

**THE EQUATIONAL THEORIES OF REPRESENTABLE POSITIVE
CYLINDRIC AND RELATION ALGEBRAS ARE DECIDABLE.**

First draft.

H. ANDRÉKA

August, 1990.

THEOREM. (i) *The equational theory of $Cs_n^+ = \{\langle \mathcal{P}(^nU), \cup, \cap, \emptyset, ^nU, C_i, D_{ij} \rangle_{i,j < n} : U \text{ is a set}\}$ is decidable for any ordinal n .*

(ii) *The equational theory of $Rs^+ = \{\langle \mathcal{P}(U \times U), \cup, \cap, \emptyset, U \times U, \circ, Id, \smile \rangle : U \text{ is a set}\}$ is decidable.*

Proof: Let $n < \omega$. To any Cs_n^+ -term τ we will associate an existential formula $\exists \bar{w}\psi$ with free variables v_0, \dots, v_{n-1} , such that \bar{v} is disjoint from \bar{w} and ψ is quantifier-free, and further

(*) the meanings of τ and $\exists \bar{w}\psi$ are the same.

By this latter we mean the following. Assume that the variables occurring in τ are $x_1, \dots, x_m \in X$. In our first-order language we will consider the elements $x \in X$ to be n -ary relation symbols. Let $\mathfrak{A} \in Cs_n^+$ with $A \subseteq \mathcal{P}(^nU)$, let $k : X \rightarrow A$ be an evaluation of the variables and let $\bar{u} = \langle u_0, \dots, u_{n-1} \rangle \in ^nU$. Then $\tau^{\mathfrak{A}}[k]$ denotes the n -ary relation on U (by evaluating the variables in τ along k), and $(U, k(x))_{x \in X}$ denotes the first-order model with universe U where the relation symbol $x \in X$ is realized by $k(x) \subseteq ^nU$. Now (*) means that for any \mathfrak{A}, k and \bar{u} we have

$$\langle u_0, \dots, u_{n-1} \rangle \in \tau^{\mathfrak{A}}[k] \quad \text{iff} \quad (U, k(x))_{x \in X} \models \exists \bar{w}\psi[u_0, \dots, u_{n-1}].$$

Let $trans(\tau)$ denote the first-order formula associated to τ . We now define $trans$ by induction on τ (we will be a little sloppy in doing so): For $x \in X$ let $trans(x) = x(v_0, \dots, v_{n-1})$. Assume that $trans(\tau) = \exists \bar{w}\psi$, $trans(\sigma) = \exists \bar{z}\varphi$. We may assume (by sloppyness) that \bar{w} and \bar{z} are disjoint. Now

$$\begin{aligned} trans(\tau + \sigma) &= \exists \bar{w}\exists \bar{z}(\psi \vee \varphi) \\ trans(\tau \cdot \sigma) &= \exists \bar{w}\exists \bar{z}(\psi \wedge \varphi) \\ trans(0) &= (v_0 \neq v_0) \\ trans(1) &= (v_0 = v_0) \\ trans(c_i\tau) &= \exists v_j \exists \bar{w}\psi(v_i/v_j) \quad \text{where } v_j \text{ is a new variable} \\ trans(d_{ij}) &= (v_i = v_j) \end{aligned}$$

Now, $Cs_n^+ \not\models \tau = \sigma$ iff $(Cs_n^+ \not\models \tau \leq \sigma \text{ or } Cs_n^+ \not\models \sigma \leq \tau)$. Thus, it is enough to decide whether $Cs_n^+ \models \tau \leq \sigma$. Let $trans(\tau) = \exists \bar{w}\psi$, $trans(\sigma) = \exists \bar{z}\varphi$ with \bar{w} and \bar{z} disjoint. Then $Cs_n^+ \not\models \tau \leq \sigma$ iff $\exists \bar{v}(\exists \bar{w}\psi \wedge \neg \exists \bar{z}\varphi)$ is satisfiable, but the latter formula is equivalent to the $\exists \forall$ -formula $\exists \bar{v}\exists \bar{w}\forall \bar{z}(\psi \wedge \neg \varphi)$ and satisfiability of $\exists \forall$ -formulas is decidable.

⁰This work was supported by the Hungarian National Foundation for Scientific Research Grant No 1810.

The case of $n \geq \omega$ follows from the result in [HMTII] saying that $\langle \mathbf{SP}C_{s_\alpha} : \alpha \in Ord \rangle$ is a strong system of varieties definable by a schema (i.e. that for an equation e in the language of C_{s_3} we have $C_{s_3} \models e$ iff $C_{s_\omega} \models e$).

The relational algebraic case is completely analogous, therefore we omit its proof. **QED.**

REMARK. (i) The above proof fails if dual cylindrifications $c_i^\partial (= -c_i -)$ are added. (Dual cylindrifications are the algebraic counterparts of universal quantifiers.)

(ii) If we omit intersection but add transitive reflexive closure $*$, then we obtain the class of Kleene relation algebras. It is proved in Fischer–Ladner[77], that the equational theory of this class, i.e. that the equational theory of $\{\langle \mathcal{P}(U \times U), \cup, \emptyset, U \times U, \circ, Id, \smile, * \rangle : U \text{ is a set} \}$ is decidable. It would be interesting to know what the situation is if we allow both intersection and transitive closure.

PROBLEM 1. *Is the equational theory $Eq\mathcal{K}$ of $\mathcal{K} = \{\langle \mathcal{P}(U \times U), \cup, \cap, \emptyset, U \times U, \circ, Id, \smile, * \rangle : U \text{ is a set} \}$ decidable? Is $Eq\mathcal{K}$ the same as the equational theory of $\{\mathfrak{A} : \mathfrak{A} \text{ is finite, } \mathfrak{A} \subseteq \mathfrak{B} \text{ for some } \mathfrak{B} \in \mathcal{K}\}$?*

PROBLEM 2. *Let $C_{s_n}^{++}$ be $C_{s_n}^+$ expanded with $\{c_i^\partial : i < n\}$. Is the equational theory $Eq(C_{s_n}^{++})$ decidable (for $n > 2$)?*

REFERENCES

HMTII. Henkin,L. Monk.J.D. and Tarski,A., *Cylindric Algebras. Part II.* North-Holland, Amsterdam, 1985.

Keywords. Relation algebras, Kleene algebras, cylindric algebras, decidability, algebraic logic.
1980 *Mathematics subject classifications:* 03G15, 06, 08

H. Andr eka : Mathematical Institute of the Hungarian Acad. of Sci., Budapest, Pf.127, H-1364, Hungary