frac{1-p(e')}{1-p(e)} = p(f)\frac{x^{n-1}(x+1)}{x^n+1}.$$

For $i = 1, 2, \ldots, d$, define a map $T_i^{(n)}$ from $(0, 1]^d$ to itself as follows. If a vector has i-th coordinate equal to $1/(x+1)$ and j-th coordinate equal to y_j for $j \neq i$, then the image will have i-th coordinate equal to $1/(x^n + 1)$ and j-th coordinate equal to $y_j x^{n-1}(x+1)/(x^n+1)$ for $j \neq i$.

Let $f_1, \ldots, f_d$ be the edges emanating from one end of e, so that $f_i = e$ for one index i. Consider the corresponding edges $f'_1, \ldots, f'_d$ emanating from one of the two corresponding vertices of G', one of these edges being the new edge $f'_i = e'$. Then

$$(p(f'_1), \ldots, p(f'_d)) = T_i^{(n)}(p(f_1), \ldots, p(f_d)).$$

We wish to construct regular bipartite graphs with irregular edge probabilities. As a warm-up, we construct a graph that has a very improbable edge. This will not be formally needed for the sequel.

Theorem 5.2. (a) *For any integers $d \geq 1$ and $n \geq 0$, there exists a d–regular bipartite graph on $2 \cdot n$ points with an edge e such that*

$$p(e) = \frac{1}{(d-1)^n + 1}.$$

(b) *For any integers $d \geq 1$ and $n \geq 0$, there exists a d–regular bipartite simple graph on $2 \cdot dn$ points with an edge e such that $p(e)$ is as in (a) above.*

Proof. Apply Construction 5.1 starting from the d–regular bipartite graph on two points for part (a) and from $K_{d,d}$ for part (b). □

We now take up the opposite task: producing a high edge probability.

Lemma 5.3. *For any integer $d \geq 3$, there exists a d–regular bipartite simple graph with an edge e such that $p(e) > 1/2$.*

Proof. It suffices to prove that there exist positive integers $n_2, n_3, \ldots, n_d$ such that the first coordinate of

$$T_2^{(n_2)} T_3^{(n_3)} \cdots T_d^{(n_d)} \left(\frac{1}{d}, \ldots, \frac{1}{d}\right)$$

is larger than $1/2$.

For this, we use induction on d. For $d = 3$, a direct calculation shows that the first coordinate of

$$T_2^{(3)} T_3^{(2)} \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right)$$

is $18/35 > 1/2$.

Suppose that for some $d \geq 3$, the positive integers $n_2, n_3, \ldots, n_d$ have the desired property. Note that

$$T_{d+1}^{(n)} \left(\frac{1}{d+1}, \ldots, \frac{1}{d+1}, \frac{1}{d+1}\right) \to \left(\frac{1}{d}, \ldots, \frac{1}{d}, 0\right)$$

as $n \to \infty$. Note also that all maps $T_i^{(n)}$ (in dimension d) are continuous. Thus, if n_{d+1} is large enough, then the sequence $n_2, n_3, \ldots, n_d, n_{d+1}$ will have the desired property for $d+1$. □

We now prove Theorem 1.7; we repeat the statement.

Theorem 1.7. *For any integer $d \geq 3$, there exists a constant $0 < c < 1$ such that for any positive integer $n \geq d$ there exists a d–regular bipartite simple graph on $2 \cdot n$ points with an edge e such that*

$$p(e) > 1 - c^n.$$

Proof. Let G_0 and e_0 be the graph and the edge given by the Lemma, with G_0 having $2 \cdot n_0$ vertices. We can write

$$p(e_0) = \frac{1}{1 + c^{n_0}},$$

where $0 < c < 1$. If $n = rn_0$ with an integer r, then we can apply Construction 5.1 with r in place of n to get a d–regular bipartite simple graph on $2 \cdot n$ vertices, having an edge e such that

$$p(e) = \frac{1}{1 + c^{n_0 r}} = \frac{1}{1 + c^n} > 1 - c^n,$$

so the statement holds for all n divisible by n_0. By changing c if necessary, we can achieve that it hold for all integers $n \geq d$. □

Starting from a graph G as in this Theorem, we shall produce two d–regular bipartite graphs (denoted $2G$ and $\tilde{G}$) that share a common vertex set and differ only in two edges, but have a very different number of perfect matchings.

Theorem 1.8. *Fix $d \geq 3$. Then, for d–regular bipartite simple graphs, the graph parameter $(\ln \mathrm{pm})/v$ is not estimable.*

Proof. For $n \geq d$, let $G = G_n$ and $e = e_n$ be the graph and the edge given by the previous Theorem, with G having $2 \cdot n$ vertices and $p(e) > 1 - c^n$. Let f be an edge adjacent to e, so that $p(f) \leq 1 - p(e) < c^n$. Let $2G = 2G_n$ denote the disjoint union of two copies of G, so that $\mathrm{pm}(2G) = \mathrm{pm}(G)^2$. Let $\tilde{G} = \tilde{G}_n$ be the graph $2G$ with e erased from the first copy of G and f erased from the second copy, and with d–regularity restored by two new edges going across. We have

$$\begin{aligned} \mathrm{pm}(\tilde{G}) &= \mathrm{pm}(G)^2 \left(p(e)p(f) + (1 - p(e))(1 - p(f))\right) \leq \\ &\leq \mathrm{pm}(2G)(p(f) + 1 - p(e)) < \mathrm{pm}(2G) \cdot 2c^n. \end{aligned}$$

Thus,

$$\frac{\ln \mathrm{pm}(2G)}{v(2G)} - \frac{\ln \mathrm{pm}(\tilde{G})}{v(\tilde{G})} > -\frac{\ln(2c^n)}{4n} \to \frac{1}{4} \ln \frac{1}{c} > 0$$

as $n \to \infty$.

Choose a Benjamini–Schramm convergent subsequence (G_{n_k}) of the sequence G_d, G_{d+1}, ... Then the graph sequence $2G_{n_1}$, $\tilde{G}_{n_1}$, $2G_{n_2}$, $\tilde{G}_{n_2}$, ... is also Benjamini–Schramm convergent. The graph parameter $(\ln \mathrm{pm})/v$ does not converge along this graph sequence and therefore is not estimable. □

6. Perfect matchings in bipartite expander graphs

In this part we prove Theorem 1.9. We say that the bipartite graph $G = (U, V, E)$ with vertex classes U and V is a δ-expander if

$$|N(U')| \geq (1+\delta)|U'|$$

holds for every $U' \subset U$ such that $|U'| \leq |U|/2$, and

$$|N(V')| \geq (1+\delta)|V'|$$

holds for every $V' \subset V$ such that $|V'| \leq |V|/2$.

A digraph $G = (V, E)$ is a δ-expander if for every $V' \subset V(G), |V'| \leq |V|/2$ the inequalities

$$|N_{in}(V')| \geq (1+\delta)|V'| \quad \text{and} \quad |N_{out}(V')| \geq (1+\delta)|V'|$$

hold. Given a graph G, a matching and an even cycle in G, we call a cycle *alternating* if every other edge of the cycle is in the matching.

Lemma 6.1. *Let $G = (U, V, E)$ be a bipartite graph and M a perfect matching of G. Consider the following digraph $\overline{G}$: $V(\overline{G}) = V$ and $(x, y) \in E(\overline{G})$ if and only if there exists $u \in U$ such that $(x, u) \in M$ and $(u, y) \in E(G)$. Then, for every $\delta > 0$, if G is a δ-expander, then $\overline{G}$ is a δ-expander.*

Proof. Let $V' \subseteq V$; assume that $V' \leq |V|/2$. Since G is a δ-expander, we have $|N(V')| \geq (1+\delta)|V'|$. And the set of vertices matched to $N(V')$ has size at least $|N(V')|$, hence $|N_{in}(V')| \geq (1+\delta)|V'|$. The size of $N_{out}(V')$ can be estimated similarly. □

Lemma 6.2. *Let $\delta > 0$. Let $\overline{G}$ be a δ-expander digraph on n vertices. Then every edge of $\overline{G}$ is contained in a directed cycle of length at most*

$$1 + 2\frac{\ln n}{\ln(1+\delta)}.$$

Proof. We may assume $n \geq 2$. Set

$$k = 1 + \left\lfloor \frac{\ln\lfloor n/2\rfloor}{\ln(1+\delta)} \right\rfloor.$$

Let (x, y) be an arbitrary edge. Consider the following sets of vertices defined recursively:

$$S_0 = \{x\}, T_0 = \{y\} \quad \text{and} \quad S_{i+1} = N_{in}(S_i), T_{i+1} = N_{out}(T_i),$$

for $i = 0, \ldots, (k-1)$. The expander property implies that

$$|S_i|, |T_i| \geq \min\{(1+\delta)^i; (1+\delta)\lfloor n/2\rfloor\} = (1+\delta)^i$$

for $i = 0, \ldots, k$. This yields $|S_k|, |T_k| > n/2$, hence the sets S_k and T_k are not disjoint. There is a closed directed walk of length $(2k+1)$ through (x, y), and so there is a directed cycle of length at most $(2k+1)$ through (x, y). Using that $n \geq (1+\delta)\lfloor n/2\rfloor$, the lemma follows. □

Lemma 6.3. *Let $n \geq 2$, $\delta > 0$, let G be a δ-expander bipartite graph on $2 \cdot n$ vertices, and M a perfect matching of G. Then every edge of G is contained by an alternating cycle of length at most*

$$2 + 4\frac{\ln n}{\ln(1+\delta)}.$$

Proof. Let $G = (U, V, E)$. Consider the following digraph $\overline{G}$: $V(\overline{G}) = V$ and $(x, y) \in E(\overline{G})$ if and only if there exists $u \in U$ such that $(x, u) \in M$ and $(u, y) \in E(G)$. Let e be an edge of G, and consider an edge of $\overline{G}$ that corresponds to a 2-path containing e. We know that $\overline{G}$ is a δ-expander, so it has a directed cycle of length at most

$$1 + 2\frac{\ln n}{\ln(1+\delta)}$$

containing this edge. This cycle will correspond to an alternating cycle of the required length. □

Now we prove Theorem 1.9. We repeat the statement.

Theorem 1.9 *Let $n \geq 2$, $\delta > 0$, let G be a δ-expander bipartite graph of maximum degree d on $2 \cdot n$ vertices, and e an edge of G. Assume that G admits a perfect matching. Then*

$$p(e) \geq \frac{1}{d} n^{-2\cdot\ln(d-1)/\ln(1+\delta)}.$$

Proof. Consider the following bipartite graph $H = (A, B, E(H))$: The elements of the set A are the perfect matchings of G not containing e, while the elements of the set B are the perfect matchings of G containing e. The pair (M, N) is in $E(H)$ if the symmetric difference of M and M' is a cycle of length at most

$$2 + 4\frac{\ln n}{\ln(1+\delta)}.$$

Note that $p(e) = \frac{|B|}{|A|+|B|}$. Every matching M in A has degree at least one, since there is an alternating cycle of length at most

$$2 + 4\frac{\ln n}{\ln(1+\delta)}$$

containing e. On the other hand, given a matching in B, the edge e is contained by at most

$$(d-1)^{1+2\cdot\ln n/\ln(1+\delta)}$$

alternating cycles of length at most

$$2 + 4\frac{\ln n}{\ln(1+\delta)}.$$

Hence the maximum degree of B can be at most $(d-1)^{1+2\cdot\ln n/\ln(1+\delta)}$. The theorem follows. □

We end this section with the following simple observation on the corresponding problem concerning all matchings.

Proposition 6.4. *Let G be a graph with maximum degree d, $e \in E(G)$ and M a matching in G chosen uniformly at random. Then*

$$\mathbb{P}(e \in M) \geq \frac{1}{d^2+1}.$$

Proof. To every matching, we assign another matching that contains e: Given an arbitrary matching we remove the edges of the matching adjacent to e and add e to the matching. The pre-image of every matching containing e consists of at most $(d^2 + 1)$ matchings. □

7. Open problems

There are two natural questions arising from the previous sections.

Problem 7.1. Is it true that if (G_n) is a sequence of d–regular bipartite graphs such that

$$\frac{\ln \mathrm{pm}(G_n)}{v(G_n)} \to \frac{1}{2}\ln\left(\frac{(d-1)^{d-1}}{d^{d-2}}\right),$$

then G_n essentially has large girth?

Given a graph G, set $p_{\min}(G) = \min_{e\in E(G)} p(e)$ and $p_{\max}(G) = \max_{e\in E(G)} p(e)$.

Problem 7.2. Can $p_{\min}(G)$ be arbitrarily close to 0, resp. to 1, for δ-expander graphs with degree bound d ? What is the expected value of $p_{\min}$ and $p_{\max}$ for random regular graphs?

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