Axiomatizability of Positive Algebras of Binary Relations

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Abstract

We consider all positive fragments of Tarski's representable relation algebras and determine whether the equational and quasiequational theories of these fragments are finitely axiomatizable in first-order logic. We also look at extending the signature with reflexive, transitive closure and the residuals of composition.

1 Introduction

Tarski defined the class RRA of *representable relation algebras*, a class of algebras of binary relations with the Boolean connectives meet $\cdot$, join $+$ and negation $-$, relation composition ;, the converse operation $\smile$ and the identity constant $1'$. RRA is a variety, but it is not finitely axiomatizable [Mo64]. An important line of research in algebraic logic is to investigate fragments of RRA from the finite axiomatizability point of view. That is, we consider less expressive languages than that of RRA, typically subsignatures of RRA, and try to figure out if these versions of algebras of binary relations admit finite axiomatizations. See [Sc91] for a somewhat outdated survey and [Mi04] for related results.

In this paper, we concentrate on *positive* similarity types Λ: the languages are restricted to subsets of $\{;, +, \cdot, \smile, 1', 1, 0\}$. Furthermore, we will assume that composition ; is always present and that there is at least one semilattice operation (either $+$ or $\cdot$) in Λ so that an ordering $\leq$ is available. We will systematically go through these similarity types and check if the equational and quasiequational theories are finitely axiomatizable. Since these fragments are usually not closed under homomorphic images, the problems of finitely axiomatizing the class and the variety generated by it become separate. Many of the results have been known (we will recall these from the literature), but fragments including the identity constant $1'$ attracted less attention. In particular, we will present maximal, finitely axiomatizable fragments involving the identity in Theorem 4.1 and prove this result using term graphs. We also include the proof of the non-finite axiomatizability of fragments involving join and composition, cf. Theorem 3.1, since this has been available only in a preprint [An88]. Finally, we will look at larger similarity types that include the residuals of composition, $\backslash$ and $/$, and/or the Kleene star, *. We prove a strengthening of Theorem 3.1 where the similarity type can include any of $\backslash$, $/$ and * as well, Theorem 5.1.

2 Main definitions

Let us recall the formal definition of positive algebras of relations.

Definition 2.1 *Let Λ be a signature such that $\Lambda \subseteq \{;, +, \cdot, \smile, 1', 1, 0\}$ and $\mathfrak{A} = (A, \Lambda)$ be an algebra. We say that $\mathfrak{A}$ is a* positive algebra of relations *if $A \subseteq \mathcal{P}(U \times U)$ for some set U, the* base *of* $\mathfrak{A}$*, and each operation in* Λ *is interpreted as follows:* $+$ *is union,* $\cdot$ *is intersection,* ; *is interpreted as* composition *of relations*

$$x\,;y = \{(u,v) \in U \times U : \exists w((u,w) \in x \text{ and } (w,v) \in y)\}$$

$\smile$ *is interpreted as* converse *of relations*

$$x^{\smile} = \{(u,v) \in U \times U : (v,u) \in x\}$$

$1'$ *is the* identity *constant*

$$1' = \{(u,v) \in U \times U : u = v\}$$

0 *is the empty set and* 1 *is a subset of* $U \times U$ *such that* 1 *is the top element of* $\mathfrak{A}$*.*

We say that an algebra $\mathfrak{A}$ *is* representable *if it is isomorphic to a positive algebra of relations. We denote the class of representable* Λ*-algebras by* $\mathsf{R}(\Lambda)$*, while* $\mathsf{V}(\Lambda)$ *stands for the variety generated by* $\mathsf{R}(\Lambda)$*.*

We note that $\mathsf{R}(\Lambda)$ is a quasivariety. Indeed, it is not difficult to see that it is closed under subalgebras and the formation of products, and it is closed under ultraproducts as well (since it is pseudo-axiomatizable using first-order logic), see, e.g., [ANS01, proof of Theorem 2, pp. 141–146], and [Né91, sections 3–5]. Since every representable algebra $\mathfrak{A}$ is (isomorphic to) a positive algebra of relations, there is an isomorphism h from $\mathfrak{A}$ into an algebra whose elements are subsets of a relation $U \times U$ and the operations are interpreted as in the definition above. In such a case, we say that $\mathfrak{A}$ is represented on $W = \bigcup\{h(a) : a \in A\} \subseteq U \times U$. Note that, in general, $\mathsf{V}(\Lambda)$ may contain non-representable algebras, since $\mathsf{R}(\Lambda)$ may not be closed under homomorphic images.

Remark 2.2 An alternative definition of representable algebras can be given by considering positive subreducts of representable relation algebras, RRA. In that case, the top element 1 must be an equivalence relation regardless of the other operations present in Λ. We feel that it is natural to be more permissive and allow a top element, say, to be irreflexive if the 1-free reduct can be represented on an irreflexive relation (cf. the case $\mathsf{R}(;,\cdot)$ below).

However, the presence of certain operations in Λ forces properties on 1:

- $; \in \Lambda$ implies that 1 is transitive
- $^{\smile} \in \Lambda$ implies that 1 is symmetric
- $1' \in \Lambda$ implies that 1 is reflexive.

Hence, if $\{;, ^{\smile}, 1'\} \subseteq \Lambda$, then our definition coincides with subreducts of RRA.

We summarize the main results in Table 1, where

- Λ is a positive RRA-subsignature containing composition ; and at least one of the lattice operations
- we do not mention the constants 0 and 1, since including them in Λ do not change the results; see below.

Note that the cases covered by the empty entries follow from other entries. We will provide more explanation in the following two sections. In Section 3 we will go through the results concerning quasivarieties $\mathsf{R}(\Lambda)$ and in Section 4 we discuss the results about varieties $\mathsf{V}(\Lambda)$. Finally, in Section **??**, we extend some of the results to reflexive and transitive closure and the residuals.

3 Quasivarieties

In this section, we look at the finite axiomatizability of the quasivarieties of the representable algebras. In most cases, we have non-finite axiomatizability — these proofs use ultraproduct constructions: there are non-representable algebras whose ultraproduct is representable.

	$\mathsf{R}(\Lambda)$	$\mathsf{V}(\Lambda)$
$\Lambda = \{;, \cdot\}$	Yes, [BS78]	Yes, [BS78]
$\Lambda = \{;, \cdot, 1'\}$	No, [HM07]	Yes, Thm. 4.1(1)
$\Lambda \supseteq \{;, \cdot, ^\smile\}$	No, [Ha91, HM00]	No, [HM00]
$\Lambda \supseteq \{;, +\}$	No, [An88, An91]	
$\Lambda \not\supseteq \{;, \cdot, ^\smile\}$		Yes, [An91, Br93], Thm. 4.1

Table 1: Finite axiomatizability of positive RRA fragments

Lower semilattice ordered semigroups The class $\mathsf{R}(;, \cdot)$ is finitely axiomatizable by equations, hence coincides with $\mathsf{V}(;, \cdot)$, see [BS78, Corollary 7]. The axioms are

- $\cdot$ is a semilattice operation

$$a \cdot b = b \cdot a \quad a \cdot a = a \quad (a \cdot b) \cdot c = a \cdot (b \cdot c) \tag{SL}$$

- ; is a semigroup operation

$$(a \,;\, b) \,;\, c = a \,;\, (b \,;\, c) \tag{SG}$$

- ; is monotonic w.r.t. the ordering $\leq$ (defined in the usual way: $x \leq y$ iff $x \cdot y = x$)

$$(a \cdot b) \,;\, (c \cdot d) \leq a \,;\, c \tag{Mon}$$

This result can be proved by a step-by-step construction of a transitive and irreflexive representation, cf. [Mi04].

Including 0 can be done by requiring normality

$$0 \,;\, a = 0 = a \,;\, 0 \tag{Nor}$$

and that 0 is the bottom element

$$0 \leq a \tag{Bot}$$

while 1 requires stating that 1 is the top element

$$a \leq 1 \tag{Top}$$

Lower semilattice ordered monoids The class $\mathsf{R}(;, \cdot, 1')$ is not finitely axiomatizable, [HM07]. The same holds if we include 0 and/or a top element 1 (whose representation is a reflexive and transitive relation).

Lower semilattice ordered involuted semigroups For $\{;, \cdot, ^\smile\} \subseteq \Lambda \subseteq \{;, +, \cdot, ^\smile, 1', 1, 0\}$, the class $\mathsf{R}(\Lambda)$ is not finitely axiomatizable, see [Ha91] and [HM00, Theorem 2.3].

Upper semilattice ordered semigroups For $\{;, +\} \subseteq \Lambda \subseteq \{;, +, ^\smile, 1', 1, 0\}$, the class $\mathsf{R}(\Lambda)$ is not finitely axiomatizable, [An88]. Since [An88] is not widely available, we will recall the proof below.

Distributive lattice ordered semigroups For $\{;, +, \cdot\} \subseteq \Lambda \subseteq \{;, +, \cdot, ^\smile, 1', 1, 0\}$, the class $\mathsf{R}(\Lambda)$ is not finitely axiomatizable, [An91, Theorem 4].

3.1 Non-finite axiomatizability of representable upper semilattice ordered semigroups

We recall the following from [An88].

Theorem 3.1 *Let* $\{;, +\} \subseteq \Lambda \subseteq \{;, +, ^\smile, 1', 1, 0\}$. *The class* $\mathsf{R}(\Lambda)$ *is not finitely axiomatizable.*

Proof: For every natural number n we construct an algebra $\mathfrak{A}_n = (A_n, +, ;, {}^\smile, 1', 0, 1)$ such that

1. the $\{+, ;\}$-reduct of $\mathfrak{A}_n$ is not representable
2. any non-trivial ultraproduct over ω, $\mathfrak{A}$, of $\mathfrak{A}_n$ is representable.

We define
$$G_n = \{a, a'_1, a''_1, \dots, a'_n, a''_n, b, b'_1, b''_1, \dots, b'_n, b''_n, o, 1', 0\}$$
Let $(A_n, +)$ be the free upper semilattice generated freely by G_n under the defining relations:
$$\{a \leq a'_i + a''_i,\ b \leq b'_i + b''_i,\ 0 + x = x : 1 \leq i \leq n, x \in G_n\}$$
Let S denote the following set of two-element subsets of A_n:
$$S = \big\{\{a, b'_1\}\big\} \cup \big\{\{a'_i, b''_i\} : 1 \leq i \leq n\big\} \cup \big\{\{a''_i, b'_{i+1}\} : 1 \leq i < n\big\} \cup \big\{\{a''_n, b\}\big\}$$
Next we define the rest of the operations on A_n as follows.
$$0 = \emptyset \qquad 1 = \sum G_n \qquad x^\smile = x$$
$$0 \,;\, x = 0 = x \,;\, 0 \qquad 1' \,;\, x = x = x \,;\, 1'$$
$$\text{if } x, y \notin \{0, 1'\}, \text{ then } x \,;\, y = \begin{cases} o & \text{if } \{x, y\} \in S \\ 1 & \text{otherwise} \end{cases}$$

1. We define the quasiequation q_n as
$$\bigwedge_{i=1}^{n} (x \leq x'_i + x''_i \wedge y \leq y'_i + y''_i) \rightarrow$$
$$x \,;\, y \leq x \,;\, y'_1 + \sum_{i=1}^{n-1} (x'_i \,;\, y''_i + x''_i \,;\, y'_{i+1}) + x'_n \,;\, y''_n + x''_n \,;\, y$$

By an induction on n one can show that q_n is valid in representable algebras. On the other hand, the evaluation ϵ given by
$$\epsilon(x) = a \quad \epsilon(x'_i) = a'_i \quad \epsilon(x''_i) = a''_i \quad \epsilon(y) = b \quad \epsilon(y'_i) = b'_i \quad \epsilon(y'_i) = b''_i$$
falsifies q_n in $\mathfrak{A}_n$ (since $a \,;\, b = 1$ and each term on the right of $\leq$ in the consequent evaluates to o). Since q_n uses only the operations ; and $+$, it follows that already the $\{+, ;\}$-reduct of $\mathfrak{A}_n$ is not representable.

2. We will build a representation of the ultraproduct $\mathfrak{A}$ in a step-by-step manner. First we describe $\mathfrak{A}$ to some extent using first-order properties of $\mathfrak{A}_n$. The set of *atoms* consists of all minimal, non-zero elements. An element c is called *join-prime* if for all $d, e \in A$, $c \leq e + d$ implies $c \leq d$ or $c \leq e$. The set of atoms of $\mathfrak{A}_n$ is precisely G_n, and exactly two of them, a and b, are not join-prime. Let At denote the set of atoms of $\mathfrak{A}$, and $\bar{x}$ denote the image (under the natural embedding, note that $A_n \subseteq A_{n+1}$) of an element x of $\mathfrak{A}_n$ in $\mathfrak{A}$. Then we have the following.

- There are exactly two elements, $\bar{a}$ and $\bar{b}$, of At which are not join-prime.
- Every element of $\mathfrak{A}$ is the supremum of the atoms below it.
- Every atom is self converse: $x^\smile = x$. The atom $\bar{1}'$ is the identity in $\mathfrak{A}$: $x \,;\, \bar{1}' = \bar{1}' \,;\, x = x$.

Observing more first-order properties of $\mathfrak{A}_n$ we will arrive at the following. There is an atom $\bar{o}$, subsets A', A'' and B', B'' of At and two functions p and s such that the following hold.

- $At = \{\bar{a}, \bar{b}, \bar{1}', \bar{o}\} \cup A' \cup A'' \cup B' \cup B''$ is a disjoint union.

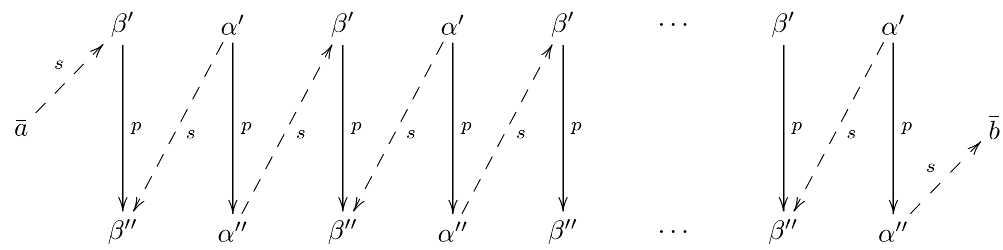

Figure 1: Functions p and s.

- The functions

  $$p : A' \cup B' \to A'' \cup B'' \qquad p \restriction A' : A' \to A'' \qquad p \restriction B' : B' \to B''$$
  $$s : \{\bar a\} \cup A' \cup A'' \to B' \cup B'' \cup \{\bar b\} \qquad s \restriction A' : A' \to B'' \qquad s \restriction A'' : A'' \to B' \smallsetminus \{s(\bar a)\}$$

  are all bijections, $s(\bar a) \in B'$ and $s^{-1}(\bar b) \in A''$. See Figure 1, where different occurrences of α', α'', β' and β'' denote distinct elements of A', A'', B' and B'', respectively.

- $\bar a \le x + p(x)$ and $\bar b \le y + p(y)$ for any $x \in A'$ and $y \in B'$. For any $x, y \in At \smallsetminus \{\bar 1'\}$,

  $$x \,;\, y = \begin{cases} \bar o & \text{if } y = s(x) \text{ or } x = s(y) \\ \bar 1 & \text{otherwise} \end{cases}$$

- There is no finite, undirected $\{s, p\}$-cycle in At. There is no finite, undirected $\{s, p\}$-path between $\bar a$ and $\bar b$.

We will define a labelled graph $\mathcal{G}$ whose edges are labelled by *prime filters* F of $\mathfrak{A}$ (i.e., F is upward closed and $x + y \in F$ implies $x \in F$ or $y \in F$). For any join-prime atom x, the principal filter $F(x)$ generated by x is a prime filter. Since $\bar a$ and $\bar b$ are not join-prime, we have to "replace" them by prime filters when representing $\mathfrak{A}$. To this end we let Y_0 be the set of those atoms from which there is an undirected $\{s, p\}$-path to $\bar b$, and define

$$\ell(\bar a) = \{\bar a\} \cup (A'' \smallsetminus Y_0) \cup (A' \cap Y_0) \qquad \ell'(\bar a) = \{\bar a\} \cup (A' \smallsetminus Y_0) \cup (A'' \cap Y_0)$$
$$\ell(\bar b) = \{\bar b\} \cup (B'' \smallsetminus Y_0) \cup (B' \cap Y_0) \qquad \ell'(\bar b) = \{\bar b\} \cup (B' \smallsetminus Y_0) \cup (B'' \cap Y_0)$$

For $X \subseteq A$, let $F(X)$ denote the filter generated by X in $\mathfrak{A}$. In case $X = \{x\}$ is a singleton, we use the notation $F(x)$. For subsets X, Y of elements, we use the notation $X;Y = \{x;y : x \in X, y \in Y\}$.

Consider the subset $\mathcal{F}$ of filters of $\mathfrak{A}$:

$$\mathcal{F} = \big\{F(x), F(\{x, \bar o\}) : x \in At \smallsetminus \{\bar a, \bar b\}\big\} \cup \big\{F(Z), F(Z \cup \{\bar o\}) : Z \in \{\ell(\bar a), \ell'(\bar a), \ell(\bar b), \ell'(\bar b)\}\big\}$$

Note that these filters are prime, $F(X)$ contains $\bar 1'$ precisely when $\bar 1' \in X$ and, for any element $x \in A' \cup A''$, we have $x \in F(\ell(\bar a))$ iff $x \in F(\ell(\bar a) \cup \{o\})$ iff $x \notin F(\ell'(\bar a) \cup \{o\})$ iff $x \notin F(\ell'(\bar a))$, and similarly for $y \in B' \cup B''$, $\ell(\bar b)$ and $\ell'(\bar b)$. Furthermore, $F(\ell(\bar a)) \,;\, F(\ell(\bar a)) = F(\ell(\bar a)) \,;\, F(\ell(\bar b)) = F(\ell(\bar b)) \,;\, F(\ell(\bar b)) = \{\bar 1\}$ and $F(\ell'(\bar a)) \,;\, F(\ell(\bar b)) = F(\ell(\bar a)) \,;\, F(\ell'(\bar b)) = \{\bar o, \bar 1\}$.

We are ready to define the labelled graph $\mathcal{G} = (U, E, \ell)$ where U is the set of vertices, $E \subseteq U \times U$ is the reflexive, transitive and symmetric set of edges and ℓ is a labelling function associating with every $(u, v) \in E$ a prime filter from $\mathcal{F}$.

Base step For every $F \in \mathcal{F}$ such that $\bar 1' \notin F$, we choose distinct points u, v such that u and v are different for distinct elements of $\mathcal{F}$. We let

$$\ell_0(u, v) = \ell_0(v, u) = F \text{ and } \ell_0(u, u) = \ell_0(v, v) = F(\bar 1')$$

see Figure 2. We define $\mathcal{G}_0 = (U_0, E_0, \ell_0)$ as the disjoint union of these graphs. Note that, by the definition of $\mathcal{F}$, for any pair (x, y) of distinct atoms, there is $(u, v) \in E_0$ such that $x \in \ell_0(u, v)$ but $y \notin \ell_0(u, v)$. Observe that $F(\bar 1') \,;\, F = F \,;\, F(\bar 1') = F$ and $F \,;\, F = \{\bar 1\} \subseteq F(\bar 1')$.

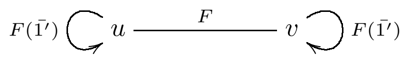

Figure 2: Base step

Inductive step Assume, inductively, that we have already defined $\mathcal{G}_k = (U_k, E_k, \ell_k)$ satisfying the following *coherency condition* of the triangles $(\ell_k(u,v), \ell_k(u,w), \ell_k(w,v))$ in E_k: for every $(u,w),(w,v) \in E_k$,

$$\ell_k(u,w) \,;\, \ell_k(w,v) \subseteq \ell_k(u,v) \tag{Coh}$$

i.e., if $x \in \ell_k(u,w)$ and $y \in \ell_k(w,v)$, then $x \,;\, y \in \ell_k(u,v)$.

We assume that there is a fair scheduling function σ of all *potential defects* $(x,y,z) \in At \times At \times At$ such that $x \,;\, y \geq z$. Assume that $\sigma_k = (x,y,z)$ and $z \in \ell_k(i,j)$ for some $(i,j) \in E_k$. If there is $w \in U_k$ such that both $x \in \ell_k(i,w)$ and $y \in \ell_k(w,j)$, then (x,y,z) is not a defect of $(i,j) \in E_k$ and we do not need to extend $\mathcal{G}_k$. Note that this includes the cases when either $x = \bar{1}'$ or $y = \bar{1}'$.

So assume otherwise: (x,y,z) is a defect of $(i,j) \in E_k$. Pick $w_{i,j}$ such that $w_{i,j} \notin U_k$. We will choose $F, G \in \mathcal{F}$ such that $x \in F$ and $y \in G$. First assume that $\bar{o} \in \ell_k(i,j)$. Consider the condition

$$\text{there is } z' \in \ell_k(i,j) \text{ such that } x \,;\, z' = \bar{o} \tag{1}$$

i.e., $s(x) \in \ell_k(i,j)$ or $s(z') = x$. If condition 1 holds, then we let $G = F(\ell(\bar{a}) \cup \{\bar{o}\})$ if $y = \bar{a}$, or $G = F(\ell(\bar{b}) \cup \{\bar{o}\})$ if $y = \bar{b}$, or $G = F(\{y, \bar{o}\})$ otherwise. If condition 1 fails, then $G = F(\ell(\bar{a}))$ or $G = F(\ell(\bar{b}))$ or $G = F(y)$ again depending on whether $y \in \{\bar{a}, \bar{b}\}$. Similarly, $F = F(\ell(\bar{a}) \cup \{\bar{o}\})$, $F = F(\ell(\bar{a}))$ or $F = F(\ell(\bar{b}) \cup \{\bar{o}\})$, $F = F(\ell(\bar{b}))$, or $F = F(\{x, \bar{o}\})$ or $F = F(x)$ depending on whether $y \in \{\bar{a}, \bar{b}\}$ and whether

$$\text{there is } z'' \in \ell_k(i,j) \text{ such that } z'' \,;\, y = \bar{o} \tag{2}$$

i.e., whether $s(y) \in \ell_k(i,j)$ or $s(z'') = y$.

Next assume that $\bar{o} \notin \ell_k(i,j)$. Then $x \,;\, y = \bar{1}$. We have different cases again according to whether conditions 1 and 2 hold. We work out the details for the case when both conditions fail, and indicate the necessary modifications when conditions 1 and/or 2 hold in square brackets [like this]. If $\{x,y\} \cap \{\bar{a}, \bar{b}\} = \emptyset$, then we choose $F, G \in \mathcal{F}$ such that $F = F(x)$ [or $F = F(\{x, \bar{o}\})$ if condition 2 above holds] and $G = F(y)$ [or $G = F(\{y, \bar{o}\})$ if condition 1 holds]. Obviously, $\bar{o} \notin \{\bar{1}\} = F \,;\, G$.

Now assume that $x = \bar{a}$. We have several cases according to the value of y.

Case $y = \bar{a}$: We choose $F = F(\ell(\bar{a}))$ [or $F(\ell(\bar{a}) \cup \{\bar{o}\})$] and $G = F(\ell(\bar{a}))$ [or $F(\ell(\bar{a}) \cup \{\bar{o}\})$].

Case $y = \bar{b}$: We let $F = F(\ell(\bar{a}))$ [or $F(\ell(\bar{a}) \cup \{\bar{o}\})$] and $G = F(\ell(\bar{b}))$ [or $F(\ell(\bar{b}) \cup \{\bar{o}\})$].

Case $y \notin \{\bar{a}, \bar{b}\} \cup B' \cup B''$: We define $F = F(\ell(\bar{a}))$ [or $F(\ell(\bar{a}) \cup \{\bar{o}\})$] and $G = F(y)$ [or $G = F(\{y, \bar{o}\})$].

Case $y \in B' \cup B''$: In this case $s^{-1}(y)$ is defined. We choose F such that $s^{-1}(y) \notin F$. In fact, $s^{-1}(y)$ is in precisely one of $F(\ell(\bar{a}))$ [or $F(\ell(\bar{a}) \cup \{\bar{o}\})$] or $F(\ell'(\bar{a}))$ [or $F(\ell'(\bar{a}) \cup \{\bar{o}\})$], hence we can choose the one that avoids $s^{-1}(y)$. We also let $G = F(y)$ [or $G = F(\{y, \bar{o}\})$].

Note that in all cases, $\bar{o} \notin \{\bar{1}\} = F \,;\, G$. The case when $x = \bar{b}$ is completely analogous.

Observe that $\bar{o}$ is a "flexible" atom, i.e., $d, e \in At \smallsetminus \{\bar{1}'\}$ implies $\bar{o} \leq d;e$ and $d \leq \bar{1} = \bar{o};e = e;\bar{o}$. Then it is not difficult to check that the triangle $(\ell_k(i,j), F, G)$ and all its permutations are coherent in each of the above cases.

We let

$$\ell_{k+1}(i, w_{i,j}) = \ell_{k+1}(w_{i,j}, i) = F \quad \ell_{k+1}(w_{i,j}, j) = \ell_{k+1}(j, w_{i,j}) = G \quad \ell_{k+1}(w_{i,j}, w_{i,j}) = F(\bar{1}')$$
$$\ell_{k+1}(q, w_{i,j}) = \ell_{k+1}(w_{i,j}, q) = \ell_{k+1}(w_{i,j}, r) = \ell_{k+1}(r, w_{i,j}) = F(\bar{o})$$

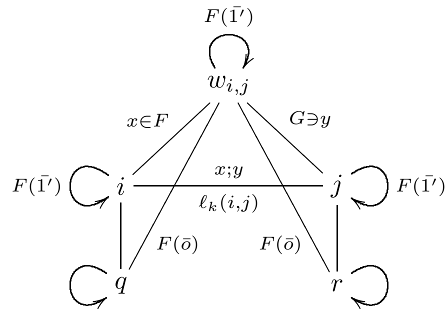

Figure 3: Inductive step

for $q, r \in U_k \smallsetminus \{i, j\}$ such that $(q, i), (r, j) \in E_k$. See Figure 3. The label $\ell_{k+1}(e)$ of "old" edges $e \in E_k$ remains $\ell_k(e)$. For every $(i, j) \in E_k$ with the defect (x, y, z), we choose a different $w_{i,j}$ and define $\mathcal{G}_{k+1}$ as the union of these graphs.

Note that, for every $(u, v) \in E_{k+1}$, we have $\bar{1}' \in \ell_{k+1}(u, v)$ iff $u = v$, since the only $F \in \mathcal{F}$ such that $\bar{1}' \in F$ are $F = F(\bar{1}')$ and $F(\{\bar{1}', \bar{o}\})$. It is not hard to verify that Coh (with k replaced by $k + 1$) holds for $\mathcal{G}_{k+1}$. We already checked this property for "primary" triangles of the form $(\ell_{k+1}(i, j), \ell_{k+1}(i, w_{i,j}), \ell_{k+1}(w_{i,j}, j))$ and their permutations when we chose the labels $F = \ell_{k+1}(i, w_{i,j}) = \ell_{k+1}(w_{i,j}, i)$ and $G = \ell_{k+1}(w_{i,j}, j) = \ell_{k+1}(j, w_{i,j})$. For all other triangles use that $\bar{o}$ is a flexible atom, whence $F' \,;\, G' \subseteq F(\bar{o})$ and $F(\bar{o}) \,;\, F' = F' \,;\, F(\bar{o}) = \{\bar{1}\} \subseteq G'$ for all $F', G' \in \mathcal{F}$ such that $\bar{1}' \notin F', G'$.

Limit step For α a limit ordinal, we let $\mathcal{G}_\alpha = \bigcup_{\kappa \le \alpha} \mathcal{G}_\kappa$. In particular, $\mathcal{G} = (U, E, \ell) = \mathcal{G}_\beta$ for an ordinal β such that all potential defects have been dealt with by step β.

It is easy to check that $\mathcal{G}$ satisfies condition Coh (with deleting the subscript k). Furthermore, we have the following saturation condition: for every $(u, v) \in E$,

$$\text{if } z \in \ell(u, v) \text{ and } z \le x \,;\, y, \text{ then there is } w \in U \text{ such that } x \in \ell(u, w) \text{ and } y \in \ell(w, v) \qquad \text{(Sat)}$$

since we assumed a fair scheduling function. We define

$$\mathsf{rep}(x) = \{(u, v) \in E : x \in \ell(u, v)\} \tag{3}$$

Then rep is an isomorphism from $\mathfrak{A}$ into a representable algebra with unit E. Indeed, rep is injective (cf. the base step), and it preserves the operations because of the following. Join $+$ is preserved because we use prime filters as labels. Composition ; is preserved by conditions Coh and Sat and by the fact that for all $c, d, e \in A$, if $c \le d \,;\, e$, then there are $x, y \in At$ such that $x \le d$, $y \le e$ and $c \le x \,;\, y$. Converse $\smile$ is preserved, since every element is self converse and we chose the same label for an edge and its inverse. The identity constant is preserved as well, by $\bar{1}' \in \ell(u, v) = F(\bar{1}')$ iff $u = v$. The bottom element is preserved, since we used only proper filters as labels, while the top element is preserved by the fact that it is in every $F \in \mathcal{F}$. ∎

We will state a strengthening, Theorem 5.1, of Theorem 3.1 in the last section when we consider larger similarity types that include residuals and/or Kleene star.

4 Varieties

Now we turn our attention to finite axiomatizability of the varieties $\mathsf{V}(\Lambda)$ generated by the representable algebras $\mathsf{R}(\Lambda)$. The picture radically changes, most of these varieties have finite axiomatizations. These proofs use graph-theoretic representations of terms that we will describe after recalling the main results.

Lower semilattice ordered semigroups We have seen above that $\mathsf{R}(;,\cdot) = \mathsf{V}(;,\cdot)$ is finitely axiomatizable by [BS78].

Lower semilattice ordered involuted semigroups We have seen that for $\{;,\cdot,\smile\} \subseteq \Lambda \subseteq \{;,+,\cdot,\smile,1',1,0\}$, the class $\mathsf{R}(\Lambda)$ is not finitely axiomatizable. The non-representability of the algebras used in the ultraproduct construction is witnessed by equations that fail in these algebras but are valid in $\mathsf{R}(\Lambda)$, hence in $\mathsf{V}(\Lambda)$. Since the ultraproduct is in $\mathsf{R}(\Lambda) \subseteq \mathsf{V}(\Lambda)$, we get that $\mathsf{V}(\Lambda)$ is not finitely axiomatizable, see [HM00] for details.

All the rest Let Λ be such that $\{;,\cdot,\smile\} \not\subseteq \Lambda \subseteq \{;,+,\cdot,\smile,1',0,1\}$ and that it contains ; and at least one of the semilattice operations $\cdot$ or $+$. Then $\mathsf{V}(\Lambda)$ is finitely axiomatizable. We will prove this result below, but mention two known cases here.

Distributive lattice ordered semigroups The class $\mathsf{V}(;,+,\cdot)$ is finitely axiomatizable, since the free distributive lattice ordered semigroup, i.e., the free algebra defined by the distributive lattice axioms DL for $\cdot$ and $+$, semigroup axioms SG for ; and the additivity

$$(a+b)\,;(c+d) = a\,;c + a\,;d + b\,;c + b\,;d \tag{Add}$$

axiom is representable, [An91, Theorem 2].

Upper semilattice ordered involuted semigroups The variety $\mathsf{V}(;,+,\smile)$ is axiomatized by the following equations: semilattice axioms SL for $+$, semigroup axioms SG for ;, additivity Add, involution

$$a^{\smile\smile} = a \text{ and } (a\,;b)^{\smile} = b^{\smile}\,;a^{\smile} \tag{Inv}$$

and

$$a \leq a\,;a^{\smile}\,;a \tag{DR}$$

see [Br93, Theorem 4].

4.1 Maximal finitely axiomatizable reducts

Next we prove finite axiomatizability for the two maximal signatures with this property, see Theorem 4.1. Finite axiomatizability for subsignatures either follows from [Br93], the results mentioned above, or by easy modifications of the proofs below (when $1'$ is in the similarity type).[1]

Distributive lattice ordered monoids Let $Ax(+,\cdot,;,1')$ be the set of the following axioms. Distributive lattice axioms DL for $+$ and $\cdot$, monoid axioms for ; and $1'$: semigroup axiom SG for ; plus

$$a\,;1' = a = 1'\,;a \tag{Ide}$$

additivity Add of ;, axiom for composition below $1'$

$$(a\cdot 1')\,;(b\cdot 1') = a\cdot b\cdot 1' \tag{CbI}$$

and an axiom expressing "functionality" below $1'$

$$[(a_1\cdot 1')\,;c\,;(a_2\cdot 1')]\cdot[(b_1\cdot 1')\,;d\,;(b_2\cdot 1')] = (a_1\cdot b_1\cdot 1')\,;(c\cdot d)\,;(a_2\cdot b_2\cdot 1') \tag{FbI}$$

[1] In [AB95], right above the Acknowledgements, it was claimed, mistakenly, that the equational theories of all positive subreducts are treated in [Br93]. In fact, all subreducts not containing $1'$ are treated in [Br93], but the subreducts containing $1'$ are not treated in that paper.

Upper semilattice ordered involuted monoids Let $Ax(+, ;, ^\smile, 1')$ be the following set of axioms. Semilattice axioms SL for $+$, monoid axioms SG and Ide for ; and $1'$, additivity Add of ; and of $^\smile$

$$(a + b)^\smile = a^\smile + b^\smile$$

involution Inv and DR, and

$$1'^\smile = 1' \tag{IC}$$

Let $\vdash$ denote derivability in equational logic.

Theorem 4.1 *1. The variety* $\mathsf{V}(+, \cdot, ;, 1')$ *is finitely axiomatizable:*

$$Ax(+, \cdot, ;, 1') \vdash \sigma = \tau \quad \textit{iff} \quad \mathsf{R}(+, \cdot, ;, 1') \models \sigma = \tau$$

2. The variety $\mathsf{V}(+, ;, ^\smile, 1')$ *is finitely axiomatizable:*

$$Ax(+, ;, ^\smile, 1') \vdash \sigma = \tau \quad \textit{iff} \quad \mathsf{R}(+, ;, ^\smile, 1') \models \sigma = \tau$$

Including 0 *requires that* 0 *is the bottom element and the normality of the operations. Including* 1 *requires that* 1 *is the top element.*

Proof: In both cases, we will show that the classes of algebraic structures defined as the $+$-free reducts augmented with an ordering $\leq$ such that each operation is monotonic w.r.t. $\leq$ have finitely based $\leq$-theories.

Ordered involuted monoids Let $Ax(;, ^\smile, 1', \leq)$ be the set of axioms given by replacing SL by the axioms stating that $\leq$ is an ordering, and additivity Add by monotonicity of ; and $^\smile$ w.r.t. $\leq$

$$a \leq b \text{ and } c \leq d \text{ imply } a ; c \leq b ; d$$
$$a \leq b \text{ implies } a^\smile \leq b^\smile$$

in $Ax(+, ;, ^\smile, 1', \leq)$.

Theorem 4.2

$$Ax(;, ^\smile, 1', \leq) \vdash \sigma \leq \tau \quad \textit{iff} \quad \mathsf{R}(;, ^\smile, 1', \leq) \models \sigma \leq \tau$$

Lower semilattice ordered monoids In the case of $\Lambda = \{\cdot, ;, 1'\}$, the ordering $\leq$ is implicit in the algebras, and its required properties (e.g., monotonicity) follow from the axioms $Ax(\cdot, ;, 1')$. Thus we can state the required result without explicitly mentioning the ordering $\leq$.

Let $Ax(\cdot, ;, 1')$ be the set of axioms given by replacing DL by the semilattice axioms SL for $\cdot$, and additivity Add by monotonicity Mon of ; in $Ax(+, \cdot, ;, 1')$.

Theorem 4.3 *The variety* $\mathsf{V}(\cdot, ;, 1')$ *is finitely axiomatizable:*

$$Ax(\cdot, ;, 1') \vdash \sigma = \tau \quad \textit{iff} \quad \mathsf{R}(\cdot, ;, 1') \models \sigma = \tau$$

Given Theorem 4.2 and 4.3, the result follows from the following, cf. [Br93, Corollary 2].

Proposition 4.4 *Let* Λ *be a positive similarity type. Then* $(A, \Lambda, +) \in \mathsf{V}(\Lambda, +)$ *iff the algebraic structure* $(A, \Lambda, \leq) \in \mathsf{V}(\Lambda, \leq)$*, each operation in* Λ *is additive, and* $(A, +)$ *is a semilattice.*

Hence, if $\mathsf{V}(\Lambda, \leq)$ is axiomatized by Ax and monotonicity, then Ax together with additivity and the semilattice axioms axiomatize $\mathsf{V}(\Lambda, +)$. The key observations are

- the semilattice operation defines the ordering $\leq$

- the quasiequations expressing that the operations are monotonic w.r.t. the ordering $\leq$ can be replaced by the axioms stating the additivity of the operations (hence an inequality $a \leq b$ can be equivalently rewritten as $a_1 + \ldots + a_n \leq b_1 + \ldots + b_m$ where all a_i and b_j are +-free subterms of a and b, respectively)
- for any +-free terms a, b_1 and b_2, $a \leq b_1 + b_2$ is valid in $\mathsf{V}(\Lambda, +)$ iff $a \leq b_1$ or $a \leq b_2$ is valid in $\mathsf{V}(\Lambda, \leq)$.

We will prove Theorems 4.2 and 4.3 below thus completing the proof of Theorem 4.1. ∎

4.2 Term graphs

To prove Theorems 4.2 and 4.3 we will work with the term graphs $G(\sigma)$ of [AB95].

Let X be a set of variables. A *labelled graph* is a structure $G = (V, E)$ where V is a set and $E \subseteq V \times X \times V$. Given two labelled graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, a *homomorphism* $h : G_1 \to G_2$ is a map from V_1 to V_2 that preserves labelled edges: if $(u, x, v) \in E_1$, then $(h(u), x, h(v)) \in E_2$. Given an equivalence relation θ on V, the *quotient graph* is $G/\theta = (V/\theta, E/\theta)$ where V/θ is the set of equivalence classes of V and

$$E/\theta = \{(u/\theta, x, v/\theta) : (u, x, v) \in E \text{ for some } u \in u/\theta \text{ and } v \in v/\theta\}$$

A *2-pointed graph* is a labelled graph $G = (V, E)$ with two (not necessarily distinct) distinguished vertices $\iota, o \in V$. We will call ι the *input* and o the *output* vertex of G, respectively, and denote 2-pointed graphs as $G = (V, E, \iota, o)$. In the case of 2-pointed graphs, we require that a homomorphism preserves input and output vertices as well.

Let $G_1 \oplus G_2$ denote the disjoint union of G_1 and G_2. For 2-pointed graphs $G_1 = (V_1, E_1, \iota_1, o_1)$ and $G_2 = (V_2, E_2, \iota_2, o_2)$, we define their *composition* as

$$G_1 \,;\, G_2 = (((V_1, E_1) \oplus (V_2, E_2))/\theta, \iota_1/\theta, o_2/\theta)$$

where θ is the smallest equivalence relation on the disjoint union $V_1 \cup V_2$ that identifies o_1 with ι_2. The *meet* of G_1 and G_2 is defined as

$$G_1 \cdot G_2 = (((V_1, E_1) \oplus (V_2, E_2))/\theta, \iota_1/\theta, o_1/\theta)$$

where θ is the smallest equivalence relation on $V_1 \cup V_2$ that identifies ι_1 with ι_2 and o_1 with o_2. When no confusion is likely we will identify an equivalence class u/θ with u, hence ι_i/θ with ι_i and o_i/θ with o_i for $i \in \{1, 2\}$.

We define *term graphs* as special 2-pointed graphs by induction on the complexity of terms. Let

$$G(1') = (\{\iota\}, \emptyset, \iota, \iota)$$

i.e., in this case $\iota = o$. For variable x, we let

$$G(x) = (\{\iota, o\}, \{(\iota, x, o)\}, \iota, o)$$

For terms σ and τ, we set

$$G(\sigma \cdot \tau) = G(\sigma) \cdot G(\tau) \text{ and } G(\sigma \,;\, \tau) = G(\sigma) \,;\, G(\tau)$$

while $G(\sigma^{\smile})$ is defined by swapping ι and o in $G(\sigma)$.

Next we recall a characterization of validities using graph homomorphisms from [AB95, Theorem 1].

Theorem 4.5 *The inequality $\tau \leq \sigma$ is valid in representable algebras iff there is a homomorphism from $G(\sigma)$ to $G(\tau)$.*

The same holds if we replace algebras with ordered algebraic structures.

The heart of our completeness proof will be the following.

The inequality $\tau \leq \sigma$ is derivable from the axioms iff there is a homomorphism from $G(\sigma)$ to $G(\tau)$.

When no confusion is likely we will omit the axioms in front of $\vdash$ and the class of algebras in front of $\models$. We leave the easy task of checking the soundness of the axiom systems with respect to the representable classes to the reader.

Proof of Theorem 4.2: Assume that we have terms σ and τ such that $\mathsf{R}(;, {}^{\smile}, 1', \leq) \models \tau \leq \sigma$. We have to show that $Ax(;, {}^{\smile}, 1', \leq) \vdash \tau \leq \sigma$.

We claim that every term ρ can be equivalently rewritten in the form $\rho_1 ; \ldots ; \rho_n$ where every ρ_i is either a variable or $1'$ or the converse of a variable. Indeed, this claim follows from the axioms Inv and IC. Furthermore, if at least one ρ_i is not $1'$, then all occurrences of $1'$ can be deleted, by using axiom Ide.

So assume that σ and τ are either $1'$ or a composition of variables and converses of variables, say, $\sigma = \sigma_1 ; \ldots ; \sigma_m$ and $\tau = \tau_1 ; \ldots ; \tau_n$. Note that if one of σ or τ is $1'$, then both of them equal $1'$, since the only valid inequality of the form $1' \leq x$ with x being a variable or the converse of a variable or $1'$ or $1'^{\smile} = 1'$ is $1' \leq 1'$. In this case, we are done, since $\vdash 1' \leq 1'$. So assume that both σ and τ are different from $1'$.

By Theorem 4.5, there is a homomorphism $h : G(\sigma_1 ; \ldots ; \sigma_m) \to G(\tau_1 ; \ldots ; \tau_n)$. Note that both graphs consist of linearly ordered finite sequences of vertices and that the directed edges between adjacent vertices are labelled by variables. Since every τ_i is different form $1'$, it follows that $n \leq m$. Also, for every τ_i, there is σ_j such that τ_i and σ_j are syntactically the same term, since the only valid inequality of the form $\tau_i \leq x$ with x being a variable or the converse of a variable is $\tau_i \leq \tau_i$. If $n = m$, then we have that τ and σ are the same sequence of the same terms, and we are done. If $n < m$, then the only way we can homomorphically map $G(\sigma)$ to $G(\tau)$ is to have "loops" of the form: $h(v_i) = u_j, h(v_{i+1}) = u_{j+1}, h(v_{i+2}) = u_j, h(v_{i+3}) = u_{j+1}$. See Figure 4, where the top line represents $G(\sigma)$, the bottom line represents $G(\tau)$ and the dashed lines represent the homomorphism h.

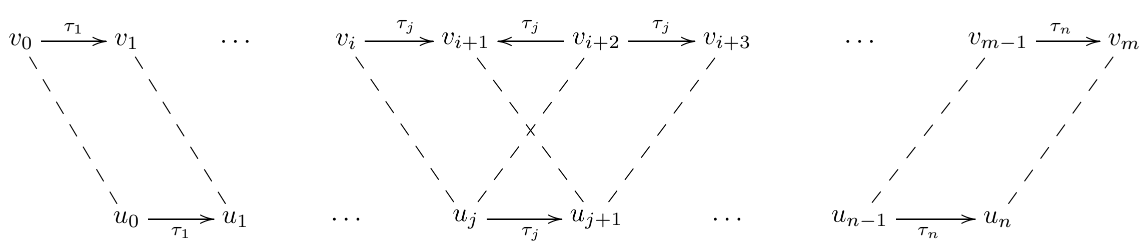

Figure 4: A loop in $h : G(\sigma) \to G(\tau)$

First assume that the directed edge (u_j, u_{j+1}) is in $G(\tau)$ and let its label be τ_j. It follows that the directed edges $(v_i, v_{i+1}), (v_{i+2}, v_{i+1}), (v_{i+2}, v_{i+3})$ are in $G(\sigma)$ and that they are labelled by τ_j as well. That is, $G(\sigma)$ restricted to (v_i, v_{i+1}) and to (v_{i+2}, v_{i+3}) is $G(\tau_j)$, while its restriction to (v_{i+1}, v_{i+2}) is $G(\tau_j{}^{\smile})$. By axiom DR we get $\vdash \tau_j \leq \tau_j ; \tau_j{}^{\smile} ; \tau_j$. Then an easy induction on the number of loops gives us $\vdash \tau \leq \sigma$. The case (u_{j+1}, u_j) is in $G(\tau)$ is similar. ∎

Proof of Theorem 4.3: Assume that $\mathsf{R}(\cdot, ;, 1') \models \tau \leq \sigma$. We have to prove $Ax(\cdot, ;, 1') \vdash \tau \leq \sigma$. We will proceed by induction on σ.

Case $\sigma = 1'$: By induction on τ.

Subcase $\tau = x$: Just note that $\not\models x \leq 1'$.

Subcase $\tau = 1'$: Obviously, $\vdash 1' \leq 1'$.

Subcase $\tau = \tau_1 \cdot \tau_2$: Assuming $\models \tau_1 \cdot \tau_2 \leq 1'$, by Theorem 4.5, we have a homomorphism $h : G(1') \to G(\tau_1 \cdot \tau_2) = G(\tau_1) \cdot G(\tau_2)$. Let $G(\tau_1) = (V_1, E_1, \iota_1, o_1)$, $G(\tau_2) = (V_2, E_2, \iota_2, o_2)$ and

$G(\tau_1) \cdot G(\tau_2) = G = (V, E, \iota, o)$. Then $\iota = o$. Hence we have either $h_1 : G(1') \to G(\tau_1)$, whence $\models \tau_1 \leq 1'$, or $h_2 : G(1') \to G(\tau_2)$, i.e., $\models \tau_2 \leq 1'$. By the IH (inductive hypothesis), we have either $\vdash \tau_1 \leq 1'$ or $\vdash \tau_2 \leq 1'$. In either case, $\vdash \tau_1 \cdot \tau_2 \leq 1'$ by the semilattice axioms.

Subcase $\tau = \tau_1 ; \tau_2$: Assuming $\models \tau_1 ; \tau_2 \leq 1'$, by Theorem 4.5, we have a homomorphism $h : G(1') \to G(\tau_1 ; \tau_2) = G(\tau_1) ; G(\tau_2)$. Let $G(\tau_1) = (V_1, E_1, \iota_1, o_1)$ and $G(\tau_2) = (V_2, E_2, \iota_2, o_2)$. Then $\iota_1 = o_1$ and $\iota_2 = o_2$. Thus, for $i \in \{1, 2\}$, we have $h_i : G(1') \to G(\tau_i)$, whence $\models \tau_i \leq 1'$, i.e., $\vdash \tau_i \leq 1'$ by the IH. Hence $\vdash \tau_1 ; \tau_2 \leq 1'$ by the monotonicity axiom Mon and the monoid axioms SG and Ide.

Case $\sigma = x$: By induction on τ.

Subcase $\tau = y$: Assuming $\models y \leq x$, we have $x = y$, whence $\vdash y \leq x$.

Subcase $\tau = 1'$: Just note that $\not\models 1' \leq x$.

Subcase $\tau = \tau_1 \cdot \tau_2$: Assuming $\models \tau_1 \cdot \tau_2 \leq x$, by Theorem 4.5, we have a homomorphism $h : G(x) \to G(\tau_1 \cdot \tau_2) = G(\tau_1) \cdot G(\tau_2)$. Let $G(\tau_1) = (V_1, E_1, \iota_1, o_1)$ and $G(\tau_2) = (V_2, E_2, \iota_2, o_2)$. Then we have either $h_1 : G(x) \to G(\tau_1)$, whence $\models \tau_1 \leq x$, or $h_2 : G(x) \to G(\tau_2)$, i.e., $\models \tau_2 \leq x$. By the IH, we have either $\vdash \tau_1 \leq x$ or $\vdash \tau_2 \leq x$. In either case, $\vdash \tau_1 \cdot \tau_2 \leq x$ by the semilattice axioms.

Subcase $\tau = \tau_1 ; \tau_2$: Assuming $\models \tau_1 ; \tau_2 \leq x$, by Theorem 4.5, we have a homomorphism $h : G(x) \to G(\tau_1 ; \tau_2) = G(\tau_1) ; G(\tau_2)$. Let $G(\tau_1) = (V_1, E_1, \iota_1, o_1)$ and $G(\tau_2) = (V_2, E_2, \iota_2, o_2)$. Then $(h(\iota), x, h(o)) \in E_1 \cup E_2$. Thus we have either $h_1 : G(x) \to G(\tau_1)$ and $h_2 : G(1') \to G(\tau_2)$, whence $\models \tau_1 \leq x$ and $\models \tau_2 \leq 1'$, or $g_1 : G(1') \to G(\tau_1)$ and $g_2 : G(x) \to G(\tau_2)$, whence $\models \tau_1 \leq 1'$ and $\models \tau_2 \leq x$. In the first case, we have $\vdash \tau_1 \leq x$ and $\vdash \tau_2 \leq 1'$ by the IH and case $\sigma = 1'$. Hence $\vdash \tau_1 ; \tau_2 \leq x ; 1' = x$ by the monotonicity axiom and the monoid axioms. The other case is completely analogous.

Case $\sigma = \sigma_1 \cdot \sigma_2$: Assuming $\models \tau \leq \sigma_1 \cdot \sigma_2$, we have $\models \tau \leq \sigma_1$ and $\models \tau \leq \sigma_2$. Then by the IH on σ, we have $\vdash \tau \leq \sigma_1$ and $\vdash \tau \leq \sigma_2$, whence $\vdash \tau \leq \sigma_1 \cdot \sigma_2$ by the semilattice axioms.

Case $\sigma = \sigma_1 ; \sigma_2$: We want to prove $\vdash \tau \leq \sigma_1 ; \sigma_2$. In this case we will not proceed by induction along the structure of τ. The reason for this is the presence of $1'$. If $1'$ does not occur in τ, σ_1, σ_2, then the usual induction w.r.t. the construction of τ works.

Motivating example Let $\tau_1 = x$, $\tau_2 = (y \cdot 1') ; w ; (z \cdot 1')$ and $\sigma_1 ; \sigma_2 = y ; x ; z$. Then $\models \tau_1 \cdot \tau_2 \leq \sigma_1 ; \sigma_2$, but neither $\models \tau_1 \leq \sigma_1 ; \sigma_2$ nor $\models \tau_2 \leq \sigma_1 ; \sigma_2$. So the problem is that, because of the presence of $1'$, the subterms can affect each other.

Definition 4.6 *Let τ be a term, $G(\tau) = (V, E, \iota, o)$ and $u, v \in V$. We define a term $\tau(u, v)$ which is intuitively the term corresponding to the subgraph of $G(\tau)$ between u and v. We define $\tau(u, v)$ by induction on τ. We will use the notation $G(\rho) = (V(\rho), E(\rho), \iota(\rho), o(\rho))$ for any term ρ.*

Case $\tau = 1'$: *In this case $V = \{\iota\} = \{o\}$, so $u = v = \iota = o$. We define $1'(\iota, \iota) = 1'$.*

Case $\tau = x$: *In this case $V = \{\iota, o\}$. We define $x(\iota, \iota) = x(o, o) = 1'$, $x(\iota, o) = x$, and $x(o, \iota)$ is undefined.*

Case $\tau = \delta \cdot \rho$: *In this case $V = V(\delta) \cup V(\rho)$ and $V(\delta) \cap V(\rho) = \{\iota, o\}$. We define $(\delta \cdot \rho)(u, v)$ by case-distinction.*

Subcase $u = v \in \{\iota, o\}$: $\delta(u, u) \cdot \rho(u, u)$.
Subcase $u = \iota$ and $v = o$: $\delta \cdot \rho$.
Subcase $u = \iota$ and $v \in G(\delta) \smallsetminus G(\rho)$: $\rho(\iota, \iota) ; \delta(\iota, v)$.
Subcase $u = \iota$ and $v \in G(\rho) \smallsetminus G(\delta)$: $\delta(\iota, \iota) ; \rho(\iota, v)$.
Subcase $u \in G(\delta) \smallsetminus G(\rho)$ and $v = o$: $\delta(u, o) ; \rho(o, o)$.
Subcase $u \in G(\rho) \smallsetminus G(\delta)$ and $v = o$: $\rho(u, o) ; \delta(o, o)$.
Subcase $u, v \in G(\delta) \smallsetminus G(\rho)$ and $\delta(u, v)$ is defined: $\delta(u, v)$.
Subcase $u, v \in G(\rho) \smallsetminus G(\delta)$ and $\rho(u, v)$ is defined: $\rho(u, v)$.
All other subcases: *undefined.*

CASE $\tau = \delta \,;\, \rho$: *In this case* $V = V(\delta) \cup V(\rho)$ *and* $V(\delta) \cap V(\rho) = \{m\}$ *for some* m *such that* $m = o(\delta) = \iota(\rho)$. *We define* $(\delta \,;\, \rho)(u, v)$ *by case-distinction.*
SUBCASE $u, v \in G(\delta) \smallsetminus G(\rho)$: $\delta(u, v)$.
SUBCASE $u \in G(\delta)$ AND $v \in G(\rho)$: $\delta(u, m) \,;\, \rho(m, v)$.
SUBCASE $u, v \in G(\rho) \smallsetminus G(\delta)$: $\rho(u, v)$.
ALL OTHER SUBCASES: *undefined.*

Lemma 4.7 *For any term* τ *we have the following:* $\tau(\iota, o) = \tau$, $\tau(u, u)$ *is defined for any* $u \in G(\tau)$ *and* $\vdash \tau(u, u) \leq 1'$.

The proof is by an easy induction. For the last case one can prove that $\models \tau(u, u) \leq 1'$, and then use our earlier case to infer $\vdash \tau(u, u) \leq 1'$. We leave the details to the reader. We will prove the following two lemmas.

Lemma 4.8 *Assume that* $\tau(u, v)$ *and* $\tau(v, w)$ *are defined. Then* $\tau(u, w)$ *is defined and* $\vdash \tau(u, w) \leq \tau(u, v) \,;\, \tau(v, w)$.

Let $h : V(\tau) \to V(\sigma)$ be a map. We call h an *endfree homo* if h is a homomorphism from the "end-free" graph $(V(\tau), E(\tau))$ to the "end-free" graph $(V(\sigma), E(\sigma))$. That is, h is a homomorphism from $G(\tau)$ to $G(\sigma)$ except that it may not preserve ι and o.

Lemma 4.9 *Let* $h : G(\tau) \to G(\sigma)$ *be an endfree homo. Let* $u, v \in G(\tau)$ *and assume that* $\tau(u, v)$ *is defined. Then* $\sigma(h(u), h(v))$ *is defined and* $h : G(\tau(u, v)) \to G(\sigma(h(u), h(v)))$ *is a homomorphism.*

From the above lemmas we will prove our case as follows. Assume that $\models \tau \leq \sigma_1 \,;\, \sigma_2$. Then there is a homomorphism $h : G(\sigma_1 \,;\, \sigma_2) \to G(\tau)$ by Theorem 4.5. Let ι and o be the beginning and endpoints of $G(\sigma_1 \,;\, \sigma_2)$, respectively, and let $m = o(\sigma_1) = \iota(\sigma_2)$ be the point "connecting" $G(\sigma_1)$ with $G(\sigma_2)$ in $G(\sigma_1 \,;\, \sigma_2)$. Then h is an endfree homo from $G(\sigma_1)$ to $G(\tau)$, and also from $G(\sigma_2)$ to $G(\tau)$ (by the definition of $G(\sigma_1 \,;\, \sigma_2) = G(\sigma_1) \,;\, G(\sigma_2)$). Then, by Lemma 4.9, $h : G(\sigma_1) \to G(\tau(\iota, h(m))$ is a homomorphism (preserving endpoints) and so $\vdash \tau(\iota, h(m)) \leq \sigma_1$ by the IH. Similarly, $\vdash \tau(h(m), o) \leq \sigma_2$. Thus $\vdash \tau(\iota, o) \leq \sigma_1 \,;\, \sigma_2$ by Lemma 4.8 and monotonicity. Hence it remains to prove Lemmas 4.8 and 4.9.

Proof of Lemma 4.8: We proceed by induction on τ.

CASE $\tau = 1'$: In this case $\iota = u = v = w = o$ and $\tau(u, w) = \tau(u, v) = \tau(v, w) = \tau(\iota, \iota) = 1'$, so we are done by $\vdash 1' \leq 1' \,;\, 1'$.

CASE $\tau = x$: In this case either $u = v = w \in \{\iota, o\}$, in which case we are done as above, or $u = v = \iota, w = o$ or $u = \iota, v = w = o$, in which cases we are done by $\vdash x \leq 1' \,;\, x$ and $\vdash x \leq x \,;\, 1'$.

CASE $\tau = \delta \cdot \rho$: We proceed by case-distinction according to the alternatives in case $\tau = \delta \cdot \rho$ in Definition 4.6. It is not hard to check that in each of the following combined situations $\tau(u, w)$ is defined.

SUBCASE $u = v = w = \iota$: We have to show that $\vdash \tau(\iota, \iota) \leq \tau(\iota, \iota) \,;\, \tau(\iota, \iota)$. By Lemma 4.7, we have that $\vdash \tau(\iota, \iota) \leq 1'$. Since $\vdash (x \cdot 1') \leq (x \cdot 1') \,;\, (x \cdot 1')$, we are done.

SUBCASE $u = v = \iota$, $w = o$: We have to show that $\vdash \tau(\iota, o) \leq \tau(\iota, \iota) \,;\, \tau(\iota, o)$, i.e., that $\vdash \tau \leq \tau(\iota, \iota) \,;\, \tau$. By the IH we have that $\vdash \delta \leq \delta(\iota, \iota) \,;\, \delta$, $\vdash \delta(\iota, \iota) \leq 1'$ and $\vdash \rho \leq \rho(\iota, \iota) \,;\, \rho$, $\vdash \rho(\iota, \iota) \leq 1'$, and $\tau(\iota, \iota) = \delta(\iota, \iota) \cdot \rho(\iota, \iota)$ by definition. Then we can use axiom FbI to conclude this case.

SUBCASE $u = v = \iota$, $w \in G(\delta) \smallsetminus G(\rho)$: We have to show that $\vdash \tau(\iota, w) \leq \tau(\iota, \iota) \,;\, \tau(\iota, w)$. Now, we have $\tau(\iota, w) = \rho(\iota, \iota) \,;\, \delta(\iota, w)$ and, by the IH, $\vdash \delta(\iota, w) \leq \delta(\iota, \iota) \,;\, \delta(\iota, w)$. We can use axiom CbI and monotonicity to conclude this case.

SUBCASE $u = v = w = o \neq \iota$: This is analogous to subcase $u = v = w = \iota$, we leave it to the reader.

SUBCASE $u = \iota,\ v \in G(\delta) \smallsetminus G(\rho),\ w = o$: We have to show that $\vdash \tau(\iota, o) \leq \tau(\iota, v) ; \tau(v, o)$, i.e., $\vdash \delta \cdot \rho \leq \rho(\iota, \iota) ; \delta(\iota, v) ; \delta(v, o) ; \rho(o, o)$. We can use that by IH we have $\vdash \delta \leq \delta(\iota, v) ; \delta(v, o) \leq 1' ; \delta(\iota, v) ; \delta(v, o) ; 1'$ and $\vdash \rho \leq \rho(\iota, \iota) ; \rho ; \rho(o, o)$ and then axiom FbI.

SUBCASE $u = \iota,\ v, w \in G(\delta) \smallsetminus G(\rho)$: We have to show that $\vdash \tau(\iota, w) \leq \tau(\iota, v) ; \tau(v, w)$. By definition, $\tau(\iota, w) = \rho(\iota, \iota) ; \delta(\iota, w)$, the same for $\tau(\iota, v)$ with v in place of w, and $\tau(v, w) = \delta(v, w)$. So we have to prove $\vdash \rho(\iota, \iota) ; \delta(\iota, w) \leq \rho(\iota, \iota) ; \delta(\iota, v) ; \delta(v, w)$, by using the IH and the axioms. By IH, $\vdash \delta(\iota, w) \leq \delta(\iota, v) ; \delta(v, w)$, hence the result follows.

SUBCASE $u \in G(\delta) \smallsetminus G(\rho),\ v = w = o$: Proceeding as before, we have to show that $\vdash \delta(u, o) ; \rho(o, o) \leq (\delta(u, o) ; \rho(o, o)) ; \delta(o, o)$. The same argument as in subcase $u = v = \iota,\ w \in G(\delta) \smallsetminus G(\rho)$ gives the result.

SUBCASE $u, v, w \in G(\delta) \smallsetminus G(\rho)$: This case is straightforward by the IH.

SUBCASE $u, v \in G(\delta) \smallsetminus G(\rho),\ w = o$: We have to show that $\vdash \delta(u, o) \cdot \rho(o, o) \leq \delta(u, v) ; (\delta(v, o) ; \rho(o, o))$. This is completely analogous to subcase $u = \iota,\ v, w \in G(\delta) \smallsetminus G(\rho)$.

SUBCASE WHERE δ AND ρ ARE INTERCHANGED: These cases are completely analogous with the earlier corresponding cases.

CASE $\tau = \delta ; \rho$: We can proceed by case-distinction according to the alternatives in case $\tau = \delta ; \rho$ in Definition 4.6. It is not hard to check that in each of the following combined situations $\tau(u, w)$ is defined. Showing that $\vdash \tau(u, w) \leq \tau(u, v) ; \tau(v, w)$ is straightforward by using the IH and the axioms. This concludes the proof of Lemma 4.8. ∎

Proof of Lemma 4.9: We proceed by induction on τ. Let $u' = h(u)$ and $v' = h(v)$.

CASE $\tau = 1'$: In this case $u = v = \iota$, hence $u' = v'$ and $\tau(u, v) = 1'$. By Lemma 4.7, $\vdash 1' \geq \sigma(u', u')$, whence $\models 1' \geq \sigma(u', u')$ by soundness. Then, by Theorem 4.5, there is $h : G(1') \to G(\sigma(u', u'))$ as desired.

CASE $\tau = x$: In this case either $u = v \in \{\iota, o\}$ in which case $\tau(u, v) = 1'$ and we are done as above, or $u = \iota$ and $v = o$ in which case $\tau(u, v) = x = \tau$. We show that $\sigma(u', v')$ is defined and $h : G(\tau) \to G(\sigma(u', v'))$ by induction on σ.

SUBCASE $\sigma = 1'$: There is no such homomorphism, so this case does not occur.

SUBCASE $\sigma = y$: In this case $y = x$, $u' = \iota$ and $v' = o$ by $h : G(x) \to G(y)$ and we are done.

SUBCASE $\sigma = \delta \cdot \rho$: In this case either $h : G(\tau) \to G(\delta)$ or $h : G(\tau) \to G(\rho)$, assume the first alternative. If $u', v' \in V(\delta) \smallsetminus V(\rho)$, then, by the IH, $h : G(\tau(u, v)) \to G(\delta(u', v')) = G(\sigma(u', v'))$ as desired. If $u' = \iota(\sigma)$ and $v' \in V(\delta) \smallsetminus V(\rho)$, then by the IH we have $h : G(\tau(u, v)) \to G(\delta(\iota(\sigma), v'))$. But $\delta(\iota(\sigma), v') \geq \sigma((\iota(\sigma), v'))$ by Definition 4.6 and lemma 4.7, whence there is $g : G(\tau(u, v)) \to G(\sigma(\iota(\sigma), v'))$ by the same argument as in case $\tau = 1'$. The case when $v' = o(\sigma)$ is completely analogous.

SUBCASE $\sigma = \delta ; \rho$: If $u', v' \in V(\delta) \smallsetminus V(\rho)$, then the case follows by the IH (for δ). If $v' = o(\delta) = \iota(\rho)$, then $\sigma(u', v') = \delta(u', v') ; \rho(v', v') \leq \delta(u', o(\delta))$, by Lemma 4.7. Hence the case follows as above. The case $u' = o(\delta) = \iota(\rho)$ is completely analogous.

CASE $\tau = \delta \cdot \rho$: There are the following cases according to the definition of $\tau(u, v)$: (1) $u, v \in g(\delta) \cap G(\rho)$, (2) $u = \iota$ and $v \in G(\delta) \smallsetminus G(\rho)$ and the symmetric case with δ and ρ interchanged, (3) $\iota \in G(\delta) \smallsetminus G(\rho)$ and $u = o$ and the symmetric case with δ and ρ interchanged, (4) $u, v \in G(\delta) \smallsetminus G(\rho)$ and the symmetric case with δ and ρ interchanged.

SUBCASE 1: By the IH we have that $h : G(\delta(u, v)) \to G(\sigma(u', v'))$ and $h : G(\rho(u, v)) \to G(\sigma(u', v'))$ are homomorphisms. Now $\tau(u, v) = \delta(u, v) \cdot \rho(u, v)$ and then, by definitions, $h : \tau(u, v) \to G(\sigma(u', v'))$ is a homomorphism.

SUBCASE 2: By the IH we have that $h : G(\delta(\iota, v)) \to G(\sigma(u', v'))$ and $h : G(\rho(\iota, \iota)) \to G(\sigma(u', u'))$ are homomorphisms. Now $\tau(\iota, v) = \rho(\iota, \iota);\delta(\iota, v)$ and then, by definitions, $h : \tau(\iota, v) \to G(\sigma(u', v'))$ is a homomorphism.

SUBCASE 3: This is completely analogous to subcase 2.

SUBCASE 4: This easily follows from the IH for δ (or for ρ in the symmetric case).

CASE $\tau = \delta\,;\rho$: The case when $u, v \in G(\delta) \smallsetminus G(\rho)$ (and its symmetric version with δ and ρ interchanged) easily follows from the IH for δ (or for ρ). So assume otherwise and let $m = o(\delta) = \iota(\rho)$ and $m' = h(m)$. By the IH, $h : G(\delta(u,m)) \to G(\sigma(u',m'))$ and $h : G(\rho(m,v)) \to G(\sigma(m',v'))$ are homomorphisms, whence we have that $h : G(\delta(u,m)\,;\rho(m,v)) \to G(\sigma(u',v'))$ is a homomorphism. By definition $G(\tau(u,v)) = G(\delta(u,m);\rho(m,v))$, and $\sigma(u',m');\sigma(m',v') \geq \sigma(u',v')$ by Lemma 4.8. Hence we have $h : G(\tau(u,v)) = G(\sigma(u,v))$ as desired, finishing the proof of Lemma 4.9. ∎

This finishes the proof of Theorem 4.3 ∎

5 Further results and problems

One might wonder if we could improve on the finite axiomatizability results by including more operations into the signature. If we go beyond the positive subsignatures of Tarski's relation algebras and do not want the full power of Boolean algebras, then arguably the most important connectives are the residuals of composition and the Kleene star * (reflexive, transitive closure). We recall the interpretation of the residuals in representable algebras:

$$x\backslash y = \{(u,v) \in U \times U : \forall w((w,u) \in x \text{ implies } (w,v) \in y)\}$$
$$x/y = \{(u,v) \in U \times U : \forall w((u,w) \in y \text{ implies } (v,w) \in x)\}$$

The operations $\backslash$, $/$ and * are not additive, thus Proposition 4.4 does not apply to them.

It is shown in [AM94] that the variety $\mathsf{R}(\cdot, ;, \backslash, /)$ is finitely axiomatizable. On the other hand, the variety $\mathsf{V}(;, \cdot, +, \backslash, /)$ generated by $\mathsf{R}(;, \cdot, +, \backslash, /)$ is not axiomatizable by finitely many equations (even if we omit composition ;), cf. [HM10]. Next we show the promised strengthening of Theorem 3.1.

Theorem 5.1 *Let* $\{;, +\} \subseteq \Lambda \subseteq \{;, +, ^\smile, 1', 1, 0, ^*, \backslash, /\}$. *Then the class* $\mathsf{R}(\Lambda)$ *is not finitely axiomatizable.*

Proof: The proof is similar to the proof of Theorem 3.1. We just mention the necessary modifications.

For the Kleene star * note the following. In $\mathfrak{A}_n$, we have $0\,;0 = 0$, $1'\,;1' = 1'$ and $x\,;x = 1$ for every $x \in A_n \smallsetminus \{0, 1'\}$. Hence we define, in $\mathfrak{A}_n$, $0^* = 1'$, $1'^* = 1'$ and $x^* = 1$ for every $x \in A_n \smallsetminus \{0, 1'\}$, and in $\mathfrak{A}$, $\bar{0}^* = \bar{1}'$, $\bar{1}'^* = 1'$ and $\bar{x}^* = \bar{1}$ for every $x \in A \smallsetminus \{\bar{0}, \bar{1}'\}$. Then it is straightforward that the representation rep defined by the equation 3 preserves * as well, whence the ultraproduct $\mathfrak{A}$ of the algebras $\mathfrak{A}_n$ augmented with * is representable.

For the residuals, first note that $\backslash$ and $/$ coincide in $\mathfrak{A}_n$ and in the ultraproduct, since these algebras are symmetric. We want to define the residual $\backslash$ in the algebras $\mathfrak{A}_n$ so that in their ultraproducts $\backslash$ will be representable. To this end we define $\backslash$ so that $x \backslash y$ is the largest element z such that $x\,;z \leq y$. Then the algebras $\mathfrak{A}_n$ are in fact closed under the operation $\backslash$ (since they are finite). Indeed, the extension of $x \backslash y$ is determined by

$$z \leq x \backslash y \text{ iff } x\,;z \leq y$$

Looking at the multiplication table of $\mathfrak{A}_n$ we can explicitly compute the extension of $x \backslash y$ as follows. For $x \in G_n \smallsetminus \{1', o\}$, let $S(x)$ denote the unique element of G_n such that $x\,;S(x) = o$, i.e., $\{x, S(x)\} \in S$. We proceed by case distinction on the form of x and y, and the value of $x \backslash y$ is determined by the first applicable case below. If $x = 1$ or $y = 1$, then

$$x \backslash 1 = 1 \qquad 1 \backslash y = \begin{cases} 1 & \text{if } y = 1 \\ 0 & \text{otherwise} \end{cases}$$

If $x = 0$ or $y = 0$, then

$$0 \setminus y = 1 \qquad x \setminus 0 = \begin{cases} 1 & \text{if } x = 0 \\ 0 & \text{otherwise} \end{cases}$$

If $x = 1'$ or $y = 1'$, then

$$1' \setminus y = y \qquad x \setminus 1' = \begin{cases} 1 & \text{if } x = 0 \\ 1' & \text{if } x = 1' \\ 0 & \text{otherwise} \end{cases}$$

If $x = o$ or $y = o$, then

$$o \setminus y = \begin{cases} 1 & \text{if } y = 1 \\ 1' & \text{if } o \leq y < 1 \\ 0 & \text{otherwise} \end{cases} \qquad x \setminus o = \begin{cases} 1' & \text{if } x = o \\ S(x) & \text{if } x \in G_n \smallsetminus \{1', o\} \\ 0 & \text{otherwise} \end{cases}$$

When $x = c \in G_n \smallsetminus \{1', o\}$ we have

$$c \setminus y = \begin{cases} 1 & \text{if } y = 1 \\ 1' + S(c) & \text{if } c + o \leq y < 1 \\ S(c) & \text{if } o \leq y < c + o \\ 1' & \text{if } c \leq y < c + o \\ 0 & \text{otherwise} \end{cases}$$

The next case is when $x = c + 1'$ for some $c \in G_n \smallsetminus \{1', o\}$:

$$(c + 1') \setminus y = \begin{cases} 1 & \text{if } y = 1 \\ 1' + S(c) & \text{if } 1' + c + S(c) + o \leq y < 1 \\ S(c) & \text{if } S(c) + o \leq y < 1' + c + S(c) + o \\ 1' & \text{if } c + 1' \leq y < 1' + c + S(c) + o \\ 0 & \text{otherwise} \end{cases}$$

In all other cases, we have

$$x \setminus y = \begin{cases} 1' & \text{if } x \leq y \\ 0 & \text{otherwise} \end{cases}$$

Thus we defined $x \setminus y$ in every $\mathfrak{A}_n$, whence the value of $\setminus$ in $\mathfrak{A}$ is determined.

Next we show that the extension of $\mathfrak{A}$ with $\setminus$ can be represented. To this end we modify the construction of the graph $\mathcal{G}$ in the proof of Theorem 3.1. We have a new type of defect: $(x, y) \in A \times A$ such that $x \setminus y \notin \ell_k(i, j)$ for some $(i, j) \in E_k$ and there is no $w \in E_k$ such that $x \in \ell_k(w, i)$ and $y \notin \ell_k(w, j)$. We assume a fair scheduling σ of all defects as in the proof of Theorem 3.1. If $\sigma_k = (x, y, z)$ is a defect of the form $z \leq x \,;\, y$, then in the $(k+1)$th step we do the construction described in the proof of Theorem 3.1. If $\sigma_k = (x, y)$ is a new type of defect of $(i, j) \in E_k$, then we do the following. We will choose prime filters $F, G \in \mathcal{F}$ such that

1. $\bar{1}' \notin F, G$,
2. $x \in F$ and $y \notin G$
3. the triangle $(F, G, \ell_k(i, j))$ and all its permutations are coherent.

Provided that such filters F and G exist, we can extend the graph $\mathcal{G}_k$ with an extra node $w_{i,j}$ and define

$$\ell_{k+1}(w_{i,j}, i) = \ell_{k+1}(i, w_{i,j}) = F \quad \ell_{k+1}(w_{i,j}, w_{i,j}) = F(\bar{1}') \quad \ell_{k+1}(w_{i,j}, j) = \ell_{k+1}(j, w_{i,j}) = G$$

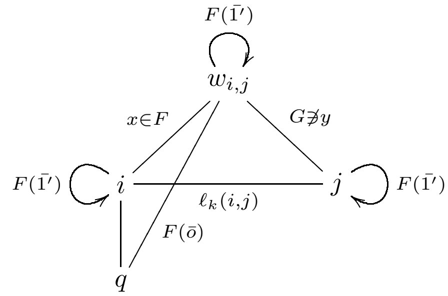

Figure 5: Residual when $x \setminus y \notin \ell_k(i,j)$

and, for every $q \in U_k \smallsetminus \{i,j\}$ with $(i,q) \in E_k$,

$$\ell_{k+1}(w_{i,j}, q) = \ell_{k+1}(q, w_{i,j}) = F(\bar{o})$$

See Figure 5 for the case when $i \neq j$. We define $\mathcal{G}_{k+1}$ as the union of these extensions of $\mathcal{G}_k$ for every $(i,j) \in E_k$ such that $x \setminus y$ is a defect of (i,j).

It remains to define the filters $F, G \in \mathcal{F}$ satisfying the conditions above. First consider the case when $i = j$. Since $x \setminus y \notin \ell_k(i,j) = F(\bar{1}')$, we have that $x \not\leq y$. Then we choose $F \in \mathcal{F}$ such that $x \in F$ and $y \notin F$, and also let $G = F$.

Now assume that $i \neq j$. We will proceed by case distinction according to the cases in the definition of $x \setminus y$. When $x \in \{\bar{1}, \bar{1}', \bar{0}\}$ or $y \in \{\bar{1}, \bar{1}', \bar{0}\}$, then either the choice of F and G is straightforward, or will be covered by one of the cases below.

First assume that $x = \bar{o}$. Then we can let $F = F(\bar{o})$ and G be any filter such that $y \notin G$. Using that $\bar{o} \setminus y \notin \ell_k(i,j)$ and that $\bar{o}$ is a flexible atom it is easy to show that $(F, G, \ell_k(i,j))$ satisfies the coherency condition above.

The next case is when $y = \bar{o}$. We worked out the situation $x = \bar{o}$ above. Next assume that $x \in At \smallsetminus \{\bar{1}', \bar{o}\}$, whence $x \setminus \bar{o} = s(x) \notin \ell_k(i,j)$ (or the symmetric case $x \setminus \bar{o} = s^{-1}(x) \notin \ell_k(i,j)$). We have to choose F so that $s(z), s^{-1}(z) \notin F$ whenever $z \in \ell_k(i,j)$, since we have the requirements that $\bar{o} \notin G$ and $F \,;\, \ell_k(i,j) \subseteq G$. We let $F = F(\{x, \bar{o}\})$ if $x \in At \smallsetminus \{\bar{1}', \bar{o}, \bar{a}, \bar{b}\}$. If $x = \bar{a}$, then one of $F(\{\ell(\bar{a}) \cup \{\bar{o}\}\})$ or $F(\{\ell'(\bar{a}) \cup \{\bar{o}\}\})$ avoids $\{s(z), s^{-1}(z) : z \in \ell_k(i,j)\}$, because of the following. Recall that $\ell_k(i,j) \in \mathcal{F}$. If $\ell_k(i,j) = F(z)$ for some $z \in At \smallsetminus \{\bar{a}, \bar{b}\}$, then z cannot be in both $\ell(\bar{a})$ and $\ell'(\bar{a})$. If $\ell_k(i,j) = F(\ell(\bar{a}))$ or $\ell_k(i,j) = F(\{\ell(\bar{a}) \cup \{\bar{o}\}\})$, then we can choose $F = F(\{\ell(\bar{a}) \cup \{\bar{o}\}\})$, and similarly with ℓ' instead of ℓ. Finally, if $\ell_k(i,j) = F(\ell(\bar{b}))$ or $\ell_k(i,j) = F(\{\ell(\bar{b}) \cup \{\bar{o}\}\})$, then we can let $F = F(\{\ell(\bar{a}) \cup \{\bar{o}\}\})$ (with the straightforward modifications for ℓ' instead of ℓ). Hence we can choose F with the above property. The case $x = \bar{b}$ is completely symmetric. Then G can be defined as an element of $\mathcal{F}$ with the property that $\bar{o} \notin G$ and, in case $\bar{o} \notin \ell_k(i,j)$, G avoids $\{s(z), s^{-1}(z) : z \in F\}$ (by the same reasoning as above). The case $x \notin At$ can be treated similarly by first picking an atom $x' \in At$ below x and then choosing F for x' and then G as above.

In the case when $x = c \in At \smallsetminus \{\bar{1}', \bar{o}\}$, F can be chosen as in the previous paragraph. The choice of G is a bit more intricate, since G has to avoid y (instead of $\bar{o}$) and $\{s(z), s^{-1}(z) : z \in F\}$ (in case $\bar{o} \notin \ell_k(i,j)$). One can either (i) choose an atom $d \in At \smallsetminus \{\bar{1}', \bar{a}, \bar{b}\}$ such that $d \not\leq y$ and $d \notin \{s(z), s^{-1}(z) : z \in F\}$, or (ii) show that $\bar{a} \not\leq y$ and either $\ell(\bar{a}) \cap \{s(z), s^{-1}(z) : z \in F\} = \emptyset$ or $\ell'(\bar{a}) \cap \{s(z), s^{-1}(z) : z \in F\} = \emptyset$, or (iii) show the same for $\bar{b}$ instead of $\bar{a}$. Then we can let G be the filter generated by $\bar{o}$ and d in case (i), or in case (ii) either $F(\{\ell(\bar{a}) \cup \{\bar{o}\}\})$ or $F(\{\ell'(\bar{a}) \cup \{\bar{o}\}\})$, and similarly in case (iii).

The case $x = c + 1'$ for $c \in At \smallsetminus \{\bar{1}', \bar{o}\}$ is completely analogous, since $\bar{1}'$ is not in any filter labelling an irreflexive edge. Finally, in the remaining case, one can choose an atom $c \leq x$ and use one of the above cases for defining F and G. It is routine to show that the above choices of F and G satisfy the requirements, in particular, that $(F, G, \ell_k(i,j))$ and its permutations are coherent. Hence, using that $\bar{o}$ is a flexible atom, $\mathcal{G}_{k+1}$ satisfies the coherency condition Coh.

We let $\mathcal{G}$ be the union of $\mathcal{G}_k$ as before. It remains to show that rep as defined by the equation 3 represents $\setminus$ as well.

First assume that $(i,j) \in \mathsf{rep}(x \setminus y)$, i.e., $x \setminus y \in \ell(i,j)$. Let w be arbitrary such that $(w,i) \in \mathsf{rep}(x)$, i.e., $x \in \ell(w,i)$. Then by coherency condition Coh, $x\,;(x \setminus y) \in \ell(w,j)$. Since $x\,;(x \setminus y) \leq y$, we have $y \in \ell(w,j)$, i.e., $\ell(w,j) \in \mathsf{rep}(y)$. Hence $(i,j) \in \mathsf{rep}(x) \setminus \mathsf{rep}(y)$ as desired.

For the other direction assume that $(i,j) \notin \mathsf{rep}(x \setminus y)$, i.e., $x \setminus y \notin \ell_k(i,j)$ for any k. Then by the construction above, we have k such that $w_{i,j} \in E_k$ and $x \in \ell_{k+1}(w_{i,j}, i)$, i.e., $(w_{i,j}, i) \in \mathsf{rep}(x)$, and $y \notin \ell_{k+1}(w_{i,j}, j)$, i.e., $(w_{i,j}, j) \notin \mathsf{rep}(y)$. Hence $(i,j) \notin \mathsf{rep}(x) \setminus \mathsf{rep}(y)$ as desired, finishing the proof of Theorem 5.1 ∎

We conclude with the following problem. It is proved in [Re64] that the variety generated by representable Kleene algebras $\mathsf{RKA} = \mathsf{R}(+, ;, 1', 0, ^*)$ is not finitely axiomatizable, while it is proved in [Ko94] that there is a finitely axiomatizable quasivariety that generates the same variety as RKA. Thus all the equations valid in RKA can be derived from a finite set of quasiequations. This motivates the following problem which asks for similar results in the extended similarity types.

Problem 5.2 Find finitely axiomatizable quasivarieties such that they generate the same varieties as those representable (semi)lattice ordered monoids/semigroups whose signatures include residuals and/or Kleene star.

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