

Surjectivity of epimorphisms in varieties of algebraic logic*

by

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Abstract

In the present paper we answer the question for a broad family of classes of algebras occurring in algebraic logic whether epimorphisms are surjective in the class in question or not. These are algebraic theorems and answer open algebraic problems stated in the literature. Using the duality theory between logics and classes of algebras elaborated in algebraic logic then we use our theorems for answering open questions posed in the literature of logic.

1 Introduction

The property of epimorphisms being surjective or not in a given class of algebras is very algebraic in nature and is a central kind of investigation in algebra, cf. e.g. [KMPT] for a survey on such investigations. Definability theory is an important and characteristic part of logic, and very logical in flavor. E.g. Beth's definability theorem states that in first-order logic, whenever a concept is implicitly definable, an explicit definition also can always be found for it.

Properties very algebraic in flavor and properties very logical in flavor often turn out to be equivalent via the so called "bridge between algebras and logic". This is a surprising and appealing discovery of algebraic logic. By the above "bridge" we intend to refer to the duality theory in algebraic logic associating classes $\text{Alg}(\mathbf{L})$ of algebras to logical systems \mathbf{L} and investigating connections of typically logical properties of \mathbf{L} with typically algebraic properties of $\text{Alg}(\mathbf{L})$. For this duality theory

*Research supported by the Hungarian National Foundation for scientific research grants No T43242 and T35192.

or “bridge” of algebraic logic cf. e.g. [P72], [BP01], [HMTII, §4.3], [ANS, Part II], [Mx], [Ho01], [AMN], [Say].

Below, AP stands for “amalgamation property” and ES stands for “epimorphisms being surjective”. Investigations of AP goes back to Schreier [Sch27] and Neumann [N48]. Implicitly, the property was used much earlier in the theory of field extensions, for it is this property that allows us to consider all the extensions of a given base field as subfields of one large extension. In universal algebraic (as well as model theoretic) setting AP appears in Fräissé [Fra54].

In Tarskian algebraic logic, Daigneault [D64], Jónsson [J665], McKenzie [Mc66], Comer [C69], Johnson [J70], Pigozzi [P72] started to investigate the AP and ES type questions. [P72] is a landmark paper in the systematic study of connections between variants of the (algebraic) AP and variants of the (logical) interpolation property. After the seminal paper [P72], many papers investigated this field. A sample of such papers is [ACNS], [BP], [C84], [CP], [M77], [Mx], [S88], [Say]. [CP] contains a more comprehensive history. Hodges [Hod, §§6.4, 7.1 (especially p.285)] writes on the importance of the AP where the reason for the importance of AP is analysed to some detail.

Via the duality theory of algebraic logic, the algebraic property of epimorphisms being surjective in the class $\text{Alg}(L)$ of algebras corresponding to a logic L is equivalent to Beth’s definability property holding for L (a result of I. Németi, cf. e.g. [HMTII, Thm.5.6.10], or [ANS, Thm.58, p.213]).

In the present paper we concentrate on ES. We answer the question for a broad family of classes of algebras occurring in algebraic logic whether epimorphisms are surjective in the class in question or not. These are algebraic theorems and answer open algebraic problems stated in the literature. Using the duality theory of algebraic logic then we use our theorems for answering open questions posed in the literature of logic.

ES is extensively investigated in algebraic logic, and in particular in cylindric algebra theory. Practically all the distinguished problems concerning the finite dimensional cases are answered, but there are only scattered results concerning the infinite dimensional case. In the present paper we answer practically all open questions concerning the infinite dimensional case.

More concretely, using standard notation in algebraic logic that will be introduced in the paper, we do the following. We prove, among others, that in the classes RCA_ω , CA_ω , Ss_ω , and Di_ω of all representable cylindric algebras, of all cylindric algebras, of all semi-simple cylindric algebras, and of all diagonal cylindric algebras of infinite dimension, respectively, there are non-surjective epimorphisms. As a contrast, in the class ${}_\kappa\text{CA}_\omega$ of cylindric algebras with positive characteristic $\kappa > 0$ and of infinite dimension, all epimorphisms are surjective. These theorems

answer open problems in [P72], [HMTAN] and [KMPT]. Actually, the results and methods in the present paper can be used to settle all open problems left open in the rather carefully written landmark paper [P72], cf. [MS]. A systematic summary of our answers to the open problems in [P72] is presented in Remark 19 at the end of the present paper.

Via the duality theory of algebraic logic, these algebraic theorems show whether Beth definability property holds in the corresponding logic. E.g. Thm.14 in the present paper implies that the logic introduced in Henkin-Tarski [HT] and discussed in Blok-Pigozzi [BP, Appendix C] and in [HMTII, §4.3] fails to have the Beth definability property. The same applies to Keisler’s logic [K]. This solves a problem implicit in [HT]. We will return to the implications of our theorems and connections with the literature at the end of the paper. In particular, we will present applications to the logic of formula-schemas (of usual first-order logic) in Corollary 18.

We will also settle some of the above kind of questions for certain multi-modal logics and multi-modal algebras in the sense of e.g. [GKWZ, e.g. §1.4, p.20] or [V]. In particular, the class Df_α of diagonal-free cylindric algebras introduced and studied below is a typical class of multi-modal algebras. E.g. Thm.1 below settles the Beth definability question for a broad class of multi-modal logics. In a similar spirit, our results have consequences to scheme-logics studied in Rybakov [Ry, §3.7, pp.361-374], where connections with cylindric algebras and their variants (like Df_ω) are discussed explicitly (hence we do not need to discuss the connections here in detail).

2 Results

Throughout the paper, α denotes an ordinal. For notational convenience, we will use that an ordinal is the set of all smaller ordinals. ω will denote the smallest infinite ordinal, hence ω is the set of all finite ordinals, and so “ $n \in \omega$ ” and “ $n < \omega$ ” are synonyms for “ n is finite”.

Let V be a set of α -sequences and let $\mathcal{P}(V) = \{X : X \subseteq V\}$ denote the powerset of V . We define some natural operations on $\mathcal{P}(V)$, as follows. Let $\Gamma \subseteq \alpha$, $E \subseteq \alpha \times \alpha$, $\tau : \alpha \longrightarrow \alpha$, and let $X \subseteq V$.

$$c_{(\Gamma)}X \stackrel{\text{def}}{=} \{s \in V : (\exists z \in X)(\forall i \in \alpha \setminus \Gamma)s_i = z_i\}$$

$$d_E \stackrel{\text{def}}{=} \{s \in V : (\forall i, j \in \alpha)[s_i = s_j \text{ iff } \langle i, j \rangle \in E]\}$$

$$s_\tau X \stackrel{\text{def}}{=} \{s \in V : s \circ \tau \in X\}.$$

In naming the above operations we sometimes indicate the set V of sequences as an upper index like $\mathbf{c}_{(\Gamma)}^V$ etc. These operations are called *cylindrifications*, *diagonal elements*, and *substitutions*, respectively. We use simplified notation for some of these operations:

$$\mathbf{c}_i \stackrel{\text{def}}{=} \mathbf{c}_{\{\{i\}\}} \quad \text{and} \quad \mathbf{d}_{ij} \stackrel{\text{def}}{=} \mathbf{d}_{\{\{i,j\}\}}.$$

${}^\alpha U$ denotes the set of all U -termed α -sequences, where s is an U -termed α -sequence iff $s : \alpha \longrightarrow U$. We call ${}^\alpha U$ a *Cartesian space*. Let $p \in {}^\alpha U$. Then

$${}^\alpha U^{(p)} \stackrel{\text{def}}{=} \{s \in {}^\alpha U : \{i \in \alpha : s_i \neq p_i\} \text{ is finite}\}.$$

We call ${}^\alpha U^{(p)}$ a *weak Cartesian space*. The set U is called the *base set* of the (weak) Cartesian space like above.

A *full set algebra* is an algebra \mathfrak{A} with universe the powerset of a Cartesian space, and with operations the Boolean ones (i.e. the operation \cup of taking union of two sets, and the operation of forming the complement of a set w.r.t. the Cartesian space) and cylindrifications \mathbf{c}_i for $i \in \alpha$ together with possibly some others from $\mathbf{c}_{(\Gamma)}$, \mathbf{d}_E , \mathbf{s}_τ where $\Gamma \subseteq \alpha$, $E \subseteq \alpha \times \alpha$ and $\tau : \alpha \longrightarrow \alpha$ is such that $\text{Rng}(\tau) = \alpha$. Thus, the operations \cup , $-$, \mathbf{c}_i ($i \in \alpha$) have to be present in a set algebra, and no operation \mathbf{s}_τ can be present as an operation where the range of τ is not α . A *full weak set algebra* is like \mathfrak{A} above but with universe the powerset of a weak Cartesian space. A (*weak*) *set algebra* is a subalgebra of a full one. The *base set* of a (weak) set algebra is the base set of its greatest element.

In the present paper all algebras will be similar to (i.e. having the same type of operations as) set algebras. The (α -dimensional) *diagonal-free reduct* of an algebra is the one which contains only the Boolean operations and cylindrifications \mathbf{c}_i for $i \in \alpha$, and the (α -dimensional) *cylindric reduct* of the algebra contains these operations together with the diagonal constants \mathbf{d}_{ij} for $i, j \in \alpha$. (The names have a historical origin.) Summing up: In the algebras \mathfrak{A} we are discussing in this paper, the Boolean operations and the cylindrifications \mathbf{c}_i ($i \in \alpha$) are *compulsory*. The other operations $\mathbf{c}_{(\Gamma)}$, \mathbf{d}_E , \mathbf{s}_τ are *optional*.

Finally we introduce three concrete classes of algebras. Df_α denotes the class of *diagonal-free cylindric algebras* as defined in [HMTI]. I.e., the operations of Df_α are the Boolean ones (which we often denote by $+$, $-$) and the cylindrifications \mathbf{c}_i for $i \in \alpha$; further, $+$, $-$ form a Boolean algebra while the \mathbf{c}_i 's are commuting complemented closure operators. In other words this means that the following equations (C0)–(C4) hold for all $i, j \in \alpha$. Below, as throughout this paper, we use $\cdot, 0, 1$ for the usual derived operations of a Boolean algebra.

(C0) the Boolean equations for $+$, $-$

(C1) $c_i 0 = 0$

(C2) $x \leq c_i x$

(C3) $c_i(x \cdot c_i y) = c_i x \cdot c_i y$

(C4) $c_i c_j x = c_j c_i x$.

For an “appetizer”, we note that Thm.1 below implies (among others) that ES fails in \mathbf{Df}_α for infinite α . Moreover, ES fails in “most” subclasses \mathbf{K} of \mathbf{Df}_α . For completeness we note that \mathbf{Df}_α is an “equivalent form” of the multi-modal logic $\mathbf{S5}^\alpha$ in the sense of [GKWZ, p.142] or $\mathbf{S5}_\infty$ in [Ry, p.363]. Hence our results have consequences for multi-modal logics, too.

\mathbf{CA}_α denotes the class of all α -dimensional *cylindric algebras* as introduced in [HMTI]. I.e., the operations of \mathbf{CA}_α are those of \mathbf{Df}_α together with constants \mathbf{d}_{ij} for $i, j \in \alpha$, and in addition to (C0) – (C4) the following equations have to be satisfied for all $i, j, k \in \alpha$:

(C5) $\mathbf{d}_{ij} \cdot \mathbf{d}_{jk} \leq \mathbf{d}_{ik}$, $\mathbf{d}_{ij} = \mathbf{d}_{ji}$, $\mathbf{d}_{ii} = 1$

(C6) $c_i \mathbf{d}_{ij} = 1$, $c_k(\mathbf{d}_{ik} \cdot \mathbf{d}_{kj}) = \mathbf{d}_{ij}$ if $k \notin \{i, j\}$ and

(C7) $\mathbf{d}_{ij} \cdot c_i(x \cdot \mathbf{d}_{ij}) = x \cdot \mathbf{d}_{ij}$.

A cylindric(-type) algebra is called *representable* if it is isomorphic to a sub-direct product of weak set algebras; \mathbf{RCA}_α denotes the class of all representable cylindric algebras.

Let $B \subseteq \mathfrak{A} \in \mathbf{K}$. Then B is called *K-dense* in \mathfrak{A} if for any two parallel homomorphisms $h, f : \mathfrak{A} \rightarrow \mathfrak{C} \in \mathbf{K}$ if h and f agree on B , then $h = f$. *ES holds* in \mathbf{K} if no \mathbf{K} -dense subalgebra is proper. This is equivalent to saying that all \mathbf{K} -epimorphisms are surjective. (ES stands for all epipmorphisms are surjective.)

The following theorem implies, among others, that ES fails in the class \mathbf{CA}_α of cylindric algebras and in the class \mathbf{RCA}_α of representable cylindric algebras for $\alpha \geq \omega$. This answers both questions in Problem II.10 of [HMTAN, p.311] for the infinite case. It also answers two questions from [KMPT, p.104]. Discussion of Thm.1 together with applications to other well-investigated classes like Halmos’ quasi-polyadic algebras will be given after the theorem. \mathbf{SK} denotes the class of all subalgebras of members of \mathbf{K} .

Theorem 1 *Assume $\alpha \geq \omega$. Let $\mathbf{K} = \mathbf{SK}$ be a class of algebras such that $\mathbf{K} \models (C0) - (C4)$ and \mathbf{K} contains a full (weak) set algebra whose greatest element contains a repetition-free sequence. Then ES fails in \mathbf{K} .*

Proof. We concentrate on the $\alpha = \omega$ case, since the general $\alpha \geq \omega$ case can be proved analogously. We will construct weak set algebras $\mathfrak{B} \subset \mathfrak{A}$ such that $B \neq A$ but B is Df_ω -dense in the diagonal-free reduct of \mathfrak{A} . (The universes of algebras $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}$ etc. are denoted by the corresponding Latin capital letters, i.e. by A, B, C etc, as usual in the literature.)

We begin with constructing a weak set algebra \mathfrak{C} . Let $U_0, U_1, U_2, \dots, U_n, \dots$ ($n \in \omega$) be mutually disjoint sets with $|U_0| = 3$ and $|U_{i+1}| = 2$ for $i \in \omega$. Let $U = \bigcup \{U_i : i \in \omega\}$. Let $T^+ \stackrel{\text{def}}{=} U_0 \times U_1 \times U_2 \times \dots \times U_n \times \dots$ (The name T^+ stands for “tunnel”.) Then $T^+ \subseteq {}^\omega U$. Let $q \in T^+$ be fixed. Let $V \stackrel{\text{def}}{=} {}^\omega U^{(q)}$ and $T \stackrel{\text{def}}{=} T^+ \cap V$. Let \mathfrak{C} be the full weak set algebra with greatest element V ; and with operations the Boolean ones, \mathbf{c}_Γ , \mathbf{d}_E , \mathbf{s}_τ where $\Gamma \subseteq \omega$, $E \subseteq \omega \times \omega$ and $\tau : \omega \rightarrow \omega$ is such that $\text{Rng}(\tau) = \omega$. Then $T \in \mathcal{C}$. Now we split the relation T to two “big” relations R and $T \setminus R$ as follows.

Let $U_0 \stackrel{\text{def}}{=} \{a, b, c\}$ and define X, Y as follows.

$$X \stackrel{\text{def}}{=} \{s \in T : s_0 = a \text{ and } |\{i \in \omega : i > 0, s_i \neq q_i\}| \text{ is even}\} \quad \text{and}$$

$$Y \stackrel{\text{def}}{=} \{s \in T : s_0 \in \{b, c\} \text{ and } |\{i \in \omega : i > 0, s_i \neq q_i\}| \text{ is odd}\}.$$

For brevity, we call $|\{i : i > 0 \text{ and } s_i \neq q_i\}|$ the q -deviation of s . So, roughly:

$$X = \{a\} \times \{s : \text{the } q\text{-deviation of } s \text{ is even}\},$$

$$Y = \{b, c\} \times \{s : \text{the } q\text{-deviation of } s \text{ is odd}\}.$$

(In the above notation we sloppily wrote “ $\{s : \dots$ ” instead of “ $\{ \text{‘tail of } s \text{’} : \dots$ ”.)

We define

$$R \stackrel{\text{def}}{=} X \cup Y, \quad R^- \stackrel{\text{def}}{=} T \setminus R.$$

Let \mathfrak{B} denote the subalgebra of \mathfrak{C} generated by R , and let \mathfrak{A} denote the subalgebra of \mathfrak{C} generated by R and X , i.e.

$$\mathfrak{B} \stackrel{\text{def}}{=} \mathfrak{Sg}^{(\mathfrak{C})}\{R\} \text{ and}$$

$$\mathfrak{A} \stackrel{\text{def}}{=} \mathfrak{Sg}^{(\mathfrak{C})}\{R, X\}.$$

Cf. Figures 1,2. We now have our algebras \mathfrak{B} and \mathfrak{A} . We will show that \mathfrak{B} is a proper subalgebra of \mathfrak{A} , and its universe B is Df_ω -dense in the diagonal-free reduct of \mathfrak{A} .

To prove that \mathfrak{B} is a proper subalgebra of \mathfrak{A} , we proceed by proving the claims below.

Claim 2 *Let $\mathfrak{B}_0 \stackrel{\text{def}}{=} \mathfrak{Sg}^{(\mathfrak{C})}\{T\}$. Then T is an atom of \mathfrak{B}_0 .*

Proof. One can prove this by the Galois theory of set algebras. I.e. one looks at base-isomorphisms of \mathfrak{C} induced by permutations of U . Then one shows that any sequence $p \in T$ can be mapped to any other sequence $t \in T$ by a base-isomorphism leaving T fixed. This proves Claim 2 ■

Let Tr denote the set of all transformations $\tau : \omega \longrightarrow \omega$ of ω which are surjective, i.e. for which $\text{Rng}(\tau) = \omega$.

Claim 3 *R, R^- are atoms of \mathfrak{B} . Moreover, B is the Boolean subalgebra of \mathfrak{C} generated by $B_0 \cup \{\mathfrak{s}_\tau R : \tau \in \text{Tr}\}$. Hence the essential difference between \mathfrak{B}_0 and \mathfrak{B} is that the atom T of \mathfrak{B}_0 is split into two new “big” atoms R, R^- in B .*

Proof. In the proof we will use (1*) below:

(1*) R, R^- are so-called “big” elements w.r.t. T below T , i.e. $\mathfrak{c}_i R = \mathfrak{c}_i R^- = \mathfrak{c}_i T$ for any $i \in \omega$.

It is easy to check that (1*) holds. Let $Z \stackrel{\text{def}}{=} \{a + \mathfrak{s}_{\tau_1} r_1 + \dots + \mathfrak{s}_{\tau_n} r_n : a \in B_0, n \in \omega, \tau_1, \dots, \tau_n \in \text{Tr} \text{ and } r_1, \dots, r_n \in \{R, R^-\}\}$. We want to show that $Z = B$. Now, $Z \subseteq B$ because R generates T by (1*) and $T = \mathfrak{c}_0 T \cap \mathfrak{c}_1 T$, so $T, R^- \in B$ and $B_0 \subseteq B$. Since $R \in Z$, to show $Z \supseteq B$ it is enough to show that Z is closed under the operations of \mathfrak{C} . Clearly, Z is closed under $+$ and $\mathfrak{s}_\tau, \mathfrak{d}_E$ if $\tau \in \text{Tr}$. Z is closed under \mathfrak{c}_i by (1*), because $\mathfrak{c}_i(a + \mathfrak{s}_{\tau_1} r_1 + \dots + \mathfrak{s}_{\tau_n} r_n) = \mathfrak{c}_i a + \mathfrak{c}_i \mathfrak{s}_{\tau_1} r_1 + \dots + \mathfrak{c}_i \mathfrak{s}_{\tau_n} r_n$, and $\mathfrak{c}_i a \in B_0$ if $a \in B_0$ (since B_0 is closed under \mathfrak{c}_i), and $\mathfrak{c}_i \mathfrak{s}_\tau r = \mathfrak{c}_i \mathfrak{s}_\tau T \in B_0$ if r is R or R^- and $\tau \in \text{Tr}$, by (1*). (In this step, $\tau \in \text{Tr}$ is an important assumption.) Actually, we showed $\mathfrak{c}_i z \in B_0$ for $z \in Z$. Thus $\mathfrak{c}_{(\Gamma)} z = \mathfrak{c}_{(\Gamma)} \mathfrak{c}_i z \in B_0$ if $i \in \Gamma$, and $\mathfrak{c}_{(\emptyset)} z = z$. To show that Z is closed under $-$, note that $-(a + \mathfrak{s}_{\tau_1} r_1 + \dots + \mathfrak{s}_{\tau_n} r_n) = -a \cdot -\mathfrak{s}_{\tau_1} r_1 \cdot \dots \cdot -\mathfrak{s}_{\tau_n} r_n$. Notice also that $-\mathfrak{s}_\tau r = (-\mathfrak{s}_\tau T) + \mathfrak{s}_\tau(T - r)$, $\mathfrak{s}_\tau T \cap \mathfrak{s}_\sigma T = 0$ if $\tau \neq \sigma$ and $a \cdot \mathfrak{s}_\tau r$ is 0 or $\mathfrak{s}_\tau r$ if $a \in B_0$ and $r \leq T$ because T , and so $\mathfrak{s}_\tau T$, are atoms of B_0 . The above show that Z is closed under $-$, and so $Z = B$ has been proved. We have shown that B is the Boolean subalgebra of \mathfrak{C} generated by $B_0 \cup \{\mathfrak{s}_\tau R : \tau \in \text{Tr}\}$.

From what we wrote in the proof, and from the form of the elements of Z it also follows that R, R^- are atoms in \mathfrak{B} . ■

Our next claim is analogous with Claim 3 and the proof is analogous, too. We omit its proof because Claim 4 is not really needed in the proof of Theorem 1. Let

$$X_0 \stackrel{\text{def}}{=} \{s \in T : s_0 = a \text{ and } |\{i \in \omega : i > 0, s_i \neq q_i\}| \text{ is odd}\} \quad \text{and}$$

$$Y_0 \stackrel{\text{def}}{=} \{s \in T : s_0 \in \{b, c\} \text{ and } |\{i \in \omega : i > 0, s_i \neq q_i\}| \text{ is even}\}.$$

Then $X_0 = c_1 X \cap c_0 Y$ and $Y_0 = c_0 X \cap c_1 Y$.

Claim 4 X, Y are atoms in \mathfrak{A} . Moreover, A is the Boolean subalgebra of \mathfrak{C} generated by $B \cup \{s_\tau X : \tau \in Tr\} \cup \{s_\tau X_0 : \tau \in Tr\}$.

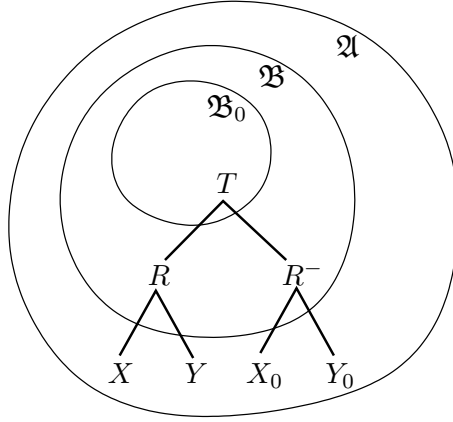


Figure 1: T is an atom of \mathfrak{B}_0 , R and R^- are atoms of \mathfrak{B} , and X, Y, X_0, Y_0 are atoms of \mathfrak{A} .

By Claim 3, \mathfrak{B} is a proper subalgebra of \mathfrak{A} , because $X \in A$ but $X \notin B$ since R is an atom in B and X is a proper nonempty subset of R . We now proceed to prove that $B \subset A$ is Df_ω -dense in the diagonal-free reduct of \mathfrak{A} . To keep notation simple, we will write \mathfrak{A} and \mathfrak{B} also for the diagonal-free reducts of \mathfrak{A} and \mathfrak{B} . That we work in the diagonal-free reducts means that we can use only the cylindric operations in the proof that follows. Also, to show the idea of the proof more clearly, first we show only that B is RCA_ω -dense. After that, we will proceed to show that B is Df_ω -dense, too.

To prove this, assume

$$\mathfrak{B} \subset \mathfrak{A} \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{g} \end{array} \mathfrak{N} \in \text{RCA}_\omega.$$

I.e. the “enemy” chooses two parallel homomorphisms h, g from \mathfrak{A} into some algebra $\mathfrak{N} \in \text{RCA}_\omega$ such that h and g agree on B and we want to show that $h = g$. We may assume that \mathfrak{N} is subdirectly irreducible. Hence we may assume that

(†) \mathfrak{N} is a weak set algebra of RCA_ω .

Next we discuss what the g -image $g^*(\mathfrak{A}) \subseteq \mathfrak{N}$ of \mathfrak{A} can be like w.r.t. the representation of \mathfrak{N} . The unit element $1^\mathfrak{N}$ of \mathfrak{N} must be of the form $W \stackrel{\text{def}}{=} \omega U^{(q')}$ for some infinite set U' and repetition free sequence q' because $T \leq -\mathbf{d}_{ij}$ for all distinct $i, j \in \omega$, and hence $g(T) \leq -\mathbf{d}_{ij}$ for all distinct $i, j \in \omega$. Further, $T' \stackrel{\text{def}}{=} g(T) = (U'_0 \times U'_1 \times \dots) \cap W$ where $|U_i| = |U'_i|$ and the U'_i 's are disjoint. This is so because $\mathbf{c}_i T \cap \mathbf{c}_j T = T$ for all distinct $i, j \in \omega$, and since in RCA_ω 's “we can count finite sets”. I.e. T' is similar to T .

Case 1. Assume $g(R) \neq 0$. We are going to show that, by using some combinatorics, one can prove that $R' \stackrel{\text{def}}{=} g(R)$ is also similar to R . I.e. there are a', b', c' and q'' such that

$$(\star) \quad R' = \left(\{a'\} \times \{s \in T' : \text{the } q''\text{-deviation of } s \text{ is even}\} \cup \right. \\ \left. \{b', c'\} \times \{s \in T' : \text{the } q''\text{-deviation of } s \text{ is odd}\} \right) \cap W.$$

The proof of (\star) goes as follows. From now on we will use the following notation. Let p be an ω -sequence, $i \in \omega$ and let u be an element of U . Then $p(i/u)$ denotes the sequence we get from p by changing its i 'th term to u . I.e. if $s = p(i/u)$, then $s(i) = u$ and $s_j = p_j$ for all $j \neq i$. Let $0 < i < \omega$. By $\mathbf{c}_i R = \mathbf{c}_i(T - R) = \mathbf{c}_i T$, $0 < R < T$, and $|U'_i| = 2$ we get that the same equations hold for R', T' and so for every $p \in R'$ the sequence $p(i/u)$ where $u \in U'_i$, $u \neq p_i$ is in $T' - R'$, and the same is true for $T' - R'$ in place of R' . By using this repeatedly for all $0 < i < \omega$, we get that

($\star\star$) for every $p \in R'$ and $s \in T'$ if $s_0 = p_0$, then $s \in R'$ if and only if the p -deviation of s is even.

Fix now $p \in T'$ arbitrarily and define

$x(p) \stackrel{\text{def}}{=} \{u \in U'_0 : p(0/u) \in R'\}$ and

$y(p) \stackrel{\text{def}}{=} \{u \in U'_0 : p(0/u) \notin R'\}$.

By $\mathfrak{c}_0 R' = \mathfrak{c}_0(T' - R')$ we have that $x(p)$, $y(p)$ form a partition of U'_0 to two nonempty sets. Thus one of $x(p)$ and $y(p)$ is a one-element set and the other one is a two-element set. If $y(p)$ is one-element, then choose p' to be $p(1/u)$ for $u \in U'_1$, $u \neq p_1$. Then by $(\star\star)$, $x(p') = y(p)$ and so $x(p')$ is one-element. Thus we may assume that $x(p)$ is one-element. Also we may “name” the elements of U'_0 so that $x(p) = \{a'\}$ and $y(p) = \{b', c'\}$. Finally, we choose q'' to be p . Then (\star) is satisfied (by $(\star\star)$). Finally, we may assume that $q' = q''$ because for q' we can choose any element of the unit of \mathfrak{N} .

But then, without loss of generality we may assume that $\mathfrak{N} = \mathfrak{C}$ and that $g = Id$ and $h : \mathfrak{A} \rightarrow \mathfrak{C}$ with h restricted to B being the identity. (Note that this is only for Case 1; we will have to return to the case when $g(R) = h(R) = 0$.) So, $h(R) = R \geq h(X), h(Y)$. We want to show $h(X) = X$. We start by collecting some equational properties of X . It is easy to check that these hold.

$$(2^*) \quad \mathfrak{c}_0 \mathfrak{c}_1 X = \mathfrak{c}_0 \mathfrak{c}_1 T$$

$$(3^*) \quad \mathfrak{c}_i(\mathfrak{c}_j X - X) \cap R \subseteq X \text{ if } i, j \neq 0$$

$$(4^*) \quad \mathfrak{c}_1(X) \cap \mathfrak{s}_1^0 \mathfrak{c}_1(X) \subseteq \mathfrak{d}_{01}$$

$$(5^*) \quad R \cap \mathfrak{c}_0 X = X.$$

We now begin to prove $X' \stackrel{\text{def}}{=}} h(X) = X$. $X' \neq 0$ by (2^*) because $h(T) = T \neq 0$. Let $s \in X'$ be arbitrary. We show that

$(*) \quad z \in X'$ for all $z \in T$ such that $z_0 = s_0$ and the s -deviation of z is even.

Indeed, let i, j be two distinct nonzero elements of ω and let $p \in X'$ be arbitrary. Let $u \in U_j$, $u \neq p_j$. Then $p(j/u) \in R^-$ by $\mathfrak{c}_j R = \mathfrak{c}_j R^-$ and $|U_j| = 2$. In particular, $p(j/u) \in (\mathfrak{c}_j X - X)$. Let $v \in U_i$, $v \neq p_i$. Then $p(j/u)(i/v) \in R$ by $\mathfrak{c}_i R^- = \mathfrak{c}_i R$ and $|U_i| = 2$. Then $p(j/u)(i/v) \in X'$ by (3^*) . This proves $(*)$.

Next we show that

$(**) \quad s_0 = z_0$ for all $s, z \in X'$.

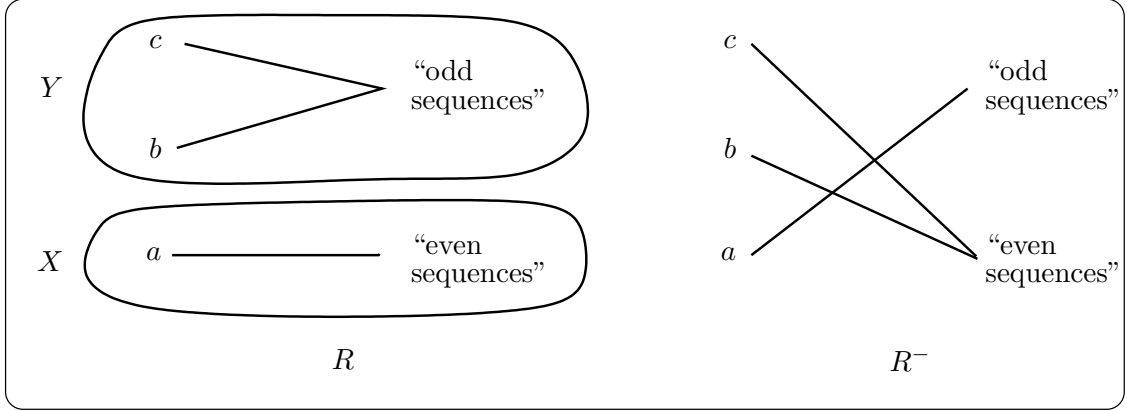


Figure 2: The definition of X, Y, R and R^- .

Indeed, let $u \in U_1$ be such that the s -deviation of $z(1/u)$ is even. Then $\langle s_0, u, z_2, z_3, \dots \rangle \in X'$ by $s \in X'$ and $(*)$. Then $w \stackrel{\text{def}}{=} \langle s_0, z_0, z_2, z_3, \dots \rangle \in \mathbf{d}_{01} \cap \mathbf{c}_1 X'$ because $z \in X'$. Thus $w \in \mathbf{c}_0(\mathbf{d}_{01} \cap \mathbf{c}_1 X') = \mathbf{s}_1^0 \mathbf{c}_1 X'$. By (4^*) then $w \in \mathbf{d}_{01}$, i.e. $s_0 = z_0$. $(**)$ has been proved.

By $(*)$, $(**)$ and by the definition of R we have that

$$(***) \quad X' = \{s \in R : s_0 = e\}$$

for some $e \in U_0$. We now want to show that $e = a$.

Let us assume $e = b$ and let $s \in X'$ be arbitrary. Then $s(0/c) \in R - X'$ by $(***)$ but $s(0/c) \in \mathbf{c}_0 X'$. This contradicts (5^*) . The case of $e = c$ is completely analogous. Thus $e = a$ must be the case and this proves that $X' = X$.

By this we already have a nonsurjective epimorphism in \mathbf{RCA}_ω , because let \mathfrak{A}' be the subalgebra of the cylindric reduct of \mathfrak{A} generated by $B \cup \{X\}$. Then the cylindric reduct of \mathfrak{B} is a proper subalgebra of \mathfrak{A}' because $X \notin B$, and we have just seen that this inclusion is an epimorphism in \mathbf{RCA}_ω . But we also have that the original inclusion $\mathfrak{B} \subset \mathfrak{A}$ is an epimorphism in \mathbf{RCA}_ω (and not only for \mathfrak{A}' in place of \mathfrak{A}) because of the following. For every $\tau \in \text{Tr}$ we can prove $h(\mathbf{s}_\tau X) = \mathbf{s}_\tau X$ exactly as we proved $h(X) = X$, and also we have by Claim 4 that \mathfrak{A} is generated as a cylindric algebra by $B \cup \{\mathbf{s}_\tau X : \tau \in \text{Tr}\}$. This proves $h = g$ for Case 1 (i.e. under assuming $h(R) \neq 0$).

Case 2 Assume $h(R) = 0$. Then $0 = h(\mathbf{s}_\tau R) = g(\mathbf{s}_\tau R) = g(T) = h(T)$. Since $X \leq R$

we also have $h(\mathbf{s}_\tau X) = g(\mathbf{s}_\tau X) = 0$. But then h and g agree on $B \cup \{\mathbf{s}_\tau X : \tau \in \text{Tr}\}$, which (as we discussed above) is enough for seeing $h = g$.

We now turn to proving that the inclusion $\mathfrak{B} \subset \mathfrak{A}$ is also a Df_ω -epimorphism. So we assume

$$\mathfrak{B} \subset \mathfrak{A} \begin{array}{c} \xrightarrow{h} \\ \xrightarrow{g} \end{array} \mathfrak{N} \in \text{Df}_\omega$$

such that h and g agree on B , and we want to show that $h(\mathbf{s}_\tau X) = g(\mathbf{s}_\tau X)$ for every $\tau \in \text{Tr}$ of ω . We may assume, as before, that \mathfrak{N} is sub-directly irreducible. We also may assume that $h(R) = g(R) \neq 0$, since the Case 2 - part of the previous proof goes through without modifications in the present situation, too.

We concentrate on proving $h(X) = g(X)$; the proof for $h(\mathbf{s}_\tau X) = g(\mathbf{s}_\tau X)$ is analogous, we just have to apply \mathbf{s}_τ everywhere in the derivation. We assume that $h(X) \neq g(X)$, and we will derive a contradiction.

Claim 5 $h(X) \cdot g(X) = 0$.

Proof. First we show that either $a \stackrel{\text{def}}{=} g(X) \cdot h(X) = 0$ or $b \stackrel{\text{def}}{=} g(X) - h(X) = 0$. Let Γ be a finite subset of ω . We show that $\mathbf{c}_{(\Gamma)} a \cdot \mathbf{c}_{(\Gamma)} b = 0$.

Assume first $0 \notin \Gamma$. We will use the following equations true in \mathfrak{A} .

$$(6^*) \quad X = R \cdot \mathbf{c}_{(\Gamma)} X \quad \text{if } 0 \notin \Gamma \quad \text{and}$$

$$(7^*) \quad R - X = R - \mathbf{c}_{(\Gamma)} X, \quad \text{if } 0 \notin \Gamma.$$

Thus

$$\begin{aligned} \mathbf{c}_{(\Gamma)} a &= \mathbf{c}_{(\Gamma)}(g(X) \cdot h(X)) \\ &= \mathbf{c}_{(\Gamma)}(g(X) \cdot h(R) \cdot \mathbf{c}_{(\Gamma)} h(X)) \\ &= \mathbf{c}_{(\Gamma)}(g(X) \cdot g(R) \cdot \mathbf{c}_{(\Gamma)} h(X)) \\ &= \mathbf{c}_{(\Gamma)}(g(X) \cdot \mathbf{c}_{(\Gamma)} h(X)) \\ &= \mathbf{c}_{(\Gamma)} g(X) \cdot \mathbf{c}_{(\Gamma)} h(X). \end{aligned}$$

Similarly,

$$\begin{aligned}
c_{(\Gamma)}b &= c_{(\Gamma)}(g(X) - h(X)) \\
&= c_{(\Gamma)}(g(X) \cdot g(R) \cdot -h(X)) \\
&= c_{(\Gamma)}(g(X) \cdot h(R) \cdot -h(X)) \\
&= c_{(\Gamma)}(g(X) \cdot (h(R) - c_{(\Gamma)}(h(X)))) \\
&= c_{(\Gamma)}(g(X) - c_{(\Gamma)}h(X)) \\
&= c_{(\Gamma)}g(X) - c_{(\Gamma)}h(X).
\end{aligned}$$

This shows $c_{(\Gamma)}a \cdot c_{(\Gamma)}b = 0$ when $0 \notin \Gamma$. Let now $i \in \Gamma$ be arbitrary (and we still assume $0 \notin \Gamma$). We will show $c_0c_{(\Gamma)}a \cdot c_0c_{(\Gamma)}b = 0$.

In the derivation below we will use the h -images of the diagonal constants \mathbf{d}_{ij} of \mathfrak{A} as special elements of \mathfrak{N} . We introduce the following notation, for any $i, j \in \omega$, $i \neq j$:

$$\begin{aligned}
\mathbf{d}'_{ij} &\stackrel{\text{def}}{=} g(\mathbf{d}_{ij}) = h(\mathbf{d}_{ij}) \\
\mathbf{s}_j^i x &\stackrel{\text{def}}{=} c_i(\mathbf{d}_{ij} \cdot x) \quad \text{and} \\
\mathbf{s}_j^{i'} x &\stackrel{\text{def}}{=} c_i(\mathbf{d}'_{ij} \cdot x).
\end{aligned}$$

Thus, for any $x \in A$ we have $g(\mathbf{s}_j^i x) = \mathbf{s}_j^{i'} g(x)$, and the same for h in place of g .

Let $a' \stackrel{\text{def}}{=} c_{(\Gamma)}a$ and $b' \stackrel{\text{def}}{=} c_{(\Gamma)}b$. We already showed that $a' = c_{(\Gamma)}gX \cdot c_{(\Gamma)}hX$ and $b' = c_{(\Gamma)}gX \cdot -c_{(\Gamma)}hX$. First we show that

$$(+)\ \mathbf{d}'_{0i} \cdot \mathbf{s}_i^{0'} b' = \mathbf{d}'_{0i} \cdot b'.$$

Indeed, by $\mathfrak{A} \models \mathbf{d}_{0i} \cdot \mathbf{s}_i^0 x = \mathbf{d}_{0i} \cdot x$ and $b' = c_{(\Gamma)}gX - c_{(\Gamma)}hX$ we have $\mathbf{d}'_{0i} \cdot \mathbf{s}_i^{0'} b' \leq \mathbf{d}'_{0i} \cdot \mathbf{s}_i^{0'} c_{(\Gamma)}gX = \mathbf{d}'_{0i} \cdot c_{(\Gamma)}gX$ and similarly $\mathbf{d}'_{0i} \cdot \mathbf{s}_i^{0'} b' \leq \mathbf{d}'_{0i} \cdot \mathbf{s}_i^{0'} c_{(\Gamma)}hX = \mathbf{d}'_{0i} \cdot c_{(\Gamma)}hX$. Thus $\mathbf{d}'_{0i} \cdot \mathbf{s}_i^{0'} b' \leq \mathbf{d}'_{0i} \cdot b'$. But $\mathbf{d}'_{0i} \cdot b' \leq \mathbf{d}'_{0i} \cdot (\mathbf{d}'_{0i} \cdot b') \leq \mathbf{d}'_{0i} \cdot c_0(\mathbf{d}'_{0i} \cdot b') = \mathbf{d}'_{0i} \cdot \mathbf{s}_i^{0'} b'$. This proves (+).

In the following we will use (8*) below.

$$(8^*)\ c_{(\Gamma)}X \cdot \mathbf{s}_i^0 c_{(\Gamma)}X \leq \mathbf{d}_{0i}, \quad \text{if } 0 \notin \Gamma.$$

Notice that $g(X) = a + b$. Hence

$$(++)\ c_{(\Gamma)}a \cdot \mathbf{s}_i^0 c_{(\Gamma)}b \leq \mathbf{d}_{0i}$$

by (8*) because both c_i and s_i^0 are additive and so monotonic. We already showed $a' \cdot b' = 0$. Notice that $a' = c_i a'$ and $b' = c_i b'$. Now

$$\begin{aligned}
a' \cdot s_i^{0'} b' &= d'_{0i} \cdot a' \cdot s_i^{0'} b' && \text{(by (++))} \\
&= a' \cdot d'_{0i} \cdot s_i^{0'} b' \\
&= a' \cdot d'_{0i} \cdot b' && \text{(by (+))} \\
&= 0.
\end{aligned}$$

Thus

$$\begin{aligned}
0 &= s_0^{i'} c_0(a' \cdot s_i^{0'} b') \\
&= s_0^{i'}(c_0 a' \cdot s_i^{0'} b') && \text{(by (C3))} \\
&= c_i(d'_{i0} \cdot c_0 a' \cdot s_i^{0'} b') \\
&= c_i(d'_{i0} \cdot c_0 a' \cdot b') && \text{(by (+))} \\
&= c_i(d'_{i0} \cdot c_i c_0 a' \cdot c_i b') \\
&= c_i d'_{i0} \cdot c_0 a' \cdot b' && \text{(by (C3))} \\
&= c_0 a' \cdot b' .
\end{aligned}$$

Hence $0 = c_0 a' \cdot c_0 b'$, i.e. $c_0 c_{(\Gamma)} a \cdot c_0 c_{(\Gamma)} b = 0$. We have proved that $c_{(\Gamma)} a \cdot c_{(\Gamma)} b = 0$ for all finite $\Gamma \subseteq \omega$.

This means that the ideals generated by a and b in \mathfrak{A} meet only in $\{0\}$. Since \mathfrak{A} is sub-directly irreducible, then it is impossible that both a and b would be nonzero. (Alternately one can use [HMTI, 2.4.44] which applies to \mathbf{Df}_ω , too.)

We have seen that one of $g(X) \cdot h(X)$ and $g(X) - h(X)$ has to be 0. Since the roles of g and h are completely symmetric, we also have that one of $h(X) \cdot g(X)$ and $h(X) - g(X)$ has to be 0. Then $h(X) \neq g(X)$ implies $h(X) \cdot g(X) = 0$. Claim 5 has been proved. ■

Claim 6 $h(X) \neq g(Y)$.

Proof. We will use the following equational property of Y in \mathfrak{A} .

$$(9^*) \quad c_1(c_1 Y \cdot s_1^0 c_1 Y - d_{01}) = c_1 Y.$$

By our assumption $g(R) \neq 0$ and thus also $g(c_1Y) \neq 0$ by $c_1Y = c_1R$. Now (8*) immediately yields $h(X) \neq g(Y)$. ■

Let $y \stackrel{\text{def}}{=} g(Y) - h(X)$. Then $y \neq 0$ and $h(Y) = g(X) + y$. In deriving our contradiction, we will use the following equations.

$$(10^*) \quad c_0X \cdot c_0Y = 0$$

$$(11^*) \quad c_1X \cdot c_1Y = 0$$

$$(12^*) \quad R^- \cdot c_0Y \leq c_1X$$

$$(13^*) \quad Y \leq c_0R^-.$$

Now, $g(R^-) \cdot c_0y \leq g(R^-) \cdot c_0g(Y) \leq c_1g(X)$ by $y \leq g(Y)$ and (12*). Similarly, $g(R^-) \cdot c_0y \leq g(R^-) \cdot c_0h(Y) \leq c_1h(X)$ by $y \leq h(Y)$ and (12*). But $h(X) \leq g(Y)$ by Claim 5, and so $c_1g(X) \cdot c_1h(X) \leq c_1g(X) \cdot c_1g(Y) = 0$ by (11*). This shows that $g(R^-) \cdot c_0y = 0$. Hence $c_0g(R^-) \cdot y = 0$. But $y \leq g(Y) \leq c_0g(R^-)$ by (13*). So $0 = c_0g(R^-) \cdot y = y$, a contradiction.

This proves Theorem 1 for the case a full weak set algebra is in \mathbf{K} . We note that in the proof we used only that $|U_0| \geq 3$, so it was not important in the proof that the base set of \mathfrak{A} is countable. The case when \mathbf{K} contains a full set algebra with infinite base set is similar: we start out with \mathfrak{C}' the full set algebra with unit ${}^\omega U$, but we use the same T, R, X . By the above, Theorem 1 has been proved. **QED**

Problem II.10 in [HMTAN, p.311] asks whether ES holds in \mathbf{CA}_α or \mathbf{RCA}_α . For finite α , the question was answered in Comer [C69] (for $\alpha < 2$ in the affirmative) and in [ACNS] (for $2 \leq \alpha < \omega$ in the negative), but the question was left open for infinite α . The same two questions were left open in [KMPT, p.104]. Corollary 7 below gives a negative answer for both parts of Problem II.10 of [HMTAN] for $\alpha \geq \omega$ and completes the table in [KMPT].¹

Corollary 7 *ES fails in any class \mathbf{K} such that $\mathbf{RCA}_\alpha \subseteq \mathbf{K} \subseteq \mathbf{CA}_\alpha$ and $\alpha \geq \omega$.*

¹For completeness we note that in Sain [S88], in Thm.10 therein, an idea of proof for a statement related to our Thm.1 was partially recalled from an unpublished lecture of Némethi. Later the authors (Sain and Némethi) withdrew their statement because they found a gap in that idea of proof which turned out to be irreparable. (Hence also the above quoted Thm.10 in [S88] became withdrawn.)

Proof. Let $\mathfrak{B} \subset \mathfrak{A}$ be the algebras we constructed in the proof of Thm.1. Let $\mathfrak{B}', \mathfrak{A}'$ be the cylindric reducts of \mathfrak{A} and \mathfrak{B} (i.e. we disregard the operations not present in cylindric algebras). Then \mathfrak{B}' is a proper CA_α -dense subalgebra of \mathfrak{A}' because the diagonal-free reduct of \mathfrak{B} is a proper Df_α -dense subalgebra of the diagonal-free reduct of \mathfrak{A} , by the proof of Thm.1. Also, $\mathfrak{B}', \mathfrak{A}' \in \mathbf{K}$ by $\mathbf{K} \supseteq \text{RCA}_\alpha$, showing that ES fails in \mathbf{K} . **QED**

[Say2] proves that G -polyadic set algebras have the strong amalgamation property, hence ES holds in them, where G is a so-called rich sub-semigroup of the transformation semigroup of ω . These rich semigroups have to contain a transformation τ such that $\text{Rng}(\tau) \neq \omega$. This shows that it is necessary that we allow in our set algebras a substitution \mathfrak{s}_τ only if the range of τ is the whole α .

The condition $\mathbf{K} \models c_i c_j(x) = c_j c_i(x)$ (i.e. (C4)) is also needed in Thm.1 above, since Némethi [N85] proves that ES holds in the class Crs_α and Crs_α satisfies all the conditions in Thm.1 except for (C4). (So, the subalgebra B constructed in the proof of Thm.1 is not Crs_α -dense in \mathfrak{A} .)

Let L_α be the logic which is called in [HMTII, p.162] the full restricted language with α many variables. Assume $\alpha \geq \omega$. By [HMTII, Thm.5.6.10], our Theorem 1 implies that L_α fails to have the Beth definability property. This applies both to the proof-theoretical version and the semantical version of L_α . It remains an interesting open question whether the weak Beth property (in the sense of, say, Hoogland [Hoo] or Barwise-Fefferman) holds for L_α . Hence, using the terminology of [Hoo], it remains open whether RCA_ω -extensible epimorphisms are surjective in RCA_ω .

Theorem 1 has similar applications both to the scheme-logics studied in [Ry, §3.7, pp.361-374] and to the multi-modal logics like S5^ω of [GKWZ, pp.142-143]. Similarly we conclude that the Beth property fails for cylindric modal logic CML_α of [V], [Mx], [MV] and for the logics discussed in [MV, pp.152-167], by Thm.1.

Corollary 8 below explores the versatility in the choice of operations present in our set algebras. For the definition of the classes occurring in Corollary 8 see [HMTII], [SayN], [ST], or [MS], [Say2]. Each case in Corollary 8 has a corollary concerning the logic corresponding to the class, we will not state all these logical consequences explicitly. For these classes and the logics corresponding to them we refer to [ANS, Part II].

Corollary 8 *Let $\alpha \geq \omega$. ES fails in the following classes of algebras: Halmos' quasi-polyadic algebras QPA_α , quasi-polyadic-equality algebras QPEA_α , their representable versions RQPA_α and RQPEA_α , the reduct of polyadic(-equality) algebras*

PEA^- where we omit those substitutions \mathfrak{s}_τ 's where τ is not surjective, Lukas' expansion of polyadic equality algebras $\text{PEA}^- + \mathbf{d}_E$, Tarski's cylindric and representable cylindric algebras CA_α , RCA_α , the axiomatic class we obtain by adding $\mathfrak{c}_{(\alpha)}$ to CA_α (E. Fried called this the discriminator-enriched cylindric algebras), Pinter's substitution cylindrification algebras and their representable members SC_α and RSC_α .

One of the assumptions in Thm.1 is that the class contain a full (weak) set algebra with a repetition-free sequence. This requirement (together with $\alpha \geq \omega$) forces the base set of the set algebra to be infinite. Our next theorem shows that this assumption in Thm.1 is also necessary. Thm.9 below answers two questions from [P72, Table 2.4.1, p.346] affirmatively, cf. Corollary 13.

To state Thm.9, we need to recall the notion of having a positive characteristic, which is the abstract notion corresponding to having a finite base-set. Let \mathfrak{A} be an infinite-dimensional cylindric algebra and let $0 < \kappa < \omega$. Now, \mathfrak{A} is said to have characteristic κ iff $a_k \stackrel{\text{def}}{=} \mathfrak{c}_{\{0,1,\dots,k\}} \prod \{-\mathbf{d}_{ij} : i, j \leq k, i \neq j\} = 1$ in \mathfrak{A} for all $k < \kappa$ and $a_\kappa = 0$. On the other hand, \mathfrak{A} is said to have characteristic 0 iff $a_k = 1$ in it for all $k < \omega$. Cf. [HMTI, Def.2.4.61, p.329]. Intuitively, \mathfrak{A} is of characteristic 0 iff \mathfrak{A} can only be represented on infinite base sets if at all, and \mathfrak{A} has characteristic $0 < \kappa < \omega$ iff in all possible representations of \mathfrak{A} the base sets have to be of size κ . If \mathbf{K} is a class of algebras, then ${}_\kappa\mathbf{K}$ denotes the class of all algebras in \mathbf{K} of characteristic κ .

Thm.9 below implies the following simple statement: ES holds in \mathbf{K} whenever $\text{CA}_\alpha \supseteq \mathbf{K} \models a_k = 0$ for some $\alpha \geq \omega > k$.

Theorem 9 *Let ${}_\kappa\text{CA}_\alpha$ denote the class of α -dimensional cylindric algebras of characteristic κ . Similarly for ${}_\kappa\text{QPEA}_\alpha$. Assume that $\alpha \geq \omega$. Then (i)-(iv) below are equivalent.*

- (i) $0 < \kappa < \omega$
- (ii) ES holds for ${}_\kappa\text{CA}_\alpha$
- (iii) ES holds for ${}_\kappa\text{RCA}_\alpha$
- (iv) ES holds for ${}_\kappa\text{QPEA}_\alpha$

Thus, ${}_\kappa\text{CA}_\omega$ has ES if $\kappa \neq 0$ and ES fails in ${}_0\text{CA}_\omega$; and similarly for RCA_ω and QPEA_ω (in place of CA_ω).

Proof. The $\kappa = 0$ case follows from Theorem 1 because the full set algebras with bases $\alpha \geq \omega$ have repetition-free sequences in their units, and at the same time they have characteristic 0. Assume that $0 < \kappa < \omega$. First we prove that ES holds in ${}_{\kappa}\mathbf{RCA}_{\omega}$, we give the proof only for $\alpha = \omega$. Let \mathbf{Gs}_{ω} denote the class of all sub-direct products of ω -dimensional cylindric set algebras. It is known that \mathbf{RCA}_{α} consists of all isomorphic copies of elements of \mathbf{Gs}_{α} , cf. e.g. [HMTII, 3.1.107].

Assume $\mathfrak{B} \subset \mathfrak{A} \in {}_{\kappa}\mathbf{Gs}_{\omega}$. We may assume that \mathfrak{A} is compact in the sense of [N85], which means that $\bigcap F \neq \emptyset$ for any filter F of \mathfrak{A} . We may assume this by Lemma 7 of [N85] (the proof of Lemma 7 applies to ${}_{\kappa}\mathbf{Gs}_{\omega}$ in place of \mathbf{Crs}_{ω} , too).

Claim 10 *There are two sequences p and q such that (i)-(iii) below hold.*

(i) $p \in x$ and $q \notin x$ for some $x \in A$.

(ii) $p \in y$ iff $q \in y$ for all $y \in B$.

(iii) $|Rng(p)| = \kappa$.

Proof. Let $x \in A \setminus B$ be arbitrary. Let $n \stackrel{\text{def}}{=} \kappa^2 + 1$, let E denote the set of all equivalence relations on n which have $\leq \kappa$ blocks. We chose n large enough so that any $e \in E$ has at least one block with $\kappa + 1$ elements. For any $e \in E$ let $\mathbf{d}(e) \stackrel{\text{def}}{=} \prod\{\mathbf{d}_{ij} : i, j \in n \text{ and } \langle i, j \rangle \in e\} \cdot \{-\mathbf{d}_{ij} : i, j \in n \text{ and } \langle i, j \rangle \notin e\}$. Then $1 = \Sigma\{\mathbf{d}(e) : e \in E\}$ because $\mathfrak{A} \in {}_{\kappa}\mathbf{Gs}_{\omega}$. So $x = \Sigma\{x \cdot \mathbf{d}(e) : e \in E\}$. Therefore $x \cdot \mathbf{d}(e) \notin B$ for some $e \in E$. Let $e \in E$ be such and let $\Delta \subseteq n$ be such that $|\Delta| = \kappa$ and Δ is a subset of a unique block of e . Let $\mathbf{d}_{\Delta} \stackrel{\text{def}}{=} \prod\{\mathbf{d}_{ij} : i, j \in \Delta\}$ and $\delta \stackrel{\text{def}}{=} \prod\{-\mathbf{d}_{ij} : i, j \in \Delta, i \neq j\}$. Then $\mathbf{d}(e) \leq \mathbf{d}_{\Delta}$ and $\mathbf{c}_{(\Delta)}\delta = 1$. Set $X \stackrel{\text{def}}{=} \mathbf{c}_{(\Delta)}(x \cdot \mathbf{d}(e))$. Then

$$\begin{aligned} & \mathbf{d}_{\Delta} \cdot \mathbf{c}_{\Delta}(\delta \cdot X) = \\ & \mathbf{d}_{\Delta} \cdot \mathbf{c}_{\Delta}(\delta \cdot \mathbf{c}_{(\Delta)}(x \cdot \mathbf{d}(e))) = \\ & \mathbf{d}_{\Delta} \cdot \mathbf{c}_{(\Delta)}\delta \cdot \mathbf{c}_{(\Delta)}(x \cdot \mathbf{d}(e)) = \\ & \mathbf{d}_{\Delta} \cdot \mathbf{c}_{(\Delta)}(\mathbf{d}_{\Gamma} \cdot x \cdot \mathbf{d}(e)) = \\ & \mathbf{d}_{\Delta} \cdot x \cdot \mathbf{d}(e) = \\ & x \cdot \mathbf{d}(e) \notin B. \end{aligned}$$

We have seen that $\mathbf{d}_{\Delta} \cdot \mathbf{c}_{\Delta}(\delta \cdot X) \notin B$. This implies that $\delta \cdot X \notin B$ because $\mathbf{d}_{\Gamma} \in B$. Thus $X \notin B$ by $\delta \in B$ and $\delta \cdot X \neq 0$. We show that $\delta \cdot -X \neq 0$ either. By $X = \mathbf{c}_{(\Delta)}X$ we have $-X = \mathbf{c}_{(\Delta)}-X$ and so

$$\begin{aligned}
c_{(\Delta)}(\delta \cdot -X) &= \\
c_{(\Delta)}(\delta \cdot c_{(\Delta)} - X) &= \\
c_{(\Delta)}\delta \cdot c_{(\Delta)} - X &= \\
c_{(\Delta)} - X &\neq 0.
\end{aligned}$$

We have seen that $c_{(\Delta)}(\delta \cdot -X) \neq 0$. Then $\delta \cdot -X \neq 0$. Now, using $X \notin B$, $X \cdot \delta \neq 0$, and $-X \cdot \delta \neq 0$, we can construct an ultrafilter F of \mathfrak{B} such that $\delta \in F$ and both $\{X\} \cup F$ and $\{-X\} \cup F$ have the finite intersection property. Since \mathfrak{A} is compact, $\bigcap(\{X\} \cup F) \neq 0$ and $\bigcap(\{-X\} \cup F) \neq 0$. Let p and q be sequences in these meets respectively. Then $p \in X$, $q \in -X$ and $p \in y$ iff $q \in y$ for all $y \in B$. Thus p, q satisfy (i) and (ii) of our claim. It remains to show that p satisfies (iii), too. This is so because $\delta \in F$ and thus $p \in \delta$. ■

We are ready to show that $B \subset \mathfrak{A}$ is not ${}_{\kappa}\text{RCA}_{\omega}$ -dense.

Let U_i, U_j be subbases of \mathfrak{A} such that $p \in {}^{\omega}U_i$ and $q \in {}^{\omega}U_j$. Then $|U_i| = |U_j| = \kappa$ by $\mathfrak{A} \in {}_{\kappa}\text{Gs}_{\omega}$. Let $i, j \in \omega$. Then $p \in \mathbf{d}_{ij}$ iff $q \in \mathbf{d}_{ij}$ because $\mathbf{d}_{ij} \in B$, i.e. p and q have the same kernel. Therefore there is a bijection $\pi : U_i \rightarrow U_j$ such that $q = \pi \circ p = \langle \pi(p_i) : i < \omega \rangle$. Let \mathfrak{C} be the full weak cylindric set algebra with unit $V \stackrel{\text{def}}{=} {}^{\omega}U_i^{(p)}$ and define $h, k : A \rightarrow C$ as follows. For all $a \in A$

$$\begin{aligned}
h(a) &\stackrel{\text{def}}{=} \{s \in V : s \in a\} \quad \text{and} \\
k(a) &\stackrel{\text{def}}{=} \{s \in V : \pi \circ s \in a\}.
\end{aligned}$$

The choice of p and q ensures that $h \neq k$, namely let $x \in A$ be such that $p \in x$ and $q \notin x$. Then $h(x) \neq k(x)$ because $p \in h(x)$ while $p \notin k(x)$ by $\pi \circ p = q \notin x$.

Claim 11 $h, k : \mathfrak{A} \rightarrow \mathfrak{C}$ are homomorphisms.

Proof. For the case of h the claim is obvious since h is relativization with V (by $h(a) = V \cap a$ for all $a \in A$) and V is a zero-dimensional element in the full set algebra containing \mathfrak{A} . For k we observe that it is a composition of the base-isomorphism induced by π with relativization with V ; in more detail: For any set a of sequences define $\pi(a) \stackrel{\text{def}}{=} \{s : \pi \circ s \in a\}$. Then $k(a) = V \cap \pi(a)$ for any $a \in A$. Hence k is a homomorphism, too. ■

Claim 12 h and k agree on B , i.e. $h(y) = k(y)$ for all $y \in B$.

Proof. We will prove by induction that

(\star) $z \in y$ iff $\pi \circ z \in y$ for all $y \in B$ and $z \in V$.

This will prove $h(y) = k(y)$ for all $y \in B$ because by definition, for all $z \in V$, $z \in h(y)$ iff $z \in y$ and $z \in k(y)$ iff $\pi \circ z \in y$. To prove (\star) first we prove ($\star\star$) below.

($\star\star$) Assume that $s \in y$ iff $z \in y$ for all $y \in B$ and let $i, j \in \omega$. Then $s(i/s_j) \in y$ iff $z(i/z_j) \in y$ for all $y \in B$.

To prove ($\star\star$) first we show that $s(i/s_j) \in y$ iff $s \in \mathbf{c}_i(\mathbf{d}_{ij} \cdot y)$. Indeed, $s(i/s_j) \in y$ iff (by $s(i/s_j) \in \mathbf{d}_{ij}$) $s(i/s_j) \in \mathbf{d}_{ij} \cdot y$ which implies that $s \in \mathbf{c}_i(\mathbf{d}_{ij} \cdot y)$. On the other hand, assume that $s \in \mathbf{c}_i(\mathbf{d}_{ij} \cdot y)$. Then $s(i/u) \in \mathbf{d}_{ij} \cdot y$ for some u , and then $u = s_j$ by $s(i/u) \in \mathbf{d}_{ij}$. We have seen $s(i/s_j) \in y$ iff $s \in \mathbf{c}_i(\mathbf{d}_{ij} \cdot y)$. The same holds for z , and this immediately yields ($\star\star$).

Clearly, $\pi \circ z(i/z_j) = (\pi \circ z)(i/\pi(z_j))$. Thus to prove (\star), it is enough to show that every $z \in V$ can be obtained from p by repeatedly changing a term of the sequence to another term of the same sequence. Here we will use $Rng(p) = U_i$.

Let $z \in V$ be arbitrary and let $\Gamma = \{i \in \omega : z_i \neq p_i\}$. For a sequence s and $\Delta \subseteq \omega$ we let $s[\Delta] \stackrel{\text{def}}{=} \{s_i : i \in \Delta\}$. Let $\Delta \subseteq \omega$, $|\Delta| = \kappa$ be such that $p[\Delta] = Rng(p) = U_i$. By our assumptions, such a Δ exists. Let $\Sigma \subseteq \omega$, $|\Sigma| = \kappa$ be disjoint from $\Gamma \cup \Delta$. Let $\Sigma_0 \subseteq \Sigma$ be such that $p[\Sigma_0] = p[\Sigma]$ and $(\forall i \in \Sigma \setminus \Sigma_0)(\exists j \in \Sigma_0)p_i = p_j$. I.e. p takes on every value of $p[\Sigma]$ exactly once on Σ_0 . Note that p and z agree on Σ by $\Sigma \cap \Gamma = \emptyset$. Let $\Sigma_1 \stackrel{\text{def}}{=} \Sigma \setminus \Sigma_0$, and let $m : \Sigma_1 \rightarrow \Delta$ be such that $p[\Sigma_0] \cup p[m[\Sigma_1]] = U_i$.

Let $p' = p(j/p(mj))_{j \in \Sigma_1}$, i.e. p' is obtained from p by changing the values in Σ_1 to the values in $\Delta_1 = m[\Sigma_1]$ (one by one). Since $\Sigma \cap \Gamma = \emptyset$, then we can change p' to p'' on Γ such that p'' agrees on Γ with z . In more detail: for all $i \in \Gamma$ let $n(i) \in \Sigma$ be such that $z_i = p'(n(i))$, and let $p'' = p'(i/p'(n(i)))_{i \in \Gamma}$. Finally, we change p'' on Σ_1 back to p , this is possible because $(\forall i \in \Sigma_1)(\exists j \in \Sigma_0)p_i = p_j$. I.e. for all $i \in \Sigma_1$ let $t(i) \in \Sigma_0$ be such that $p_i = p_{t(i)}$ and let $p''' = p''(j/p''(t(j)))_{j \in \Sigma_1}$. Then $p''' = z$, and this finishes the proof of Claim 12. ■

We have proved that ES holds in ${}_\kappa\text{RCA}_\omega$. The ${}_\kappa\text{CA}_\omega$ -case follows from the fact that infinite-dimensional cylindric algebras of positive characteristic are all representable (i.e. $0 < \kappa \implies {}_\kappa\text{CA}_\omega = {}_\kappa\text{RCA}_\omega$), cf. [HMTII, Thm.3.2.11]. The proof for ${}_\kappa\text{QPEA}_\omega$ is analogous, we omit the details. **QED**

The following corollary settles two questions from [P72, Table 2.4.1] for infinite α (and also settles the question in [P72, Remark 2.2.9 on p.336]). The corresponding questions for finite α were answered affirmatively in Comer [C84].

Corollary 13 *Let $0 < \kappa < \omega \leq \alpha$. The variety ${}_{\kappa}\mathbf{CA}_{\alpha}$ of cylindric algebras of characteristic κ has the strong amalgamation property SAP, both w.r.t. \mathbf{CA}_{α} and w.r.t. \mathbf{RCA}_{α} .*

Proof. Let $0 < \kappa < \omega \leq \alpha$. ES holds in ${}_{\kappa}\mathbf{CA}_{\alpha}$ by Thm.9. AP holds in ${}_{\kappa}\mathbf{CA}_{\alpha}$ by Comer [C84, Thm's 4.2,4.3, p.781]. In varieties, the strong AP holds iff both ES and AP holds (i.e. SAP=AP+ES), by Ringel [R], cf. [KMPT, Prop.6.3, p.93]. **QED**

In Remark 19 at the end of this paper we will outline that even the stronger so-called super-amalgamation property SUPAP can be established for ${}_{\kappa}\mathbf{CA}_{\alpha}$.

Let \mathfrak{A} be a cylindric algebra and let $x \in A$. The dimension set $\Delta^{\mathfrak{A}}(x)$ is defined as $\Delta^{\mathfrak{A}}(x) \stackrel{\text{def}}{=} \Delta(x) \stackrel{\text{def}}{=} \{i \in \alpha : c_i(x) \neq x\}$. \mathfrak{A} is called *locally finite dimensional* if $\Delta(x)$ is finite for all $x \in A$, \mathbf{Lf}_{α} denotes the class of all locally finite dimensional cylindric algebras of dimension α . A set $x \subseteq {}^{\alpha}U$ of α -sequences is called *regular* iff $x = c_{(\alpha \setminus \Delta(x))}x$, and a cylindric set algebra is called regular iff all of its elements are regular. $\mathbf{Cs}_{\alpha}^{\text{reg}}$ denotes the class of all regular α -dimensional cylindric set algebras.

Regular and locally finite dimensional algebras play an important role in algebraic logic. Full set algebras of infinite dimension are neither regular nor locally finite dimensional, hence we need to refine the proof of Thm.1 to settle ES for classes related to regular or locally finite dimensional algebras. This is what we do in Thm.14 below. The following theorem answers four open questions from [P72]: ES, and hence also the strong amalgamation property, fail both in the class \mathbf{Ss}_{ω} of semi-simple \mathbf{CA}_{ω} 's and in the class \mathbf{Di}_{ω} of diagonal \mathbf{CA}_{ω} 's, cf. Corollary 17.

If \mathbf{K} is a class of \mathbf{CA}_{ω} 's and $0 \leq \kappa < \omega$, then ${}_{<\omega}\mathbf{K} \stackrel{\text{def}}{=} \bigcup \{ {}_{\kappa}\mathbf{K} : 0 < \kappa < \omega \}$ denotes the class of members of \mathbf{K} of positive characteristic, and ${}_{\infty}\mathbf{K} \stackrel{\text{def}}{=} {}_0\mathbf{K}$. \mathbf{PK} and \mathbf{SK} denote the classes of all direct products and of all subalgebras of elements of \mathbf{K} , respectively.

Theorem 14

- (i) $\mathbf{SP}({}_{<\omega}\mathbf{Cs}_{\omega}^{\text{reg}} \cap \mathbf{Lf}_{\omega}) \subseteq \mathbf{K} \models (C0) - (C4) \implies ES \text{ fails in } \mathbf{K}.$
- (ii) $\mathbf{SP}({}_{\infty}\mathbf{Cs}_{\omega}^{\text{reg}} \cap \mathbf{Lf}_{\omega}) \subseteq \mathbf{K} \models (C0) - (C4) \implies ES \text{ fails in } \mathbf{K}.$

Proof. Let U, U_0, U_1, \dots, q be as in the proof of Theorem 1. Let $n \in \omega$. We define

$$W_n \stackrel{\text{def}}{=} U_0 \cup U_1 \cup \dots \cup U_{n-1}$$

$$T_n \stackrel{\text{def}}{=} \{s \in {}^{\omega}W_n : (\forall i < n) s_i \in U_i\}$$

$$X_n \stackrel{\text{def}}{=} \{s \in T_n : s_0 = a \text{ and } |\{0 < i < n : s_i \neq q_i\}| \text{ is even}\}$$

$$Y_n \stackrel{\text{def}}{=} \{s \in T_n : s_0 \in \{b, c\} \text{ and } |\{0 < i < n : s_i \neq q_i\}| \text{ is odd}\}$$

$$R_n \stackrel{\text{def}}{=} X_n \cup Y_n$$

\mathfrak{C}_n is the full cylindric set algebra with base set W_n

$$\mathfrak{C} \stackrel{\text{def}}{=} \Pi \langle \mathfrak{C}_{n+2} : n \in \omega \rangle$$

$$\mathbb{R} \stackrel{\text{def}}{=} \langle R_{n+2} : n \in \omega \rangle$$

$$\mathbb{X} \stackrel{\text{def}}{=} \langle X_{n+2} : n \in \omega \rangle$$

$$\mathfrak{B} \stackrel{\text{def}}{=} \mathfrak{Sg}^{(\mathfrak{C})} \{\mathbb{R}\}$$

$$\mathfrak{A} \stackrel{\text{def}}{=} \mathfrak{Sg}^{(\mathfrak{C})} \{\mathbb{R}, \mathbb{X}\}.$$

We will show that $\mathfrak{B}, \mathfrak{A} \in \mathbf{SP}(<_{\omega} \mathbf{Cs}_{\omega}^{\text{reg}} \cap \mathbf{Lf}_{\omega})$ and the diagonal-free reduct of \mathfrak{B} is a proper Df_{ω} -dense subalgebra of the diagonal-free reduct of \mathfrak{A} .

Claim 15 $\mathfrak{A} \in \mathbf{SP}(<_{\omega} \mathbf{Cs}_{\omega}^{\text{reg}} \cap \mathbf{Lf}_{\omega})$.

Proof. R_n, X_n are regular locally finite dimensional elements in \mathfrak{C}_n . Let \mathfrak{A}_n be the subalgebra of \mathfrak{C}_n generated by $\{R_n, X_n\}$. Then $\mathfrak{A}_n \in <_{\omega} \mathbf{Cs}_{\omega}^{\text{reg}} \cap \mathbf{Lf}_{\omega}$ by [HMTII, 3.1.63]. But $\mathfrak{A} \subseteq \Pi \langle \mathfrak{A}_{n+2} : n \in \omega \rangle$. ■

Claim 16 $\mathbb{X} \notin B$, hence $\mathfrak{B} \neq \mathfrak{A}$.

Proof. For any $n \in \omega$ let \mathfrak{B}_n denote the subalgebra generated by R_n in the n -dimensional full cylindric set algebra with base set W_n . For any $b \in B_n$ let $\bar{b} \stackrel{\text{def}}{=} \{s \in {}^{\omega}W_n : \langle s_0, \dots, s_{n-1} \rangle \in b\}$ and let $\overline{B_n} \stackrel{\text{def}}{=} \{\bar{b} : b \in B_n\}$. Now, the proof of Claim 3 proves $X_n \notin B_n$, since in the proof we did not use $\alpha = \omega$. Let

$$Z \stackrel{\text{def}}{=} \{a \in C : \{i \in \omega : a_i \notin \overline{B_{i+2}}\} \text{ is finite}\}.$$

Then $\mathbb{R} \in Z$ and $\mathbb{X} \notin Z$. Therefore it is enough to show that Z is closed under the cylindric operations. Let $a, b \in Z$ and let $i \in \omega$ be such that $a_i, b_i \in \overline{B_{i+2}}$. Let $s|n \stackrel{\text{def}}{=} \langle s_0, \dots, s_{n-1} \rangle$. Now, $\{s|n : s \in a_i\} \in B_{i+2}$ and $\{s|n : s \in b_i\} \in B_{i+2}$. Thus $\{s|n : s \in a_i + b_i\} \in B_{i+2}$, showing $a + b \in Z$. The proof for $-a \in Z$ is similar. The proof for $c_i a \in Z$ is similar, too, we let $i < j < \omega$ and show that $c_i a_j \in \overline{B_{j+2}}$

by $a_j \in \overline{B_{j+2}}$. The proof of $d_{ij} \in Z$ is also similar: we show that $d_{ij} \in \overline{B_{k+2}}$ for all $k \geq i, j$. ■

To show that B is a Df_ω -dense subset of \mathfrak{A} , we only have to check that equations (6*)-(13*) hold for \mathbb{X}, \mathbb{R} in place of X, R (when we replace Y, R^- with $R \setminus X$ and $c_0R \cdot c_1R \cdot -R$ respectively). This finishes the proof of (i). The proof of (ii) is almost the same, the only change is that we define the base set of our algebras to be U in place of W_n . In more detail, we define $T_n \stackrel{\text{def}}{=} \{s \in {}^\omega U : (\forall i < n) s_i \in U_i\}$, we define \mathfrak{C}_n to be the full cylindric set algebra with base set U , and the rest of the construction, as well as the proof, is the same. **QED**

As in the case of Thm.1, the assumption $\mathbf{K} \models (C4)$ is also essential for Thm.14 to hold (the same reason applies).

Recall that an algebra is called simple if it has no non-trivial congruences, and an algebra is called *semi-simple* if it is a sub-direct product of simple algebras. Ss_α denotes the class of all semi-simple members of CA_α , cf. [HMTI, Def.2.4.51].

An algebra $\mathfrak{A} \in \text{CA}_\alpha$ is called *dimension-complemented* iff $\alpha \setminus \Delta(x)$ is infinite for all $x \in A$, and \mathfrak{A} is called *diagonal* iff for every finite $\Gamma \subseteq \alpha$ and every non-zero $x \in A$ there exist distinct $i, j \in \alpha \setminus \Gamma$ such that $x \cdot d_{ij} \neq 0$; cf. [HMTI, p.231 and p.461] and [M61]. Dc_α and Di_α denote the classes of all dimension-complemented and all diagonal cylindric algebras respectively.

Let α be infinite. Then $\text{Lf}_\alpha \subseteq \text{Dc}_\alpha \subseteq \text{Dc}_\alpha \cup \text{Ss}_\alpha \subseteq \text{Di}_\alpha \subseteq \text{RCA}_\alpha$, cf. [M61] and [HMTI, Thm.2.6.50]. Daigneault [D64] proved that the strong AP holds in Lf_ω and Pigozzi [P72] proved that the strong AP holds in Dc_ω , too. It was asked in [P72, Table 2.4.1, p.346] whether the strong AP, either w.r.t. RCA_α or w.r.t. CA_α , holds in Ss_ω or in Di_ω . Corollary 17(ii) below answers these four questions negatively.

Corollary 17 *Let α be infinite.*

(i) *ES fails both in Ss_α and in Di_α .*

(ii) *Strong AP both w.r.t. CA_α and w.r.t. RCA_α fail both for Ss_α and for Di_α .*

Proof. (i): $\text{Ss}_\alpha = \text{SPSs}_\alpha$ by [HMTI, Thm.2.4.54], and every member of $\text{Cs}_\alpha^{\text{reg}} \cap \text{Lf}_\alpha$ is simple by [HMTII, Thm.3.1.70] or equivalently by [HMTAN, Thm.I.5.2(i)]. Hence $\text{SP}(\text{Cs}_\alpha^{\text{reg}} \cap \text{Lf}_\alpha) \subseteq \text{Ss}_\alpha$. By $\text{Ss}_\alpha \subseteq \text{Di}_\alpha \subseteq \text{CA}_\alpha$ then both Ss_α and Di_α satisfy the hypotheses of Thm.14, hence ES fails in both of them. (ii): Strong AP implies ES, cf. e.g. [KMPT, p.96] or [MSL]. Since $\text{Ss}_\alpha, \text{Di}_\alpha \subseteq \text{RCA}_\alpha$, (i) then implies that strong AP w.r.t. RCA_α fails for Ss_α as well as for Di_α . In the proof of Thm.14 we constructed $\mathfrak{B}, \mathfrak{A} \in \text{SP}(\text{Cs}_\alpha^{\text{reg}} \cap \text{Lf}_\alpha)$ such that \mathfrak{B} is a proper Df_α -dense subalgebra

of \mathfrak{A} . By $\text{CA}_\alpha \models (C0) - (C4)$ this shows that strong AP w.r.t. CA_α fails for Ss_α and for Di_α . **QED**

Theorem 14 above implies that the logic of formula-schemas of classical first-order logic, introduced e.g. in Henkin-Tarski [HT, p.108], has neither the Beth definability property nor the Craig interpolation property. Before stating this as a corollary, we briefly recall the *logic of formula-schemas*. A *formula-variable* is the “atomic” symbol Φ_i for $i \in \omega$. A *formula-schema* is a formula built up from formula-variables Φ_i ($i \in \omega$) and atomic formulas $v_i = v_j$ ($i, j \in \omega$) by using the usual connectives of first-order logic. (I.e. Φ_i and $v_i = v_j$ for $i, j \in \omega$ are formula-schemas, and if σ, δ are formula-schemas then $\sigma \vee \delta$, $\neg\sigma$ and $\exists v_i\sigma$ are also formula-schemas.) E.g. $\neg\Phi_0 \vee \exists v_1\Phi_0$ is a formula-schema. We say that a formula-schema is *valid* if no matter how we substitute concrete first-order formulas in place of the formula-variables, we obtain a valid formula of classical first-order logic. E.g. $\Phi_0 \vee \neg\Phi_0$ is a valid formula-schema while $\Phi_0 \vee \neg\Phi_1$ is not a valid formula-schema. By a *model* of the logic of formula-schemas we understand a pair $\langle \mathfrak{M}, f \rangle$ where \mathfrak{M} is a traditional (finitary) model of classical predicate logic and f is a function mapping the formula-variables to formulas in the usual first-order language of \mathfrak{M} . Then f induces a mapping of all formula-schemas to usual formulas in the language of \mathfrak{M} . Now we define for a formula-schema σ

$$\langle \mathfrak{M}, f \rangle \models \sigma \quad \text{iff} \quad \mathfrak{M} \models f(\sigma).$$

It is easy to check that a formula-schema is valid iff it is valid in all models in the above sense. (The present notion of formula-schemas and their validity is used throughout the works of Tarskian tradition as well as in e.g. [MV, Def.5.7.17, p.167] or [Ry, §3.7, pp.361-374].) The definition when a logic has Beth-definability property and Craig-interpolation property can be found e.g. in [HMTII, p.259], [Hoo], and in [MSL], [ANS, D.56, p.212].

Corollary 18

- (i) *The logic of formula-schemas does not have the Beth-definability property, i.e. there is an implicit definition in this logic that cannot be made explicit.*
- (ii) *The logic of formula-schemas does not have the Craig-interpolation property, i.e. there are formula-schemas σ, δ such that $\sigma \rightarrow \delta$ is a valid formula-schema but there is no formula-schema γ such that every formula-variable occurring in γ occurs both in σ and in δ and at the same time both $\sigma \rightarrow \gamma$ and $\gamma \rightarrow \delta$ are valid formula-schemas.*

Proof. (i) follows from [HMTII, 5.6.10] and from Theorem 14, because the logic of formula-schemas corresponds to the class $\mathbf{K} = \mathbf{SP}(\mathbf{Cs}_\omega^{reg} \cap \mathbf{Lf}_\omega)$ of algebras. (ii) follows from (i) because in [MSL] it is proved, among others, that failure of Beth-definability property implies failure of Craig-interpolation property. **QED**

We note that, by analyzing the proof in the present paper, one can construct concrete formulas showing that Beth's definability property does not hold for the logic of formula-schemas, and concrete formula schemas σ, δ as in Cor.18(ii) showing that the Craig interpolation property does not hold for the logic of formula-schemas.

Remark 19 On problems in Pigozzi [P72] solved herein (or in strongly related work of the author). Below we refer to parts of [P72] by page numbers.

(1) pp.330-331, item 2.2.21. It is asked there whether SAP holds for the class \mathbf{Di}_α of diagonal cylindric algebras, or whether the same holds w.r.t. \mathbf{RCA}_α . We answer both questions negatively in Corollary 17(ii) herein.

(2) p.336, lines 3-4 (Remark 2.2.29). It is asked there if SAP holds for ${}_\kappa\mathbf{CA}_\alpha$ for $0 < \kappa < \omega$. We give an affirmative answer in Corollary 13 for infinite α (the corresponding question for finite α was answered in [C84]). Moreover, with the same method we can obtain the stronger property of super-amalgamation SUPAP for ${}_\kappa\mathbf{CA}_\alpha$. The super-amalgamation property SUPAP was introduced and investigated by Maksimova and is strictly stronger than SAP, cf. e.g. [M91], [MSL] for a discussion of SUPAP and its logical counterparts. The reason why we obtain SUPAP from the proof of Corollary 13 is the following. In the proof of Corollary 13 we used results from Némethi [N85]. We could have used [N85] to prove ES directly by checking that the class ${}_\kappa\mathbf{CA}_\alpha$ satisfies the conditions needed in [N85]. This way [N85] would yield not only ES but also SAP directly. Moreover, as it was observed in [MSL] and in Marx [Mx], the method of [N85] yields SUPAP as well as SAP.

(3) pp.311, line 3 bottom up - p.312, line 1. The question is asked there of what amalgamation type property could be the natural counterpart of the strong interpolation property IP (analogously to the AP - IP connection established in items 1.2.8, 2.1.11 therein). Our answer is that the desired "AP-style" property is SUPAP and the SUPAP - strong IP connection was established by the present author in [MSL].

(4) Table 2.4.1 on p.346. The $\alpha \geq \omega$ part of Table 2.4.1 contains 10 questions indicated by question marks. All these questions can be answered if we use the results proved in the present paper and a joint result of the author with András Simon [Sim, Chapter 6, pp.75-81] settling the embedding property (EP) and strong EP for $\mathbf{RCA}_\omega \subseteq \mathbf{K} \subseteq \mathbf{CA}_\omega$ negatively. The just quoted joint result (on EP and strong EP) settles two questions. So we are left with 8 questions. Of these, the six questions

in the first two columns of Table 2.4.1 are immediately settled by the results of the present paper via settling the ES question for the classes of algebras involved. In more detail, for ${}_{\kappa}\mathbf{CA}_{\alpha}$ ($0 < \kappa < \omega \leq \alpha$) Corollary 13 gives a positive answer, settling 2 questions, and the remaining 4 questions (in the first 2 columns) are settled negatively by Corollary 17. This leaves us with 2 questions, both concerning the strong EP (for \mathbf{Ss}_{α} and \mathbf{Di}_{α}). These are settled as follows. The proof of strong EP for \mathbf{Ss}_{α} and \mathbf{Di}_{α} ($\alpha \geq \omega$) is based on observing that both of these classes are contained in \mathbf{RCA}_{α} and uses the upward Löwenheim-Skolem type theorems (and the “change-of-base” type ones) in [HMTAN] for \mathbf{RCA}_{α} ($\alpha \geq \omega$). The proof does not fit into the style of the present paper, therefore we omit it.

(5) Table 2.4.2, p.347. The $\alpha \geq \omega$ part of this table contains 5 questions indicated by question marks. The results in item (4) above settle all these five questions via the results in Pigozzi’s paper [P72]. More concretely, the questions in row 4 are answered negatively (these concern $\mathbf{RCA}_{\alpha} \subseteq \mathbf{K} \subseteq \mathbf{CA}_{\alpha}$), the rest are answered positively.

(6) p.348, lines 4-6. Here the questions like SAP are asked about arbitrary classes \mathbf{K} of cylindric algebras in place of the 7 kinds of algebras ($\mathbf{Lf}_{\alpha}, \dots, {}_{\kappa}\mathbf{CA}_{\alpha}$) investigated in Table 2.4.1. In the present work, Theorems 1,14 are formulated in terms of (almost) arbitrary classes \mathbf{K} of cylindric algebras, hence they represent a step in the direction of answering this general type of question.

(7) p.342, Remark 2.3.5. It is asked there whether the 2-generated free \mathbf{K} -algebra (in Thm.2.3.4) has the Interpolation Property IP, where $\mathbf{RCA}_{\alpha} \subseteq \mathbf{K} \subseteq \mathbf{CA}_{\alpha}$. We believe that for infinite α , our proof method for failure of ES in Thm.1 solves the second question in Remark 2.3.5 affirmatively, because our counterexample-algebras are one-generated.

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