

# Reducing Predicate Logic to Propositional Logic

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GOAL: Reduce predicate logic  $L_\omega$  to a very simple modal propositional logic  $M_3$ .

$M_3$  = “propositional modal logic with 3 commuting S5 modalities”.

### Theorem 1

*$L_\omega$  can be reduced to  $M_3$ , i.e.,*

*There is a computable meaning preserving translation mapping*

$$\text{Tr} : L_\omega \rightarrow M_3$$

*such that*

$$Ax \vdash \varphi \quad \Leftrightarrow \quad \text{Tr}(Ax) \vdash_m \text{Tr}(\varphi)$$

*for all  $Ax \subseteq L_\omega, \varphi \in L_\omega$ .*

*Here,  $\vdash$  and  $\vdash_m$  are provability of  $L_\omega$  and  $M_3$  respectively.*

Stronger theorems will follow.

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## SYNTAX

1. Propositional variables. We will need only one propositional variable  $p$ . These are formulas.
2. Connectives: if  $\varphi, \psi$  are formulas, then  $\neg\varphi, \varphi \vee \psi, \diamond_1\varphi, \diamond_2\varphi, \diamond_3\varphi$  are formulas, too.
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**SEMANTICS** A **model** consists of a set with three commuting equivalence relations on it:  $\langle V, R_1, R_2, R_3 \rangle$ . The elements of  $V$  are called possible worlds or situations. An equivalence relation  $R_i$  means classifying (or partitioning) the situations according to some point of view.

An **evaluation** in this model is a function  $\text{val} : M_3 \rightarrow \mathcal{P}(V)$  satisfying the following:

1.  $\text{val}(p) \subseteq V$  is arbitrary.
2.  $\text{val}(\neg\varphi) = V \setminus \text{val}(\varphi)$ ,  $\text{val}(\varphi \vee \psi) = \text{val}(\varphi) \cup \text{val}(\psi)$ ,
3.  $\text{val}(\diamond_i\varphi) = R_i[\text{val}(\varphi)]$ , where if  $X$  is a subset of  $V$ , and  $R$  is an equivalence relation on  $V$ , then  $R[X]$  denotes the  $R$ -image of  $X$ , i.e., the  $R$ -granulated version of  $X$ .

$\varphi$  is valid in  $\langle V, R, S, Q \rangle$ , in symbols  $\langle V, R, S, Q \rangle \models \varphi$ , if  $\text{val}(\varphi) = V$  for all evaluations  $\text{val}$  in this model.  $\varphi$  is **valid**, in symbols  $\models \varphi$  iff  $\varphi$  is valid in all models.

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**PROOF SYSTEM** We use  $\Box_i \stackrel{d}{=} \neg\Diamond_i\neg$  and  $\rightarrow$  as derived connectives as usual. The **axioms** are the following (where  $\varphi, \psi \in M_3$  and  $i, j \in \{1, 2, 3\}$ ):

((B))  $\varphi$ , if  $\varphi$  is a propositional tautology,

((K))  $\Box_i(\varphi \rightarrow \psi) \rightarrow (\Box_i\varphi \rightarrow \Box_i\psi)$ ,

((S5))  $\Diamond_i\varphi \rightarrow \Box_i\Diamond_i\varphi$ ,

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The **rules** are Modus Ponens and Generalization (or, in other word, Necessitation, i.e.,  $\varphi \vdash \Box_i\varphi$ ). This modal logic is complete w.r.t. the above semantics (i.e., the frames consisting of three commuting equivalence relations as accessibility relations for the three modalities). See the book [GKWZ].

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## Three-variable logic without equality or substitutions $\mathcal{L}d_3$ .

### SYNTAX

1. Atomic formula:  $P(x, y, z)$ . This is a formula.

Note: the formula  $P(y, x, z)$  is not available in this language. No substitutions!

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An **evaluation** in this model is a function  $\text{val} : \{x, y, z\} \rightarrow M$ . A formula  $\varphi$  is satisfied in  $\langle M, P \rangle$  under  $\text{val}$ , in symbols

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$\varphi$  is valid in  $\mathfrak{M}$ , in symbols  $\mathfrak{M} \models \varphi$ , if  $\mathfrak{M} \models \varphi[\text{val}]$  for all evaluations  $\text{val}$  in this model.  $\varphi$  is **valid**, in symbols  $\models \varphi$  iff  $\varphi$  is valid in all models.

**PROOF SYSTEM**  $Fmd_3$  is inherently incomplete w.r.t. the above semantics, i.e., there is no finite, Hilbert-style proof system which would be sound and complete w.r.t. this semantics. We will use the following sound but incomplete proof system  $\vdash$ :

The logical **axioms** are the following. Let  $\varphi, \psi \in Fmd_3$  and  $v, w \in \{x, y, z\}$ .

- ((1))  $\varphi$ , if  $\varphi$  is a propositional tautology.
- ((2))  $\forall v(\varphi \rightarrow \psi) \rightarrow (\exists v\varphi \rightarrow \exists v\psi)$ .
- ((3))  $\varphi \rightarrow \exists v\varphi$ .
- ((4))  $\exists v\exists v\varphi \rightarrow \exists v\varphi$ .
- ((5))  $\exists v(\varphi \vee \psi) \leftrightarrow (\exists v\varphi \vee \exists v\psi)$ .
- ((6))  $\exists v\neg\exists v\varphi \rightarrow \neg\exists v\varphi$ .
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The inference **rules** are Modus Ponens ((MP), or detachment), and Generalization ((G)).

We will state a **partial completeness theorem** for  $\mathcal{L}d_3$  which is as good as it can be.

## The equational theory of $Df_3$ (our third form)

This is equational logic as the background logic, and the defining axioms of  $Df_3$  as logical axioms. (We refer to this logic informally as “ $Df_3$ -logic”.)

**SYNTAX** The language consists of equations  $\tau = \sigma$  where  $\tau, \sigma$  are terms built up from (arbitrarily many) variables by the use of the function symbols  $+, -, \diamond_i$  for  $i = 1, 2, 3$ , where  $+$  is binary and the rest are unary.

**SEMANTICS** If  $\mathfrak{A}$  is an algebra of the above similarity type, then  $\mathfrak{A} \models \tau = \sigma$  is the usual one. We say that  $\tau = \sigma$  is **valid** in  $Df_3$ -logic if it is valid in all Boolean algebras with three commuting complemented closure operators (i.e., in all  $Df_3$ s).

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## PROOF SYSTEM

The **axioms** are those of Boolean algebras with three commuting complemented closure operators  $\Diamond_i$ : Let  $x, y, z$  be variables and  $i, j = 1, 2, 3$ .

$$((B1)) \quad x + y = y + x,$$

$$((B2)) \quad x + (y + z) = (x + y) + z,$$

$$((B3)) \quad -(- (x + y) + -(x + -y)) = x,$$

$$((D1)) \quad x + \Diamond_i x = \Diamond_i x,$$

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The **rules** are those of the equational logic:

Rules of equivalence:

$$\tau = \tau, \quad \tau = \sigma \vdash \sigma = \tau, \quad \tau = \sigma, \sigma = \rho \vdash \tau = \rho,$$

Rules of congruence:

$$\tau = \sigma, \rho = \delta \quad \vdash \quad -\tau = -\sigma, f\tau = f\sigma, g\tau = g\sigma, h\tau = h\sigma, \\ \tau + \rho = \sigma + \delta,$$

Rule of invariance:

$\tau = \sigma \vdash \tau' = \sigma'$  where  $\tau', \sigma'$  are obtained from  $\tau, \sigma$  by replacing the variables simultaneously with arbitrary terms.

The above proof system is complete w.r.t.  $Df_3$ -logic.

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The above proof system is **complete** w.r.t.  $Df_3$ -logic.

## Theorem 2

*These three logics are equivalent with each other.*

Two logics  $\langle L, \vdash \rangle$  and  $\langle M, \vdash^a \rangle$  are said to be **equivalent** if there is a computable bijection (translation function)  $\text{tr}$  between their formulas which respects their proof systems:

$\text{tr} : L \rightarrow M$  such that for all  $T \subseteq L$  and  $\varphi \in L$  we have

$$T \vdash \varphi \quad \Leftrightarrow \quad \text{tr}(T) \vdash^a \text{tr}(\varphi).$$

## Theorem 3

*There is a computable meaning preserving translation mapping  $\text{Tr} : L_\omega \rightarrow \text{Fmd}_3$  with a decidable range such that the following three statements hold for all  $Ax \subseteq L_\omega, \varphi \in L_\omega$ .*

$$Ax \models \varphi \quad \Leftrightarrow \quad \text{Tr}Ax \models \text{Tr}\varphi$$

$$\text{Tr}Ax \models \text{Tr}\varphi \quad \Leftrightarrow \quad \text{Tr}Ax \models \text{Tr}\varphi$$

$$\Delta \models \varphi \leftrightarrow \text{Tr}\varphi,$$

*where  $\Delta$  is an explicit definition of the connection between  $L_\omega$  and  $\mathcal{L}d_3$ :  $\Delta$  defines the atomic formulas of  $L_\omega$  in terms of  $\mathcal{L}d_3$ .*

The first statement is reducing  $L_\omega$  to  $\mathcal{L}d_3$  (equivalently, to  $M_3$ ).  
 The second statement is a partial completeness theorem for  $\mathcal{L}d_3$ .  
 The third statement says that  $\text{Tr}$  preserves meanings.

There is a uniform translation mapping  $\text{Tr}$  which works for all possible theories  $T$  (and  $Ax$ ) of predicate logic.

## Theorem 4

*The conclusion of Gödel's incompleteness theorem holds for modal propositional logic  $[S5, S5, S5]$  as well as for three-variable equality or substitution-free logic  $\mathcal{L}d_3$ .*

The theorem says that though these logics seem to be very weak, they are “rich enough”.

## Theorem 5

*The free  $Df_3$  algebras are not atomic (unless they have no free generators).*

There are other consequences for free algebras.

## HISTORY, MOTIVATION

Tarski 1953 “epaté the logique burgeois”.

Tarski-Givant: Formalizing Set Theory without variables. 1987.

Reduction of Set theory to equational theory of relation algebras (RA), to 4-variable logic with equality ( $CA_4$ ), and to 3-variable semantical logic with equality ( $RCA_3$ ).

Németi 1986: Decidability and free algebras in algebraic logic. Academic Doctoral Dissertation.

Reducing Set theory to 3-variable logic with equality, to the equational theory of  $CA_3$ .

Gyenis, Z.: On atomicity of free algebras in certain cylindric-like varieties. Logic Journal of the IGPL 19,1 (2011), 44-52.

3-variable logic without equality but with substitutions has Gödel's incompleteness property, and the free  $SCA_3$  algebras are not atomic.

Now: Predicate logic to 3-variable logic without equality or substitutions, to the equational theory of  $Df_3$ . Implies all above.

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## DISCUSSION OF THE CONDITIONS OF THE THEOREMS

Three modalities are needed. Two are not enough.

Commutativity of the three modalities is needed.

Ternary relation in  $\mathcal{L}d_3$  is needed. Reduction to  $\mathcal{L}d_3$  cannot be done with a binary relation  $P$ . Reduction to  $L_3$  without equality but substitutions can be done with a binary relation  $P$ .

### Open Problem

*Is it necessary that the closure operators be **complemented** in our theorems? Do the theorems hold for Boolean semigroups?*

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