Warning!

From this point on the rest of this material is only in very-very draft form. It will be polished later. It is included only for giving a more complete perspective on the approach.

7 Gödel incompleteness

8 Toward general relativity: accelerated observers

8.1 Accelerated observers

In this chapter we wish to expand our language and theory to handle non-inertial observers as well. We start general relativity theory by waiving our old assumptions that:

- all observers are inertial bodies (**Ax2**), and
- all geodesics are Euclidean lines (**Ax1**).

The physical intuition is that we do not exclude the existence of observers whose velocity changes in time; and such observers can see inertial bodies moving on geodesics different from Euclidean lines. By waiving (**Ax2**) we allow the existence of *"accelerating observers"*, besides inertial ones. Technically speaking, in a model, *Obs* may contain elements outside of *Ib*. We denote the set of *inertial observers* (that is, observers which are inertial bodies) by *IOb*, that is,

$$IOb \stackrel{\text{def}}{=} Obs \cap Ib \,.$$

Besides waiving old axioms, we need to postulate new ones. More concretely,

- we keep a part of **Newbasax** such that we replace *Obs* by *IOb* in them;
- we postulate a set of axioms referring to *all* observers, including the "new" ones ($Obs \setminus IOb$) as well;
- we postulate some more axioms for treating "real" relativistic effects.

To execute our plan above, we need to modify our first order language and frame models used so far, as follows. In our old frame models, all observers "shared" the same set G of geodesics. In our new frame models, every observer m will have a set G_m of its own geodesics. This means that we have to change our first order language as well. Throughout this chapter, G is a sort containing the geodesics and there is a new binary relation symbol Go defined between the "sort" of observers and the sort G. Intuitively, if m is an observer (in some model $\mathfrak{M}$), then the set $G_m = \{\ell : \ \langle m, \ell \rangle \in Go^{\mathfrak{M}}\}$ is the collection of the geodesics of m. The reason for this decision is the following. The set of geodesics of m is intended to represent the trace of inertial bodies. Since m is not necessarily inertial, it really depends on m, how (s)he observes the movements of inertial bodies. Thus if m_1 is another observer, then the set of traces of inertial bodies from the point of view of m is not necessarily the same as the set of traces of inertial bodies from m_1's point of view. In order to keep the notation simpler, we will use the notation G_m in the formulas below.

Another modification will be that we will introduce *metrics* d_m for each observer $m \in Obs$. This is motivated by Theorem 8.1.15 below. Before the formulation and proof of this theorem here we include an intuitive explanation of this theorem.

Before discussing Theorem 8.1.15 in a precise language, we discuss it on a very informal, intuitive level. Theorem 8.1.15 can be interpreted as predicting certain things that happen in a gravitational field. Roughly, the theorem implies that

(*) In a gravitational field, like that of the Earth, clocks (and hence processes in general) higher up in the field run slower than clocks deeper down.

E.g. in a very high tower, clocks in the attic (on top of the tower) run *faster* than clocks in the basement. Similarly, clocks closer to a black hole run slower than distant clocks[1274]. Statement (*) above is called the *tower paradox*. Below we turn to a more precise, more logical and at the same time stronger (but still intuitive) formulation of the tower paradox, i.e. to Thm.8.1.15. Theorem 8.1.15 will be stronger than (*) in the sense that it says that no matter how the accelerating observer chooses his coordinatization (of space-time), (*) will remain true.

On the proof. A statement much easier than Thm.8.1.15 would say something like the following. In the "most natural" or " simplest" models of our theory *Acc* of accelerating observers, statement (*) is true. This can be checked the following way. (1) Look at the examples we give (later) for the world-view transformation f_{mk} for the case when m is inertial and k is accelerating. (2) Try to interpret (*) formally in this context. (3) Try to check that the so interpreted version of (*) holds for the particular example of f_{mk}. Next we turn to discussing Thm.8.1.15.

Let us choose an observer m and let e_1 and e_2 be events having the same location (that is, having the same space-like coordinates) from the point of view of m. In Special Relativity the time passed between e_1 and e_2 (observed by m) is simply

[1274]Actually, for a *distant* observer, clocks on the surface of the event horizon of the black hole stop moving (they "freeze"). We did not yet check how much of this (about black holes) follows directly from Theorem 8.1.15, or whether extra considerations are needed for deriving this from Thm.8.1.15. However, we are under the impression that derivations of this effect start out from (basically, something like) Thm.8.1.15, cf. e.g. Kenyon's book on general relativity (Oxford U.P., 1990).

the difference between the time like coordinates of the events in question. By definition, a time like coordinate line connects simultaneous events. So, in particular, the time passed between two time like coordinate lines does not depend on the location of the "measurement" of m. The same (more precisely, the dual) applies to the space-like coordinate lines too. In the case of accelerating observers the situation is more complicated. Roughly speaking, in Theorem 8.1.15 below we will prove[1275] that in every model whose field reduct is isomorphic to the field of reals, if the distance of any two coordinate lines of an observer m is constant, then the speed of m is constant as well, thus m is not accelerating. This can be illustrated in the following way. Suppose m is an accelerating observer and e_0, e'_0, e_1 and e'_1 are events such that

e_0 is simultaneous with e'_0 and
e_1 is simultaneous with e'_1 and
e_0 has same location as e_1 and
e'_0 has same location as e'_1

(from the point of view of m, of course). Then the distance between e_0 and e'_0 differs from the distance between e_1 and e'_1. This suggest the following experiment. Let us move, say 1 km in a certain direction and then let us wait, say 1 year. Then our space-time coordinates will be different from our partner's space-time coordinates who first waited a year then went 1 km in the same direction as we did. Summing up, for an accelerating observer, simultaneity is not the same as "waiting the same time". Thus, if we want to speak about both of these notions then we have to expand our language distinguishing simultaneities and lengths of time intervals. This is the reason for introducing the metrics d_m. Let us notice that the result of the above mentioned physical experiment coincides with the conclusion of Theorem 8.1.15, however our decision about expanding the language is motivated by Theorem 8.1.15.

Turning to the new language, let Do be a new $1 + 2n$ – placed function symbol $Do : Obs \times {}^nF \times {}^nF \to F$. Intuitively, if m is an observer (in some model $\mathfrak{M}$) then the value of the function $d_m(x, y) := Do(m, x, y)$ is the distance between the space-time points x and y observed by m. Note that x and y are points in the space-time (not simply points in the space). As usual, in the formulas below, we will use the notation d_m.

[1275] At the moment we have a proof only in the two dimensional case, but this result has implications in higher dimensions as well.

Thus $\mathfrak{M}$ is a model for our expanded language iff it is of the form

$$\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, \ \mathrm{E}\ , W, G, Go, Do \rangle .$$

Using the above introduced notation we define our new frame models as follows. A model $\mathfrak{M}$ is a *frame model for accelerating observers* iff conditions (1) and (2) below hold.

(1)

$$\mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, \ \mathrm{E}\ , W, G_m, d_m \rangle_{m \in Obs}$$

where

- $B, Obs, Ph, Ib, \mathfrak{F}, \ \mathrm{E}\ , W$ are the same as in Definition 2.1.1;
- for arbitrary $m \in Obs$,
  $G_m \subseteq \mathcal{P}({}^nF)$, and
  $d_m : {}^nF \times {}^nF \longrightarrow F^+$, where F^+ denotes the non-negative part of F.

(2) Frame theory:

- We keep our frame theory used so far. More precisely, we assume $\mathbf{Ax}_{\mathrm{OF}}$ and the straightforward generalization $\mathbf{Ax}_{\mathrm{G}}(m)$ of $\mathbf{Ax}_{\mathrm{G}}$ from G to G_m, for every $m \in Obs$.
- To these we add the new assumptions $\mathbf{Ax}_{\mathrm{met}}(m)$ saying that d_m is a Lorentz–metric[1276] for every $m \in Obs$. We note that a Lorentz–metric is not a metric in the usual sense.

We will denote the class of our *frame models for accelerating observers* by $\mathsf{FM}_{\mathrm{acc}}$. That is,

$$\mathsf{FM}_{\mathrm{acc}} \stackrel{\mathrm{def}}{=} \{ \mathfrak{M} = \langle (B, Obs, Ph, Ib), \mathfrak{F}, \ \mathrm{E}\ , W, G_m, d_m \rangle_{m \in Obs} : \\ \mathfrak{M} \models \mathbf{Ax}_{\mathrm{OF}} \cup \{ \mathbf{Ax}_{\mathrm{G}}(m) : m \in Obs \} \cup \{ \mathbf{Ax}_{\mathrm{met}}(m) : m \in Obs \} \} .$$

[1276] For completeness, we include here the set $\mathbf{Ax}_{\mathrm{met}}$ of axioms of a Lorentz–metric: We call $d : {}^nF \times {}^nF \longrightarrow F^+$ a *Lorentz–metric* on the ordered field $\mathfrak{F}$ iff d satisfies the conditions below, for each $p, q, s \in {}^nF$.

$d(p,q) \geq 0$;
$d(p,q) = d(q,p)$;
$d(p,q) + d(q,s) \leq d(p,s)$ whenever $p_t - q_t$ and $q_t - s_t$ have the same sign (reversed triangle inequality).

According to our Convention 2.2.1, $\mathsf{FM}_{acc}(2,\mathfrak{R})$ denotes that subclass of FM_{acc} where $n = 2$ and $\mathfrak{F} = \mathfrak{R}$. In this chapter, unless stated otherwise, we tacitly assume that $\mathfrak{M} \in \mathsf{FM}_{acc}$ whenever $\mathfrak{M}$ is of the form

$$\langle (B, Obs, Ph, Ib), \mathfrak{F}, \ \mathrm{E}\ , W, G_m, d_m\rangle_{m\in Obs} .$$

Now we start to postulate our set **Acc** of axioms for accelerating observers. Though we will be able to state meaningful theorems for particular dimensions n only, we are trying to formulate the axioms for arbitrary $n \in \omega$.

As we already indicated, our first group $\mathbf{Acc}_1$ of axioms consists of an appropriate part of **Newbasax** "restricted" to inertial observers (IOb). Concretely,

$$\mathbf{Acc}_1 \stackrel{\text{def}}{=} \{\mathbf{Ax1}_g, \mathbf{Ax2}_g, \mathbf{Ax5}_g, \mathbf{AxE}_{0g}\} ,$$

where,

$\mathbf{Ax1}_g$ $m \in IOb \Rightarrow (G_m = \mathsf{Eucl}(n, \mathbf{F})$ and d_m is the Minkowski metric[1277]).

$\mathbf{Ax2}_g$ $Ph \subseteq Ib$.
(Notice that $IOb \subseteq Ib$ automatically holds by the definition of IOb.)

$\mathbf{Ax5}_g$ $m \in IOb \Rightarrow (\forall \ell \in G_m)(ang^2(\ell) < 1 \Rightarrow (\exists k \in Obs)\ell = tr_m(k)$ and

$$ang^2(\ell) = 1 \Rightarrow (\exists ph \in Ph)\ell = tr_m(ph)).$$

$\mathbf{AxE}_{0g}$ $[(m \in IOb \wedge tr_m(ph) \neq \emptyset) \Rightarrow (tr_m(ph) \in G_m \ \wedge \ v_m(ph) = 1)] \ \wedge \ (\exists k \in Obs)tr_k(ph) \neq \emptyset)$.

Our second group $\mathbf{Acc}_2$ of axioms for accelerating observers is defined as follows.

$$\mathbf{Acc}_2 \stackrel{\text{def}}{=} \{\mathbf{Ax3}_g, \mathbf{Ax4}_g, \mathbf{Ax6_{00}}, \mathbf{Ax6_{01}}\} ,$$

where, for each $k, m \in Obs$,

$\mathbf{Ax3}_g$ $(\forall h \in Ib)(tr_m(h) \in G_m \cup \{\emptyset\} \ \wedge \ (\exists k \in Obs)tr_k(h) \neq \emptyset)$.

$\mathbf{Ax4}_g$ $tr_m(m) = F \times {}^{n-1}\{0\}$.

$\mathbf{Ax6_{00}}$ $w_m[tr_m(k)] \subseteq Rng(w_k)$.

[1277] $d_m(p, q) = \mu(p, q)$ for every $p, q \in {}^nF$ (where μ will be defined later in this chapter, see Definition 8.1.5).

$\mathbf{Ax6_{01}}$ $Dom(\mathsf{f}_{mk}) \in Open(n, \mathfrak{F})$,
where $Open(n, \mathfrak{F})$ denotes the set of all open[1278] subsets of ${}^nF$.

PROPOSITION 8.1.1 *Suppose $\mathfrak{M} \in \mathsf{FM}_{\mathrm{acc}}, \mathfrak{M} \models Acc_2$ and $m \in Obs^{\mathfrak{M}}, k \in IOb^{\mathfrak{M}}$. Then the world view transformation f_{mk} is a function. Note, that f_{km} is not necessarily a function.*

Proof. Let $p \in {}^nF^{\mathfrak{M}}$ be arbitrary, and assume $q_0, q_1 \in {}^nF^{\mathfrak{M}}$ are such that $q_0, q_1 \in f_{mk}(p)$ hold. If $q_0 \neq q_1$ would hold, then by $\mathbf{Ax5}_g$ there would be an inertial observer or a photon, whose trace incident with q_0 but not incident with q_1 in the world view of k. In that case $w_m(p) = w_k(q_0) \neq w_k(q_1) = w_m(p)$ would hold, which is a contradiction. Therefore $q_0 = q_1$. ∎

In our third group of axioms, we will assume that all the world view transformations between a non inertial and an inertial observer are <u>differentiable</u> functions. In order to save space, we omit to write down everywhere the first order formula requiring differentiability. Instead of this, we introduce the following convention. Throughout, in any axiom, whenever we refer to a derivative of a function, we implicitly assume that the axiom postulates the derivability of the function in question as well.

Our third group $\mathbf{Acc}_3$ of axioms for accelerating observers is defined as follows.

$$\mathbf{Acc}_3 \stackrel{\text{def}}{=} \{\mathbf{Ax_g i} : 1 \leq \mathbf{i} \leq 5\},$$

where $\mathbf{Ax_g1}$–$\mathbf{Ax_g5}$ are defined below. Since the axioms $\mathbf{Ax_g1}$–$\mathbf{Ax_g5}$ are based on new ideas, below we will explain the intended meanings of some of these axioms.

The most important idea what we wish to formalize is that "general relativity locally behaves in the same way as special relativity does". Particularly, suppose m is an accelerating observer and k is an inertial one such that the locations and the velocities of m and k are the same at a certain moment. Then we want to require that the coordinatizations of m and k are the same (at least in a little neighborhood around their common location). One possibility to express this is taking the derivative of the life-line (trace) of m, this is a straight line (from the point of view

[1278] Recall from chapter 3 that $Q \subseteq {}^nF$ is an <u>*open set*</u> iff $(\forall q \in Q)(\exists \varepsilon \in F^+)\big(S(q,\varepsilon) \subseteq Q\big)$, where the <u>*$\varepsilon$-neighborhood $S(p,\varepsilon)$ of p*</u> was defined as

$$S(p,\varepsilon) = \{q \in {}^nF : (q_0 - p_0)^2 + (q_1 - p_1)^2 + \ldots + (q_{n-1} - p_{n-1})^2 < \varepsilon\}.$$

of a third inertial observer), therefore there is an inertial observer k on that line and requiring that the world-view transformation of k coincides with that of m. Carefully analyzing the above heuristic argument one finds that this means the following: for each point $p \in {}^nF$ there are other inertial observers k and k_1 such that $f'_{mk_1}(p) = f_{kk_1}$. Here $f'_{mk_1}(p)$ is the derivative of the function f_{mk_1} at point p, which is a linear transformation. We found this axiom too restrictive (see Theorem 8.1.2 below), therefore we will require only that the simultaneities of k and m would be the same (and the same for the space-like parts of their world views as well) in a sufficiently small neighborhood around their common location. This can be formulated in the following way:

$\mathbf{Ax_g1}$ $(\forall m \in Obs)(\forall q \in {}^nF)(\forall \varepsilon \in F^+)$
$(\exists p \in S(p, \varepsilon))(\exists m_1 \in IOb)(\exists k \in IOb)(\exists a_0, \ldots, a_{n-1} \in F)$
$(\mathsf{f}_{km_1}(\bar{0}) = \mathsf{f}_{mm_1}(p) \ \wedge\ \bigwedge\{\partial_i \mathsf{f}_{mm_1}(p) = a_i \cdot \mathsf{f}_{km_1}(1_i) : 0 \le i < n\})$.

This axiom expresses that, as seen by an inertial observer $m_1 \in IOb$, every (possibly accelerating) observer $m \in Obs$ "behaves" in every (small) neighborhood of each point $q \in {}^nF$ of space-time the same way as *some* inertial observer $k \in IOb$ with its $\bar{0}$-point at p. ($\partial_i \mathsf{f}_{mm_1}(p)$ is the i-th partial derivative of f_{mm_1} at point p .)

THEOREM 8.1.2 *If $n = 2$ then the modified version of $\mathbf{Ax_g1}$ where all the constants a_i are just 1 is inconsistent. (More precisely, this version of $\mathbf{Ax_g1}$ excludes uniformly accelerating observers over the field of reals.) In particular, it cannot be the case that both $\mathbf{Ax_g1}^+$ below and $Dom(f_{mm_1}) = {}^2F$ hold.*

$\mathbf{Ax_g1}^+$ $\quad \partial_t \mathsf{f}_{mm_1}(p) = \mathsf{f}_{km_1}(1_t)$ *and* $\partial_x \mathsf{f}_{mm_1}(p) = \mathsf{f}_{km_1}(1_x)$.

(Now, $\mathbf{Ax_g1}^+$ is the same as requiring $f'_{mm_1}(p) = f_{km_1}$ where f' is the derivative of f.)

Proof. The proof can be found in [230]. ∎

Remark 8.1.3 *Without the condition $Dom(f_{mm_1}) = {}^2F$ the above theorem is* <u>not</u> *true. In fact, there is a model $\mathfrak{M}_K$* [1279] *satisfying the remaining conditions of the*

[1279] K stands for Kruskal.

above theorem. In this model f_{mk} is uniquely determined by the life-line $tr_m(k)$ of k (which is not the case if we use $\mathbf{Ax_g1}$ instead of the axiom in Theorem 8.1.2). This model is based on the coordinate transformation described e.g. in [224] on page 156. Actually,

$$\mathfrak{M}_K \models \mathbf{Acc}_0 + \mathbf{Acc}_1 + \mathbf{Ax_g1} + \mathbf{Ax_g2} + \mathbf{Ax_g4} + \mathbf{Ax_g5},$$

where $\mathbf{Ax_g2}$ etc will be defined soon. However, this model doesn't satisfy $\mathbf{Ax_g3}$ (see below)

The strong version $\mathbf{Ax_g1}^+$ of $\mathbf{Ax_g1}$ (together with $\mathbf{Acc}_0$, $\mathbf{Acc}_1$) implies that the speed of light is the same for all observers i.e. even if m is an accelerating observer we have $(\forall ph \in Ph)(v_m(ph) = 1)$. On the other hand, our axiom system $\mathbf{Acc}$ (see below) does not imply anything like this for accelerating observers. In fact, in the section "Constructing models for accelerating observers" there is a model $\mathfrak{M}$ of the quoted axioms such that $v_m(ph) \neq 1$ for some $m \in Obs^{\mathfrak{M}}$ and $ph \in Ph^{\mathfrak{M}}$. Moreover, the velocity of ph as seen by m is not constant (ph accelerates) in that $\mathfrak{M}$. We conjecture that this behavior of photons might be strongly connected to the present version of $\mathbf{Ax_g3}$.

Conjecture 8.1.4 *Theorem 8.1.2 is provable from Theorem 8.1.15 below.*

Our following axiom, $\mathbf{Ax_g2}$, is concerned with properties of the distance d_m of (possibly accelerating) observers $m \in Obs$. We will express d_m via the so called Minkowski distance μ of some inertial observer $m_1 \in IOb$. So let us define the Minkowski distance first.

Definition 8.1.5 Assume $\mathfrak{M} \models \mathbf{Newbasax} \wedge \mathbf{Ax\triangle2}$ (see section 3.9) and assume that *square roots exist in* $\mathfrak{F}$. Then the Minkowski distance $\mu : {}^nF \times {}^nF \longrightarrow F^+$ of two points $p, q \in {}^nF$ is defined as follows. Consider the world view of some inertial observer $m_1 \in IOb$.

- If the (Euclidean) straight line $\overline{pq}$ connecting p and q is parallel with the time coordinate axis or $p_t = q_t$ then
  $\mu(p,q) \stackrel{\text{def}}{=}$ (the Euclidean distance of p and q) $= \sqrt{\sum_{i=0}^{n-1} (p_i - q_i)^2}$.
- If both p and q are on the light cone, that is, if $ang^2(\overline{pq}) = 1$, then $\mu(p,q) \stackrel{\text{def}}{=} 0$.
- If $ang^2(\overline{pq}) < 1$ then, by $\mathbf{Ax5}$, there is an observer $k \in Obs$ such that $\overline{pq} = tr_{m_1}(k)$. Then $\mu(p,q) \stackrel{\text{def}}{=} \mu(f_{m_1k}(p), f_{m_1k}(q))$.

- If $ang^2(\overline{pq}) > 1$ then, by **Ax5** again, there is an observer $k \in Obs$ such that the reflection of $\overline{pq}$ to the light cone is $tr_{m_1}(k)$, and $\mu(p,q) \stackrel{\text{def}}{=} \mu(f_{m_1 k}(p), f_{m_1 k}(q))$ for this k.

Exercise 8.1.6 Prove that μ is a Lorentz – metric (that is, μ satisfies $\mathbf{Ax}_{\text{met}}$).

Exercise 8.1.7 Working over the field of reals, prove that for any two points $p, q \in {}^n\Re$, the distance between p and q according to the Lorentz – metric is

$$\sqrt{(p_0 - q_0)^2 - \sum_{i=1}^{n-1} (p_i - q_i)^2}$$

when $ang^2(\overline{pq}) < 1$. What changes when $ang^2(\overline{pq}) \geq 1$?

The *Minkowski length* $\mu(\overline{pCq})$ of some curve C connecting p and q is defined to be the length of C from p to q according to the Minkowski distance. Suppose we are working over the field of reals. Then one can express the length of C in terms of an integral.

Now we intend to postulate the following.

$\mathbf{Ax_g 2'}$ $(\forall m \in Obs)(\forall m_1 \in IOb)(\forall p, q \in {}^nF)$
$((p_0 = q_0 \text{ or } (\forall 0 < i < n) p_i = q_i) \wedge ([p,q] \subseteq Rng(f_{m_1 m}))) \Rightarrow d_m(p,q) = \mu(\mathsf{f}_{m m_1}[\overline{pq}]))$.

There are two problems to be solved here.

Item 8.1.8 **$\mathbf{Ax_g 2}$'** should function as the definition of d_m, therefore we need that

$$(\forall m \in Obs)(\forall m_1, m_2 \in IOb)\mu(\mathsf{f}_{m m_1}[\overline{pq}]) = \mu(\mathsf{f}_{m m_2}[\overline{pq}])\,.$$

Item 8.1.9 **$\mathbf{Ax_g 2}$'** should be translated to our first order frame language (one way or another).

The problem in item 8.1.9 is that we haven't yet described integration in first order logic. First suppose that we are working over the field of reals and let us try to postulate "a first order approximation of the notion of integral". This can be done by the following two steps. First we postulate that the integral is additive, that is, its value does not changes when we cut the curve in question into two (or equivalently, finitely many) parts and compute the sum of the integrals taken in the new (small) curves. Second, we postulate, that in a sufficiently small curve, the value of the integral and the value of μ almost coincide (this can be done by a standard, first order limit process). After this, one can derive $\mathbf{Ax_g2'}$ over the field of reals as follows. Choose an arbitrary (small) positive number, say ε. Using the first condition ($\mathbf{Ax_g2_0}$ below) the curve can be cut into sufficiently small parts, in which $\mathbf{Ax_g2_1}$ below guarantees that the value of the integral differs from the value of μ with at most ε (relative to the length of the small parts). Thus, for any positive ε the difference between the value of the integral and the value of μ is at most εc (c is an appropriate constant depending on the curve in question). So the difference is smaller than any positive number, thus it is equal to 0. Now we formalize $\mathbf{Ax_g2_0}$ and $\mathbf{Ax_g2_1}$.

Below we use the following notation: if $p, q \in {}^nF$ then $[p,q]$ is defined to be $[p,q] = \{x \in \overline{pq} :\ \text{x is between } p \text{ and } q\}$

$\mathbf{Ax_g2_0}$ $(\forall m \in Obs)(\forall p, q \in {}^2F)(p_0 = q_0 \text{ or } (\forall 0 < i < n)(p_i = q_i)) \Rightarrow (\forall r \in [p,q])(d_m(p,r) + d_m(r,q) = d_m(p,q))$,

$\mathbf{Ax_g2_1}$ $(\forall m \in Obs)(\forall m_1 \in IOb)(\forall p, q \in {}^nF)(\forall \varepsilon \in F^+)(\exists \delta \in F^+)$
$(\forall a, c \in [p,q])(|a - c| \leq \delta \ \wedge\ [a,c] \subseteq Rng(f_{m_1 m}) \Rightarrow$
$(\forall c \in S(a,\delta)(|(d_m(a,c) - \mu(f_{mm_1}(a), f_{mm_1}(c)))/(a-c)| \leq \varepsilon)$.

Let us notice that these formulas are first order formulas in our frame language. However, considering these formulas over some non-archimedean field, one can find that the behaviour of $\mathbf{Ax_g2_0}$ and $\mathbf{Ax_g2_1}$ can deviate from their behaviour in the "standard" case of $\mathbf{F} = \mathrm{R}$. We will illustrate this by constructing a non-archimedean field and a frame model over it in which the function d can be defined in two different ways such that both definitions satisfy $\mathbf{Ax_g2_0}$ and $\mathbf{Ax_g2_1}$ (see Example 8.1.11 below).

In order to prove $\mathbf{Ax_g2'}$ over the field of reals, we formulated $\mathbf{Ax_g2_1}$ such that δ (the size of the neighbourhood of the point in question) does not depend on "a" (the point in question). Thus, $\mathbf{Ax_g2_1}$ requires a kind of uniform approximability of the integral. Therefore we found $\mathbf{Ax_g2_1}$ too restrictive.

The reader who doesn't want to "worry" too much at this point on finding another formalization of $\mathbf{Ax_g2}$ (improved) may skip the following items $\mathbf{Ax_g2(2)}$ and $\mathbf{Ax_g2}$ at the first reading of this chapter. Such a reader is advised to consider axiom $\mathbf{Ax_g2}$ to be $\{\mathbf{Ax_g2_0}, \mathbf{Ax_g2_1}\}$.

Another possibility for expressing $\mathbf{Ax_g2'}$ (over the field of reals) is to "switch" the equation containing an integral to its derivative form: the latter can be expressed in first order logic. This "switch" will not give us a form completely equivalent to $\mathbf{Ax_g2}$', but $\mathbf{Ax_g2}$' will follow from it, see our next theorem to come.

First suppose $n = 2$. Let $p, q \in {}^2F$ and $m \in Obs$ be arbitrary but fixed. Consider the *unary* function $d_m(p, p+a\cdot q) : F \longrightarrow F$. (In λ-calculus notation, this is $\lambda a.d_m(p, p+a\cdot q)$.) We denote the derivative of $d_m(p, p+a\cdot q)$ by $\partial_a d_m(p, p+a\cdot q)$. Thus

$$\partial_a d_m(p, p+a\cdot q) = b \Longleftrightarrow$$
$$(\forall\varepsilon > 0)(\exists\delta > 0)(\forall a_1 \in S(a,\delta))\left(\left|\frac{d_m(p, p+a_1\cdot q) - d_m(p, p+a\cdot q)}{a_1 - a} - b\right| < \varepsilon\right).$$

Now we let

$\mathbf{Ax_g2(2)}$ $\quad(\forall m \in Obs)(\forall p \neq q \in {}^nF)(\exists m_1 \in IOb)(\exists\varepsilon \in F^+)(\forall a \in F, -\varepsilon \leq a \leq \varepsilon)$

$$\Big[([p,q] \subseteq Rng(f_{m_1 m})) \wedge$$
$$\Big((ang^2(\overline{pq}) \leq 1 \Longrightarrow$$
$$[(\partial_a d_m(p, p+a\cdot(q-p)))^2 =$$
$$(\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p))_t)^2 - ((\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p)))_x)^2])$$
$$\wedge$$
$$(ang^2(\overline{pq}) > 1 \Longrightarrow$$
$$[(\partial_a d_m(p, p+a\cdot(q-p)))^2 =$$
$$((\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p)))_x)^2 - (\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p))_t)^2])\Big)\Big].$$

The generalization of $\mathbf{Ax_g2}(2)$ for arbitrary n might be:

$\mathbf{Ax_g2}$ $\quad(\forall m \in Obs)(\forall p \neq q \in {}^nF)(\exists m_1 \in IOb)(\exists\varepsilon \in F^+)(\forall a \in F, -\varepsilon \leq a \leq \varepsilon)$

$$\Big[([p,q] \subseteq Rng(f_{m_1 m})) \wedge$$
$$\Big((ang^2(\overline{pq}) \leq 1 \Longrightarrow$$
$$[(\partial_a d_m(p, p+a\cdot(q-p)))^2 =$$
$$(\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p))_t)^2 - \Sigma_{i=1}^{n-1}((\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p)))_i)^2])$$
$$\wedge$$
$$(ang^2(\overline{pq}) > 1 \Longrightarrow$$
$$[(\partial_a d_m(p, p+a\cdot(q-p)))^2 =$$
$$\Sigma_{i=1}^{n-1}((\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p)))_i)^2 - (\partial_a \mathsf{f}_{mm_1}(p+a\cdot(q-p))_t)^2])\Big)\Big].$$

THEOREM 8.1.10 $\mathsf{FM}_{\mathrm{acc}}(2,\mathfrak{R}) \models \mathbf{Ax_g2}(2) \Rightarrow \mathbf{Ax_g2'}(2)$.

Proof. The proof can be found in [230]. ∎

The further axioms are the following.

$\mathbf{Ax_g3}$ $(\forall 0 \le i < n)(\forall p \in \bar{i})(d_m(\bar{0},p) = p_i)$, where $\bar{i}$ denotes the i-th coordinate axis.[1280]

$\mathbf{Ax_g4}$ $(\forall m \in Obs)(\forall a \in {}^{n-1}F)(\exists k \in Obs)$
$$(tr_m(k) = F \times \{a\} \ \wedge \ \mathsf{f}_{mk} \text{ is a bijection } \ \wedge$$
$$(\forall p, q \in {}^nF)(p_t = q_t \to \mathsf{f}_{mk}(p)_t = \mathsf{f}_{mk}(q)_t) \ \wedge$$
$$(\textstyle\bigwedge_{0<i<n} p_i = q_i \to \bigwedge_{0<i<n} \mathsf{f}_{mk}(p)_i = \mathsf{f}_{mk}(q)_i)).$$

$\mathbf{Ax_g5}$ $(\forall m \in Obs)(\forall p \in {}^nF)(\forall \varepsilon \in F^+)(\exists q \in S(p,\varepsilon))(\exists k \in IOb)(k \in w_m(q))$.

Now
$$\mathbf{Acc} \stackrel{\text{def}}{=} \mathbf{Acc}_1 \cup \mathbf{Acc}_2 \cup \mathbf{Acc}_3\,,$$
$$Acc' = Acc - \{\mathbf{Ax_g2}\} \cup \{\mathbf{Ax_g2_0}, \mathbf{Ax_g2_1}\}.$$

Of course, **Acc** is consistent, because no axiom requires the existence of a non inertial observer, so the model over $\mathfrak{R}$ described in Theorem **??** can be extended to a model for **Acc**. Our next main goal is to build models containing non inertial (accelerating) observers. We will do this in two steps. First of all, we will restrict ourself to dimension $n = 2$ and we will always work over the field of reals. First we will study some functions of form ${}^2\mathfrak{R} \to {}^2\mathfrak{R}$ which describe "potential" world view transformations between an inertial and an accelerating observer. These investigations can be seen simply as "playing" with differential geometry (similarly to section 2.4, where we played with linear algebra). After this, we show the relative consistency of $\mathbf{Acc} \cup \{(\exists m)(m \in Obs - IOb)\}$ over **Newbasax**. That is, we will choose a model of **Newbasax** and will "put into" that model an accelerating observer. When we construct the world view transformations of the new observer, we will use our results

[1280] We are not sure that we want to keep this axiom because it fails in one of the important models of **Acc**, namely in the model discussed in 8.1.3. On the other hand, some weaker version of $\mathbf{Ax_g3}$ can be retained. For example we could require that the numbers of the time-like coordinate lines follow in the same order as $\mathbf{Ax_g3}$ would require, i.e., for example
$$(\forall p, q \in \bar{t})(p_t \le q_t \Leftrightarrow d_m(\bar{0},p) \le d_m(\bar{0},q)).$$

obtained by playing with differential geometry. After this, we will study fancy accelératings, for example such acceleratings that the speed of an accelerating observer m converges to the speed of light in such a way, that an inertial observer m_1 never sees the event when m's clock reaches say, the time instant 1. Thus, the events seen by m and the events seen by m_1 must be different (that is, $\mathbf{Ax_6}$ fails).

Discussing the axiom system *Acc*.

First we give an example showing that over some non-archimedean field, the axioms $\mathbf{Ax_g2_0}, \mathbf{Ax_g2_1}$ do not define the metric. Namely, we will construct a non-archimedean field and a frame model over it in which exists an observer, whose function d can be defined in two different ways such that both definitions satisfy $\mathbf{Ax_g2_0}$ and $\mathbf{Ax_g2_1}$.

Example 8.1.11 *Let $\mathfrak{M}$ be a frame model over the field of reals such that*

$$\mathfrak{M} \models \mathbf{Newbasax} \ \wedge \ \mathbf{Ax_g2_0} \ \wedge \ \mathbf{Ax_g2_1}.$$

(It is not hard to see at this point that such a model exists, because no axiom forces the existence of an accelerating observer.) Let $\mathfrak{M}_1$ be an ultrapower of $\mathfrak{M}$ with respect to a non-principal ultrafilter. First we concentrate on the field reduct of $\mathfrak{M}_1$. We will identify the real numbers with their images according to the natural (or diagonal) embedding into the corresponding ultrapower of $\mathfrak{R}$.

We define an equivalence relation ϱ on $\mathfrak{F}^{\mathfrak{M}_1}$ as follows: for any two elements $a, b \in F^{\mathfrak{M}_1}$ $\ a\varrho b\ $ iff there exists a real number r such that $a - r \leq b \leq a + r$. This is an equivalence relation and we will denote the equivalence class of a by $[a]$. In fact, ϱ is a congruence relation of the abelian group $\langle F^{\mathfrak{M}_1}; +\rangle$, that is, for any elements $a, a', b, b' \in F^{\mathfrak{M}_1}$ we have $(a\varrho a' \ \wedge \ b\varrho b') \ \Rightarrow (a+b)\varrho(a'+b')$. Note that $[0]$ is a group itself (that is, the set [0] is closed under addition $+^{\mathfrak{M}_1}$). To keep notation simpler, we will denote this group by $[0]$ as well. The abelian group $\langle F^{\mathfrak{M}_1}; +\rangle$ is decomposable into the direct product of $[0]$ and $\langle F^{\mathfrak{M}_1}; +\rangle/\varrho$ (this is possibly familiar to the reader, but for completeness we include here a proof).

Claim 8.1.12 The abelian group $\langle F^{\mathfrak{M}_1}; +\rangle$ is isomorphic to the direct product $[0] \times \langle F^{\mathfrak{M}_1}; +\rangle/\varrho$. We will denote by φ the isomorphism

$$\langle F^{\mathfrak{M}_1}; +\rangle \cong [0] \times \langle F^{\mathfrak{M}_1}; +\rangle/\varrho$$

constructed below.

Proof. We say, that a subgroup H of $\langle F^{\mathfrak{M}_1};+\rangle$ is a (ϱ, Q) subgroup, iff the following two conditions hold for H:

- for all rational numbers q and for all $h \in H$ $qh \in H$,
- for all $h_1, h_2 \in H$ we have $h_1 \varrho h_2 \Rightarrow h_1 = h_2$.

Clearly, there exists a (ϱ, Q) subgroup, for example the subgroup $\{0\}$ containing only one element is such. The set of all (ϱ, Q) subgroups of $\langle F^{\mathfrak{M}_1};+\rangle$ is closed under directed unions, so applying Zorn's lemma we conclude that there exists a maximal (ϱ, Q) subgroup G.

Now we show that for all $a \in F^{\mathfrak{M}_1}$ there exists $g \in G$ such that $a\varrho g$. Suppose for contradiction that there exists an element $a \in F^{\mathfrak{M}_1}$ whose ϱ equivalence class $[a]$ is disjoint from G. Let $G_0 = \{g+qa : q \text{ is rational and } g \in G\}$. It is easy to see that G_0 is a subgroup of $\langle F^{\mathfrak{M}_1};+\rangle$, strictly bigger than G and satisfies the first condition of being a (ϱ, Q) subgroup. We will show that G_0 is a (ϱ, Q) subgroup. To do this, suppose q_1 and q_2 are rational numbers, $g_1, g_2 \in G$ such that $(g_1 + q_1 a)\varrho(g_2 + q_2 a)$. We have to show $(g_1 + q_1 a) = (g_2 + q_2 a)$.

Suppose first $q_1 \neq q_2$. Observe that ϱ is compatible with multiplying by a rational number, that is, if q is a rational number and $u\varrho v$ then $qu\ \varrho\ qv$. Therefore from $(g_1 + q_1 a)\varrho(g_2 + q_2 a)$ we conclude $(g_1 - g_2)/(q_2 - q_1)\varrho a$. The left hand side belongs to G (because G is a (ϱ, Q) subgroup) contradicting to the choice of a. Therefore we have $q_1 = q_2$.

If $q_1 = q_2$ then $g_1 \varrho g_2$ since ϱ a congruence relation. But G is a (ϱ, Q) subgroup so $g_1 = g_2$ and therefore $g_1 + q_1 a = g_2 + q_1 a = g_2 + q_2 a$.

Therefore G_0 is a (ϱ, Q) subgroup, so G cannot be a maximal (ϱ, Q) subgroup, which is a contradiction. Therefore G contains an element ϱ-equivalent to a.

Summing up, G is such a subgroup that for all $a \in F^{\mathfrak{M}_1}$ we have $|G \cap [a]| = 1$. We will denote the element in $G \cap [a]$ by $[a]_G$. Finally we define the function $\varphi : F^{\mathfrak{M}_1} \to [0] \times F/\varrho$ as follows:

$$(\forall a \in F^{\mathfrak{M}_1})\ \varphi(a) = \langle a - [a]_G, [a]\rangle.$$

Note that for any two elements $a, b \in F^{\mathfrak{M}_1}$ we have $[a+b]_G = [a]_G + [b]_G$ because $a\ \varrho\ [a]_G$ and $b\ \varrho\ [b]_G$ so both $[a+b]_G$ and $[a]_G + [b]_G$ are the unique element of $G \cap [a+b]$.

Using this fact, it is easy to see that φ is an isomorphism as required. ∎

Using the notation of the above Claim we define a function $\psi : F^{\mathfrak{M}_1} \to F^{\mathfrak{M}_1}$ as follows:

$$(\forall a \in F^{\mathfrak{M}_1})\ \psi(a) = \varphi^{-1}(\langle a - [a]_G, [2a]\rangle).$$

Now we show that ψ preserves addition of $F^{\mathfrak{M}_1}$. Indeed, for all $a, b \in F^{\mathfrak{M}_1}$ we have $\psi(a+b) = \varphi^{-1}(\langle (a+b) - [a+b]_G, [2(a+b)]\rangle) = \varphi^{-1}(\langle a - [a]_G, [2a]\rangle + \langle b - [b]_G, [2b]\rangle) = \varphi^{-1}(\langle a - [a]_G, [2a]\rangle) + \varphi^{-1}(\langle b - [b]_G, [2b]\rangle) = \psi(a) + \psi(b)$.

Moreover ψ leaves the set $[0]$ fixed pointwise, that is, $(\forall a \in [0])(\psi(a) = a)$.

Now we are ready to turn to finish our frame model construction. Let m be an inertial observer of $\mathfrak{M}_1$ and let m' be a new symbol. We define a new frame model $\mathfrak{M}_1'$ as follows. The universe of $\mathfrak{M}_1'$ is the same as that of $\mathfrak{M}_1$ expanded with a new observer m', and $B^{\mathfrak{M}_1'} = B^{\mathfrak{M}_1} \cup \{m'\}$, $Obs^{\mathfrak{M}_1'} = Obs^{\mathfrak{M}_1} \cup \{m'\}, Ph^{\mathfrak{M}_1'} = Ph^{\mathfrak{M}_1}, Ib^{\mathfrak{M}_1'} = Ib^{\mathfrak{M}_1}, \mathfrak{F}^{\mathfrak{M}_1'} = \mathfrak{F}^{\mathfrak{M}_1}$, $\mathrm{E}^{\mathfrak{M}_1'} = \mathrm{E}^{\mathfrak{M}_1}$ and for all $k \in Obs^{\mathfrak{M}_1}$ we define $G_k^{\mathfrak{M}_1'} = G_k^{\mathfrak{M}_1}, d_k^{\mathfrak{M}_1'} = d_k^{\mathfrak{M}_1}, G_{m'}^{\mathfrak{M}_1'} = G_m^{\mathfrak{M}_1}, d_{m'}^{\mathfrak{M}_1'} = d_m^{\mathfrak{M}_1} \circ \psi$. Finally $W^{\mathfrak{M}_1'}$ is defined as follows:

$$(\forall k \in Obs^{\mathfrak{M}_1})(\forall x \in {}^n\mathfrak{F})\ \ w_k^{\mathfrak{M}_1'}(x) = \begin{cases} w_k^{\mathfrak{M}_1}(x) & \text{if } m \notin w_k^{\mathfrak{M}_1}(x), \\ w_k^{\mathfrak{M}_1}(x) \cup \{m'\} & \text{otherwise.} \end{cases}$$

and finally $(\forall x \in {}^n\mathfrak{F})(w_{m'}^{\mathfrak{M}_1'}(x) = w_m^{\mathfrak{M}_1'}(x))$.

By this we have defined the frame model $\mathfrak{M}_1'$ which is, intuitively, nothing more than the result of "putting m' into $\mathfrak{M}_1$". Moreover, m and m' are basically the same, except of their metrics. We now verify that $\mathfrak{M}_1' \models \mathbf{Ax_g2_0} \wedge \mathbf{Ax_g2_1}$. To do this, let us choose an observer $m \in Obs^{\mathfrak{M}_1'}$. By the construction, if $m \in Obs^{\mathfrak{M}_1}$ then the axioms $\mathbf{Ax_g2_0} \wedge \mathbf{Ax_g2_1}$ indeed hold, because here $\mathfrak{M}_1'$ and $\mathfrak{M}_1$ coincide, and $\mathfrak{M}_1$ is an ultrapower of $\mathfrak{M}$ which was chosen such that $\mathfrak{M} \models \mathbf{Ax_g2_0} \wedge \mathbf{Ax_g2_1}$. So we may suppose $m = m'$.

$\mathbf{Ax_g2_0}$ holds because ψ preserves addition. More concretely, for any points

p_1, p_2, p_3 satisfying the conditions of $\mathbf{Ax_g2_0}$ we have $d_{m'}(p_1,p_3) = \psi(d_m(p_1,p_3)) = \psi(d_m(p_1,p_2)+d_m(p_2,p_3)) = \psi(d_m(p_1,p_2))+\psi(d_m(p_2,p_3)) = d_{m'}(p_1,p_2)+d_{m'}(p_2,p_3)$.

To show $\mathbf{Ax_g2_1}$ let us choose $m_1 \in IOb^{\mathfrak{M}'_1}, p, q \in {}^nF$ and $\varepsilon \in F^+$. In $\mathfrak{M}_1$ axiom $\mathbf{Ax_g2_1}$ gives a δ for that choice. Let $\delta' = min\{\delta, 1\}$. We claim that for all $a, c \in {}^nF$, if $|a-c| \leq \delta'$ then $d_m(a,c) = d_{m'}(a,c)$. This is true, because

$$\mathfrak{M} \models (\forall m \in IOb)(\forall p, q \in {}^nF)(d_m(p,q) \leq |p-q|),$$

and therefore $\mathfrak{M}_1 \models (\forall m \in IOb)(\forall p, q \in {}^nF)(d_m(p,q) \leq |p-q|)$. Hence, $d_m(a,c) \leq \delta' \leq 1$, so $d_{m'}(a,c) = \psi(d_m(a,c))) = d_m(a,c)$ since φ leaves [0] fixed pointwise.

Finally observe that in $\mathbf{Ax_g2_1}$ the metric $d_{m'}$ is used in such a way that (using the notation of $\mathbf{Ax_g2_1}$) $|a-c| \leq \delta'$. So $d_{m'}(a,c) = d_m(a,c)$ and $\mathfrak{M}_1 \models \mathbf{Ax_g2_1}$ therefore $\mathfrak{M}'_1 \models \mathbf{Ax_g2_1}$ as desired.

This completes the example. ∎

Now we turn to the proof of Theorem 8.1.15 announced at the beginning of this chapter. From now on we restrict ourself considering only two dimensional models having field reduct isomorphic to the field of reals. We start with introducing some notation.
By definition, for any observer m the time like coordinate-lines $\ell_t := \{(t,x) : x \in \Re\}$ of m connects simultaneous events[1281] (as seen by m). Let k be another observer. Applying the world view transformation f_{mk} to the sets ℓ_t one gets curves in the coordinate system of k connecting the events simultaneous wrt m. The same applies to space like coordinates.

Definition 8.1.13 Let m and k be observers. The f_{mk}-lines, or f_{mk}-coordinates are the sets $f_{mk}[\ell_t], f_{mk}[\ell_x]$, where $\ell_t := \{(t,x) : x \in \Re\}$ and $\ell_x := \{(t,x) : t \in \Re\}$.

Using the notation of the previous definition, the coordinate line ℓ_x (for some x) of observer m can be seen as a trace of an observer m' who is at rest as seen by m. The existence of m' is guarantied by axiom $\mathbf{Ax_g4}$. We define the velocity (or speed) of the f_{mk}-line $f_{mk}[\ell_x]$ to be the velocity of m' as seen by k. It is easy to show that this definition does not depend on the particular choice of m'.

[1281]two events e_0, e_1 are simultaneous wrt m iff e_0, e_1 have the same time-coordinate in m's coordinate system.

Recall that if (t_1, x_1) and (t_2, x_2) are two points in space-time, then the set of points between them is denoted by $[(t_1, x_1), (t_2, x_2)]$, that is,

$$[(t_1, x_1), (t_2, x_2)] = \{s(t_1, x_1) + (1 - s)(t_2, x_2) : \ 0 \le s \le 1\}.$$

Recall moreover, that if $f : {}^2F \to {}^2F$ is a function, then the coordinate functions of f are denoted by f_t, f_x respectively, that is, for every $(t, x) \in Dom(f)$ we have

$$f(t, x) = (f_t(t, x), f_x(t, x)).$$

Now we are ready to formulate the first version of our theorem.

THEOREM 8.1.14 *Let $\mathfrak{M} \in \mathsf{FM}_{\text{acc}}$ be a frame model satisfying the symmetry axiom* **Ax△2** *(see in section 3.9). Suppose $\mathfrak{M} \models Acc$ and the field reduct of $\mathfrak{M}$ is isomorphic to the field of reals. Let k be an inertial observer in $\mathfrak{M}$ and let m be another (arbitrary, i.e., possibly non-inertial) observer in $\mathfrak{M}$. Suppose $f : {}^2\Re \to {}^2\Re$ is the world view transformation from m to k, thus f is a partial function. Assume moreover:*

(a) *the domain of f is open and convex,*
(b) *f is twice continuously differentiable,*
(c) *the velocity of every space-like f_{mk}-line is slower than the speed of light,*
(d) *the μ-distance between parallel f-coordinate lines is constant, that is*

$$(\forall t_0, t_1, x_0, x_1 \in \Re)((t_0, x_0), (t_0, x_1), (t_1, x_0), (t_1, x_1) \in Dom(f) \Rightarrow$$
$$\mu(f[(t_0, x_0), (t_0, x_1)]) = \mu(f[(t_1, x_0), (t_1, x_1)]) \ \wedge$$
$$\mu(f[(t_0, x_0), (t_1, x_0)]) = \mu(f[(t_0, x_1), (t_1, x_1)])).$$

Then m does not accelerate.

Proof. It follows from Theorem 8.1.15, see below. ∎

Seeing the above theorem, one can argue, that o.k., then probably, the conditions of Theorem 8.1.14 are too strong, so let us try to find models, in which an accelerating observer whose trace is not differentiable twice from any inertial observers view, and probably this strange accelerating observer can coordinatize the spacetime such that the distances between his/her coordinate lines are constant. However, we **don't want to exclude** the model of simplest uniformly accelerating observers over a real field, for example. This, and many other possibly interesting frame model do satisfy the conditions of Theorem 8.1.14.

Theorem 8.1.14 is easily seen to be equivalent with Theorem 8.1.15 below. The connections between the conditions of these theorems are as follows: $(a), (b), (c), (d)$ correspond to $(0), (1), (2), (4)$ respectively, while (3) corresponds to the assumption of $\mathfrak{M} \models \mathbf{Ax_g1}$.

THEOREM 8.1.15 *Suppose $f : {}^2\Re \to {}^2\Re$ is a partial function such that*

(0) *The domain of f is open and convex,*
(1) *f is twice continuously differentiable,*
(2) $(\forall (t,x) \in Dom(f))((\partial_t f_x(t,x))^2 < (\partial_t f_t(t,x))^2 \wedge (\partial_x f_t(t,x))^2 < (\partial_x f_x(t,x))^2)$,
(3) *f is locally Lorentz, that is,* $(\forall (t,x) \in Dom(f))(\ \dfrac{\partial_t f_x(t,x)}{\partial_t f_t(t,x)} = \dfrac{\partial_x f_t(t,x)}{\partial_x f_x(t,x)}\ )$,
(4) *The μ-distance between parallel f-coordinate lines is constant, that is* $(\forall t_0, t_1, x_0, x_1 \in \Re)((t_0,x_0),(t_0,x_1),(t_1,x_0),(t_1,x_1) \in Dom(f) \Rightarrow$
$\mu(f[(t_0,x_0),(t_0,x_1)]) = \mu(f[(t_1,x_0),(t_1,x_1)]) \wedge$
$\mu(f[(t_0,x_0),(t_1,x_0)]) = \mu(f[(t_0,x_1),(t_1,x_1)]))$.

Then f is a partial Lorentz transformation.

Proof. First we introduce functions measuring the distances between parallel coordinate lines induced by f. Let $t_0 \leq t_1 \in \Re$ be such that the set

$$D_{t_0,t_1} := \{x \in \Re :\ (t_0,x),(t_1,x) \in Dom(f)\}$$

is non empty. Then the function $g_{t_0,t_1} : D_{t_0,t_1} \to \Re$ is defined to be

$$(\forall x \in D_{t_0,t_1})(g_{t_0,t_1}(x) = \int_{t_0}^{t_1} \sqrt{(\partial_t f_t(s,x))^2 - (\partial_t f_x(s,x))^2} ds).$$

Note that this is a correct definition because of our conditions. Intuitively, the function g_{t_0,t_1} measures the time passed between the time-like coordinate lines labeled by t_0 and t_1 at location x.

Similarly, for the space-like coordinate lines, we define the functions h_{x_0,x_1} as follows. Suppose $x_0 \leq x_1 \in \Re$ are such, that the set $D_{x_0,x_1} := \{t \in \Re :\ (t,x_0),(t,x_1) \in$

$Dom(f)\}$ is non empty. Then

$$(\forall t \in D_{x_0,x_1})(h_{x_0,x_1}(t) = \int_{x_0}^{x_1} \sqrt{(\partial_x f_x(t,s))^2 - (\partial_x f_t(t,s))^2} ds).$$

Now condition (4) states, that the above defined functions g_{t_0,t_1} and h_{x_0,x_1} are constants. All of these functions are differentiable because of condition (1). Thus, their derivatives are identically zero:

$$(5) \quad (\forall t_0, t_1)(\forall x \in D_{t_0,t_1}) \quad 0 = g'_{t_0,t_1}(x) = \partial_x \int_{t_0}^{t_1} \sqrt{(\partial_t f_t(s,x))^2 - (\partial_t f_x(s,x))^2} ds,$$

$$(6) \quad (\forall x_0, x_1)(\forall t \in D_{x_0,x_1}) \quad 0 = h'_{x_0,x_1}(t) = \partial_t \int_{x_0}^{x_1} \sqrt{(\partial_x f_x(t,s))^2 - (\partial_x f_t(t,s))^2} ds.$$

By condition (1) the integration and the derivation in (5) (and in (6) as well) commute, so executing the derivation in (5), one has

$$(8) \quad (\forall t_0, t_1)(\forall x \in D_{t_0,t_1}) \; 0 = \int_{t_0}^{t_1} \frac{2\partial_t f_t(s,x)\partial_x\partial_t f_t(s,x) - 2\partial_t f_x(s,x)\partial_x\partial_t f_x(s,x)}{2\sqrt{(\partial_t f_t(s,x))^2 - (\partial_t f_x(s,x))^2}} ds.$$

It follows, that for all $(s,x) \in Dom(f)$

$$(9) \quad 0 = \frac{2\partial_t f_t(s,x)\partial_x\partial_t f_t(s,x) - 2\partial_t f_x(s,x)\partial_x\partial_t f_x(s,x)}{2\sqrt{(\partial_t f_t(s,x))^2 - (\partial_t f_x(s,x))^2}}$$

because of the following. Let us denote the right hand side of (9) by $j(s,x)$. Suppose for contradiction that there exists a point $(s,x) \in Dom(f)$ such that $j(s,x) \neq 0$. By condition (1) the function j is continuous, therefore there exists a neighbourhood $N(s,x)$ of (s,x) such that $N(s,x) \subseteq Dom(f)$ and j does not vanish on $N(s,x)$. Now choose $t_0 < t_1 \in \Re$ such that $(t_0,x),(t_1,x) \in N(s,x)$ hold. Clearly, $0 \neq \int_{t_0}^{t_1} j(s,x)ds$ contradicting to (8). Therefore (9) is true; which can be reduced to the following partial differential equation for f:

$$(10) \quad 0 = \partial_t f_t \partial_x \partial_t f_t - \partial_t f_x \partial_x \partial_t f_x.$$

Similarly to the previous paragraph, from (6) one obtains the following partial differential equation for f:

$$(11) \quad 0 = \partial_x f_x \partial_t \partial_x f_x - \partial_x f_t \partial_t \partial_x f_t.$$

Note that by condition (2) we have $(\forall (t,x) \in Dom(f))(\partial_t f_t(t,x) \neq 0 \wedge \partial_x f_x(t,x) \neq 0)$. By condition (1) $\partial_t\partial_x f_t = \partial_x\partial_t f_t$ and $\partial_t\partial_x f_x = \partial_x\partial_t f_x$ also hold. From (10) expressing $\partial_t\partial_x f_t$ and substituting the result into (11) we have

$$(12)\quad \partial_x f_x \partial_t \partial_x f_x = \partial_x f_t \frac{\partial_t f_x}{\partial_t f_t} \partial_x \partial_t f_x.$$

Similarly, from (11) expressing $\partial_x \partial_t f_x$ and substituting it into (10) we have

$$(13)\quad \partial_t f_t \partial_x \partial_t f_t = \partial_t f_x \frac{\partial_x f_t}{\partial_x f_x} \partial_t \partial_x f_t.$$

Now we show that $(\forall (t,x) \in Dom(f))(\partial_t \partial_x f_x(t,x) = 0 \;\wedge\; \partial_t \partial_x f_t(t,x) = 0)$. Suppose for contradiction, that there exists a point $(t,x) \in Dom(f)$ such that $\partial_t \partial_x f_x(t,x) \neq 0$. Now it follows from (12) that

$$(14)\quad \partial_x f_t \partial_t f_x = \partial_x f_x \partial_t f_t$$

holds in a sufficiently small neighbourhood of (t,x). Using condition (3) one concludes from (14) that $(\partial_t f_x(t,x))^2 = (\partial_t f_t(t,x))^2$ contradicting condition (2).

Similarly, if $\partial_t \partial_x f_t(t,x) \neq 0$ would hold for some $(t,x) \in Dom(f)$ then using (13) one can deduce (14) yielding the above contradiction.

So, we proved $(\forall (t,x) \in Dom(f))(\partial_t \partial_x f_x(t,x) = 0 \;\wedge\; \partial_t \partial_x f_t(t,x) = 0)$. This implies, that there exist differentiable, real valued functions $C_1, C_2, D_1, D_2 : \Re \to \Re$ such that

$$(15)\quad (\forall (t,x) \in Dom(f))(f_t(t,x) = C_1(t) + C_2(x) \;\wedge\; f_x(t,x) = D_1(t) + D_2(x))$$

and therefore $(\forall (t,x) \in Dom(f))$

$$\partial_t f_t(t,x) = C_1'(t),\; \partial_t f_x(t,x) = D_1'(t),\; \partial_x f_t(t,x) = C_2'(x),\; \partial_x f_x(t,x) = D_2'(x).$$

Using condition (3) it follows that

$$(16)\quad \forall (t,x) \in Dom(f) \quad \frac{D_1'(t)}{C_1'(t)} = \frac{C_2'(x)}{D_2'(x)}$$

which is possible only when there exists a constant $\lambda \in \Re$ such that both sides of (16) are equal with λ, that is,

$$(17)\quad \forall (t,x) \in Dom(f) \quad \lambda = \frac{D_1'(t)}{C_1'(t)} = \frac{C_2'(x)}{D_2'(x)}.$$

But (17) means that the tangent lines of the f-coordinate lines are the same in every point, that is, the f-coordinate lines are straight lines themselves, so (using again condition (3) one can conclude that) f is a partial Lorentz transformation, as desired. ∎

Now we turn to constructing models in which there exists a non-inertial observer. To do this, first we develop a little differential geometry.

Preliminaries from Differential Geometry

From now on we are working over the field of reals and in dimension $n = 2$ and when speaking about relativity models, we always assume the symmetry axiom **Ax△2**.

Definition 8.1.16 By a curve we mean a twice continuously differentiable function $f : \Re \to {}^2\Re$.

Remark 8.1.17 *Recall that if f is a curve, then the coordinate functions of f are denoted by f_t, f_x respectively. Moreover, if the function f_t' satisfies the condition $(\forall t \in Dom(f))(f_t'(t) \neq 0)$ then f_t is strictly monotone therefore $(f_t(t), f_x(t))$ is the unique point in $Ran(f)$ having first coordinate $f_t(t)$, so $Ran(f)$ can be considered as a graph of the one variable function $f_t(t) \mapsto f_x(t)$, we will denote this function by f_{*t}. In this case the first and second derivatives of f_{*t} can be computed as follows:*

$$f_{*t}'(t) = f_x'(t)/f_t'(t) \text{ and}$$
$$f_{*t}'' = (f_x''(t)f_t'(t) - f_t''(t)f_x'(t))/(f_t')^3.$$

Our intuition about the uniform acceleration is that the uniformly accelerating observer observes his/her change of speed in such a way that this change does not depend on the time instant when the observation is made. However, when velocity is changing, it is not clear, what are the unit vectors of the accelerating observer at a certain moment. By $\mathbf{Ax_g1}$ locally we can approximate the trace of an accelerating observer by an inertial one, and we can ask this new inertial observer about the acceleration of the accelerating observer.

To be more concrete let us fix a model $\mathfrak{M} \in \mathsf{FM}_{\text{acc}}$ such that $\mathfrak{M} \models Acc \wedge$ **Ax△2**. Suppose m is an uniformly accelerating observer, k is an inertial observer, and for any $t \in \Re$ k_t is such an inertial observer, whose location and velocity coincide with that of m at time instant t as seen by k. Suppose moreover that the function $f : \Re \to \Re$ is such that $f = tr_k(m)$ (as set of pairs). Then

$$tr_{k_t}(m) = \{\langle \frac{1}{\sqrt{1 - v_k(k_t)}} \ (t - v_k(k_t)f(t), \ f(t) - v_k(k_t)t)\rangle : \ t \in \Re\}.$$

So the trace of m as seen by k_t is the range of a curve. Note that $v_k(k_t)$ is equal to the velocity of m at time instance t, as seen by k. The acceleration of m observed by k_t is simply the second derivative of the above curve. Thus, the acceleration is uniform (in relativistic sense) iff this second derivative at (time instant) t does not depend on t. After a straightforward computation based on Remark 8.1.17 this motivates the following definition.

Definition 8.1.18 A twice derivable function $f : \Re \to \Re$ is called UAT iff it satisfy the differential equation

$$f'' = \alpha(1 - (f')^2)^{3/2}$$

for some $0 \neq \alpha \in \Re$. UAT stands for uniform accelerating trace.

Thus, the range of an UAT function can be the trace of an uniformly accelerating observer from an inertial observer's view.

THEOREM 8.1.19 *Let $f : \Re \to \Re$ be a function. Then f is UAT iff there are constants $\alpha, \beta, \gamma \in \Re$ such that* $(\forall t \in \Re)$

$$f(t) = \frac{\sqrt{1 + \alpha^2(t + \beta)^2} + \gamma}{\alpha}$$

Pfoof. By solving the differential equation given in Definition 8.1.18. ∎

Remark 8.1.20 The constants α, β, γ in Theorem 8.1.19 coincide with the "initial" acceleration, velocity and location of the accelerating observer m as seen by k. For instance, when $m \in w_k(\bar{0})$ and the velocity of m at time instant 0 is equal to 0 (as seen by k) then the trace of m as seen by k is the range of the function

$$t \mapsto \frac{\sqrt{1 + \alpha^2 t^2} - 1}{\alpha}.$$

Now we have computed the trace of an accelerating observer. In order to build a model, the next step is to determine the coordinate system of the accelerating observer m. Let k be an inertial observer. By axiom $\mathbf{Ax4_g}$, the events in $tr_k(m)$ are the same as the events in the time axis of m. The other coordinate lines of m are also visualizable in the coordinate system of k: these are the f_{mk} lines, and this is what we will use.

If m would be an inertial observer, then the time like f_{mk}-lines were parallel to $tr_k(m)$. Therefore, in the case of an accelerating observer, we define the time like f_{mk}-lines to be parallel to $tr_k(m)$. More precisely, we would like to find a world view transformation f_{mk} which satisfies the above property. Let us notice, that this is an ad hoc decision, that is, no axiom forces us to go this way. We will return to this question after constructing a model of Acc.

Thus, we already know what events have same location from the viewpoint of m. To determine simultaneities, we will use $\mathbf{Ax_g1}$.

Definition 8.1.21 Let $g : \Re \to \Re$ be a twice differentiable function such that $(\forall t \in Dom(g))(g'(t) < 1)$. Then by an inverse simultaneity of g we mean a function h which satisfies the differential equation

$$(\forall x)(h'(x) = g'(h(x))).$$

By a simultaneity we mean an invertible function whose inverse is an inverse simultaneity.

An inverse simultaneity is intended to be the function h associating a spacetime point $h(p)$ to each space point p. Intuitively, this function tells us "when" showed the clock at p a certain (fixed) value. A simultaneity parametrizes the range of an inverse simultaneity, but wrt the time axis. Both of the notions of inverse simultaneity and simultaneity will be useful.

THEOREM 8.1.22 *The simultaneities of the UAT function*

$$f(t) = \frac{\sqrt{1+\alpha^2(t+\beta)^2}+\gamma}{\alpha}$$

are the functions:

$$\frac{\sqrt{1+\alpha^2(t+\beta)^2}+0.5ln(\frac{e^{arsh(\alpha(t+\beta))}-1}{e^{arsh(\alpha(t+\beta))}+1})}{\alpha}.$$

Proof. By solving the differential equation in Definition 8.1.21. ∎

COROLLARY 8.1.23 For every UAT function f and for each point $p \in {}^2\Re$ there is a simultaneity (or inverse simultaneity) of f incident with p.

By this we determined all the f_{mk} lines of the uniformly accelerating observer m wrt inertial observer k.

Before building a model we describe here another acceleration motivated by the following. In special relativity, if an observer m moves faster and faster relative to k, then m's clock ticks slower and slower (relative to k's clock). So is it possible to accelerate so fast that k doesn't see the event when the internal clock of m shows, say, the time instant 1 ? The answer is yes. Intuitively, such a movement can be constructed as follows. For each $n \in \omega$ there is a velocity v_n such that the clock of an inertial observer m_n moving velocity v_n relative to k shows $1/2^n$ when the clock of k shows 1. Now if the accelerating observer m, moves in the first second with velocity v_1 (relative to k) and then with velocity v_2,... and so on, then k never observes the event when m's clock reaches 1. The problem is that the above constructed trace of m is not differentiable.

Definition 8.1.24 We say that a function $f : \Re \rightarrow \Re$ is CCT if the Minkowski lenght of (the range of) f between 0 and the infinity is finite. (CCT stands for convergent clock trace.)

THEOREM 8.1.25 *For any positive $\alpha \in \Re$ the function $f_\alpha : \Re \rightarrow \Re$,*

$$(\forall t)(f_\alpha(t) = arsh(\alpha e^x))$$

is CCT. For any point $p \in {}^2\Re$ there is an f_α simultaneity incident with p.

Proof. The proof can be found in [230]. ∎

LEMMA 8.1.26 *If f is a CCT function and there is a function g such that*

$$(\exists a \in \Re)(\forall x \geq a)(f(x) = g(x))$$

then g is a CCT function as well.

Proof. Obvious. ∎

Now we are ready to build a model in which there exists an accelerating observer. Instead of giving one model, we are giving a construction which expands certain special relativity models to a model of *Acc*. Thus, we will show the relative consistency of $Acc \cup \{(\exists m)(m \in Obs - IOb)\}$.

Let $\mathfrak{M}$ be a two dimensional special relativity model such that $\mathfrak{F}^{\mathfrak{M}} \cong \Re$ and $\mathfrak{M} \models \textbf{Newbasax} \wedge \textbf{Ax}\triangle\textbf{2}$. Let $n \in Obs^{\mathfrak{M}}$ be an (inertial) observer. Intuitively, we will put into the world view of n the new accelerating observers according to an UAT function g. So let us choose an UAT function g such that $g(0) = 0$. We will denote the new model $\mathfrak{M}^g$ (in fact, this model depends on the choice of n,$\mathfrak{M}$ and g). Fix a set of new symbols $\{s_r : r \in \Re\}$ disjoint from the universes of $\mathfrak{M}$. This set consists of the new, accelerating observers.

Let $\mathfrak{F}^g = \Re$.
Let $B^g = B^{\mathfrak{M}} \cup \{s_r : r \in \Re\}$.
Let $Obs^g = Obs^{\mathfrak{M}} \cup \{s_r : r \in \Re\}$.
Let $Ib^g = Ib^{\mathfrak{M}}$.
Let $Ph^g = Ph^{\mathfrak{M}}$.
Let $\mathrm{E}^{g} = \mathrm{E}^{\mathfrak{M}} = \in \cap {}^2\Re \times \mathcal{P}({}^2\Re)$.

To define the world view relation W^g first we introduce the functions $f_{s_r n} : {}^2\Re \to {}^2\Re$ as follows (see the figure below). Choose an arbitrary point $p = (t, x) \in {}^2\Re$. Let us denote by h the g-(inverse) simultaneity incident with $\bar{0}$. Find the point q such that the Minkowski length of h between $\bar{0}$ and q is equal to r[1282]. Find another point q_1 such that the Minkowski length of h between q and q_1 is equal to x. Let g_0 and g_1 be curves parallel to g and incident with q,q_1 respectively. Let q_2 be the point incident with g_0 such that the Minkowski length of g_0 between q and q_2 is equal to t. Let g_3 be the g (inverse-)simultaneity incident with q_2. Finally, let p' be the intersection of g_3 and g_1. We define $f_{s_r n}(p) = p'$. It is easily seen that the above defined points exist, are unique, and $f_{s_r n}$ is a bijection.

Let $w_n^g = w_n^{\mathfrak{M}} \cup \{\langle f_{s_r n}(t, 0), s_r\rangle : \ t, r \in \Re\}$.
For all $m \in Obs^{\mathfrak{M}} - \{n\}$ let $w_m^g = w_m^{\mathfrak{M}} \cup \{\langle f_{nm}^{\mathfrak{M}}(f_{s_r n}(t, 0)), s_r\rangle : t, r \in \Re\}$..

[1282] the sign of r determines the direction of the measurement

Figure 356:

For all $r \in \Re$ let $w^g_{s_r}(t,x) = w^g_n(f_{s_r n}(t,x))$.

By this we defined the world view relation W^g of the new model.

For all $m \in Obs^{\mathfrak{M}}$ let $G^g_m = G^{\mathfrak{M}}_m$.
For all $r \in \Re$ let $G^g_{s_r} = \{ f^{-1}_{s_r n}[\ell] : \ \ell \in G^{\mathfrak{M}}_n \ \}$.

For all $m \in Obs^{\mathfrak{M}}$ let $d^g_m = \mu^{\mathfrak{M}}$ (where $\mu^{\mathfrak{M}}$ is the Minkowski metric in $\mathfrak{M}$).
For all $r \in \Re$ let $d^g_{s_r}(p,q)$ be the Minkowski length of $f_{s_r n}[p,q]$.

Finally, let $\mathfrak{M}^g = \langle B^g, Obs^g, Ph^g, Ib^g, \mathfrak{F}^g, \ \mathrm{E}\ ^g, W^g, G^g_m, d^g_m \rangle_{m \in Obs^g}$.

THEOREM 8.1.27 $\mathfrak{M}^g \models$ *Acc.*

Recall that $\mathfrak{M} \models$ **Newbasax** $\wedge$ **Ax$\triangle$2**.
Proof. By checking the axioms. ∎

Let us notice that this construction is ad hoc in the sense, that no axiom forces us to go this way. Indeed, instead of the trace of the "new" observer, if we would know another function which can be seen as a simultaneity of the "new" accelerating observer, then using an analogue idea to Definition 8.1.21 we would determine the space-like f_{mk} lines and would repeat the above construction. This construction will be called the dual construction.

Exercise 8.1.28 Prove that the formula TwP (see section 2.1) describing the twin paradox is valid in $\mathfrak{M}^g$ (for arbitrary $\mathfrak{M}, g$ and $n \in Obs^{\mathfrak{M}}$ which does not excluded by the construction).

Let us notice that using the dual construction, the "reason" for the twin paradox is much more complicated.

Remark 8.1.29 *It is interesting to investigate how the speed of light behaves in models of* **Acc**. *In particular, if $m \in Obs$ is not inertial then there is no axiom requiring that m would observe the speed of light to be similar to what an inertial observer would see (the latter is "constantly" 1 by $\mathbf{AxE}_{0g}$). Let us consider the particular models $\mathfrak{M}^g$. Let us assume that $k, m \in Obs$ and k is inertial. Throughout, we assume that m accelerates with a uniform acceleration. First, assume that the speed of m at time 0 is 0 when observed by k. For simplicity assume $\bar{0} \in tr_k(m)$ too. Let $ph \in Ph$ be such that $\bar{0} \in tr_k(ph)$. Then the speed of ph as observed by m is not constant, actually the photon ph is accelerating (increasing its velocity) as seen by m. This behaviour does not depend on the direction in which ph is moving (i.e. ph may be "falling in the direction in which gravity[1283] is pulling it" or may be moving in the other direction). We do not prove this here but we refer to [230].*

Next, let us assume that the speed of m at time 0 is large i.e. it is close to 1 (e.g. it may be 0.8 or 0.9). Now we may investigate 6 kinds of photons $ph \in Ph$. First assume that /ph moves in the direction against that of gravity. Assume that ph starts out (sometime) on the lifeline of m. Now, if ph starts out sometime > 0 then we conjecture that it may accelerate. However if ph starts out at time 0 then we think that it will decelerate (as opposed to accelerating). Moreover if ph starts out sometime in the sufficiently distant past then we think that it will "strongly decelerate", in some sense. If ph moves in the other direction (i.e. it is "falling") then we do not know whether it accelerates, decelerates or what exactly it does.

[1283] We apologize for mentioning gravity in an undefined way. In the present approach gravity is a carefully defined notion, but for lack of time we do not recall the definition here. Intuitively, what we call gravity is the usual side-effect of acceleration "experienced" by the accelerating observer m, and to every point p of the spacetime ${}^nF$ of m the gravity at p experienced by m is a vector pointing in the direction opposite to that of the acceleration of m.

All these seem to point in the direction that the speed of light in models like $\mathfrak{M}^g$ behaves in somewhat complicated way. As a contrast we note that in the kind of "goldfish" models (or Kruskal-models) referred to in Remark 8.1.3 the absolute value of the speed of light seems to be constant (and is 1*) for even the accelerating observers.*

We close this remark by mentioning that what we said here is somewhat tentative and that we did not check all the details.

Now we turn to constructing a model, containing an observer, whose trace as seen by another inertial observer is the range of some CCT function. Roughly, we repeat the previous construction, but we have to do something more, because a CCT function does not determine the whole world view of the new observer m, i.e. no conditions about events seen by m in a time instant which is bigger than the Minkowski length of (the range of) the CCT function in question. Therefore we start with two models.

Let us choose a CCT function g described in Lemma 8.1.26 such that g is twice continuously differentiable, $(\forall x \le 0)(g(x) = 0)$ and $g'(0) = 0$ (clearly, there exists such a function). Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be two models over the field of reals, such that $\{\mathfrak{M}_1, \mathfrak{M}_2\} \models$ **Newbasax** $\wedge$ **Ax**$\triangle$**2** and $B^{\mathfrak{M}_1} \cap B^{\mathfrak{M}_2} = \emptyset$. Choose an observer $n \in Obs^{\mathfrak{M}_1}$ and $n_0 \in Obs^{\mathfrak{M}_2}$. Fix a set of new symbols $\{s_r : r \in \Re\}$ disjoint from the universes of $\mathfrak{M}_1, \mathfrak{M}_2$. This set consists of the new, accelerating observers. We will denote the new model constructed below by $\mathfrak{M}^g$. In fact, $\mathfrak{M}^g$ depends on the choice of $g, \mathfrak{M}_1, \mathfrak{M}_2, n$ and n_0.

Let $\mathfrak{F}^g = \Re$.
Let $B^g = B^{\mathfrak{M}_1} \cup B^{\mathfrak{M}_2} \cup \{s_r : r \in \Re\}$.
Let $Obs^g = Obs^{\mathfrak{M}_1} \cup Obs^{\mathfrak{M}_2} \cup \{s_r : r \in \Re\}$.
Let $Ib^g = Ib^{\mathfrak{M}_1} \cup Ib^{\mathfrak{M}_2}$.
Let $Ph^g = Ph^{\mathfrak{M}_1} \cup Ph^{\mathfrak{M}_2}$.
Let $\mathrm{E}^{g} = \mathrm{E}^{\mathfrak{M}} = \in \cap {}^2\Re \times \mathcal{P}({}^2\Re)$.

Let τ be the Minkowski length of g between 0 and infinity. Let $T = \{(t,x) \in {}^2\Re : \ t < \tau\}$. Analogously to the previous construction, to define the world view relation W^g first we introduce the functions $f_{s_r n} : {}^2\Re \to {}^2\Re$ as follows (see the figure below). Choose an arbitrary point $p = (t,x) \in T$. Let us denote by h the inverse simultaneity of g incident with $\bar{0}$. By the choice of g this is simply the x axis. Find the point q such that the Minkowski length of h between $\bar{0}$ and q is equal to r[1284]. Find another point q_1 such that the Minkowski length of h between q and q_1 is equal

[1284] Again, the sign of r determines the direction of the measurement.

to x. Let g_0 and g_1 be curves parallel to g and incident with q,q_1 respectively. Let q_2 be the point incident with g_0 such that the Minkowski length of g_0 between q and q_2 is equal to t. Let g_3 be the g (inverse-) simultaneity incident with q_2. Finally, let p' be the intersection of g_3 and g_1. We define $f_{s_r n}(p) = p'$. It is easily seen that the above defined points exist, are unique, and $f_{s_r n}$ is injective.

Figure 357:

Let $w_n^g = w_n^{\mathfrak{M}_1} \cup \{\langle f_{s_r n}(t,0), s_r\rangle : (t,r) \in T\}$.
For all $m \in Obs^{\mathfrak{M}_1} - \{n\}$ let $w_m^g = w_m^{\mathfrak{M}_1} \cup \{\langle f_{nm}^{\mathfrak{M}_1}(f_{s_r n}(t,0)), s_r\rangle : (t,r) \in T\}$.

$$(\forall m \in Obs^{\mathfrak{M}_2})(\forall (t,r) \in {}^2\Re) w_m^g(t,r) = \begin{cases} w_m^{\mathfrak{M}_2}(t,r) & \text{if } t < \tau \\ w_m^{\mathfrak{M}_2}(t,r) \cup \{s_r\} & \text{if } t > \tau. \end{cases}$$

In order to define the world view of s_r, for all $r \in \Re$ let us fix an observer $n_r \in Obs^{\mathfrak{M}_2}$ such that $tr_{n_0}^{\mathfrak{M}_2}(n_r) = \Re \times \{r\}$.

$$(\forall r \in \Re)(\forall (t,x) \in {}^2\Re) w^g_{s_r}(t,x) = \begin{cases} w^g_n(f_{s_r}(t,x)) & \text{if } (t,x) \in T \\ w^{\mathfrak{M}_2}_{n_r}(t,x) \cup \{s_{r+x}\} & \text{if } t > \tau \\ \emptyset & \text{if } t = \tau, x \neq r \\ \{s_r\} & \text{if } t = \tau, x = r. \end{cases}$$

By this we defined the world view relation W^g of the new model.

For all $m \in Obs^{\mathfrak{M}_1}$ let $G^g_m = G^{\mathfrak{M}_1}_m$.
For all $m \in Obs^{\mathfrak{M}_2}$ let $G^g_m = G^{\mathfrak{M}_2}_m$.
For all $r \in \Re$ let $G^g_{s_r} = \{f^{-1}_{s_r n}[\ell] : \ \ell \in G^{\mathfrak{M}}_n\} \cup \{f^{-1}_{s_r n_r}[\ell] \cap ({}^2\Re \setminus T) : \ \ell \in G^{\mathfrak{M}_2}_{n_r}\}$.

For all $m \in Obs^{\mathfrak{M}_1}$ let $d^g_m = \mu^{\mathfrak{M}_1}$
For all $m \in Obs^{\mathfrak{M}_2}$ let $d^g_m = \mu^{\mathfrak{M}_2}$

$$(\forall r \in \Re)(\forall p,q \in {}^2\Re) d^g_{s_r}(p,q) = \begin{cases} \text{the Minkowski length of } f_{s_r n}[p,q] & \text{if } p,q \in T \\ \text{the Minkowski length of } f_{s_r n_r}[p,q] & \text{if } p,q \notin T \\ 0 & \text{Otherwise.} \end{cases}$$

Finally, let $\mathfrak{M}^g = \langle B^g, Obs^g, Ph^g, Ib^g, \mathfrak{F}^g, \mathrm{E}^g, W^g, G^g_m, d^g_m \rangle_{m \in Obs^g}$.

THEOREM 8.1.30 $\mathfrak{M}^g \models Acc - \{\mathbf{Ax_g2}, \mathbf{Ax_g3}\}$.

Proof. By checking the axioms. ∎

The axioms $\mathbf{Ax_g2}, \mathbf{Ax_g3}$ fail to hold because there are points separated with the border of T (which we will call an event horizon), and somehow the observers $s_r, r \in \Re$ "changed their universe". The same situation appears around black holes. Finally we start to study such situations.

Definition 8.1.31 Let $\mathfrak{M} \in \mathsf{FM}_{\text{acc}}$ be a model, $m \in Obs^{\mathfrak{M}}$ and $r \in \Re$. We say that in $\mathfrak{M}$ observer m has an event horizont at r iff

$$(\exists n \in IOb^{\mathfrak{M}})(\forall \varepsilon, k \in (F^{\mathfrak{M}})^+)(\exists p \in {}^2F^{\mathfrak{M}})(|p_x| \geq k \wedge r - \varepsilon \leq p_t \leq r + \varepsilon \wedge n \in w^{\mathfrak{M}}_m(p)).$$

THEOREM 8.1.32 *Let* $\mathfrak{M} \in \mathsf{FM}_{\rm acc}$ *be a two dimensional model such that* $\mathfrak{F}^{\mathfrak{M}} \cong \Re$ *and* $\mathfrak{M} \models Acc - \{\mathbf{Ax_g2}, \mathbf{Ax_g3}\}$*. Let* $m \in Obs^{\mathfrak{M}}$ *such that there are two points* $p_0, p_1 \in {}^2\Re$ *and an inertial observer* $k \in IOb^{\mathfrak{M}}$ *satisfying* $k \in w_m^{\mathfrak{M}}(p_0) = w_m^{\mathfrak{M}}(p_1)$*. Suppose moreover that the set* $tr_m^{\mathfrak{M}}(k)$ *is closed (in the topology induced by the ordering),* f_{mk} *is locally injective and whenever* $q_0 \neq q_1 \in {}^2\Re$ *we have*

$$k \in w_m^{\mathfrak{M}}(q_0) \cap w_m^{\mathfrak{M}}(q_1) \Rightarrow {q_0}_t \neq {q_1}_t.$$

Then there exists an $r \in \Re$ *between* ${p_0}_t$ *and* ${p_1}_t$ *such that* m *has an event horizon at* r*.*

Proof. The proof can be found in [230]. ∎

Appendix: Why first-order logic?

Here we address the question why we want to stick in this work with first-order logic FOL, and why we do not want to use standard second-order logic L_2 or anything stronger than L_2.[1285] (To be precise, we do allow extensions of FOL called many-sorted FOL, which can be regarded as a step in the direction of higher-order logic in that many-sorted FOL is more flexible than FOL and some ideas expressible in higher-order logic can be translated to many-sorted FOL.)

After Hilbert axiomatized Euclidean geometry in L_2, why did Tarski feel it important to find an axiomatization for Euclidean geometry in FOL? What is insufficient in axiomatizing something in higher-order logic?[1286] Below we will try to sketch a brief and easily understandable answer to these questions, too.

Let $\models_2$ denote the semantic consequence relation of L_2. I.e. if φ, ψ are formulas of L_2, then $\varphi \models_2 \psi$ means that ψ is a *logical consequence* of φ in L_2.

(1) It is easy to write up a finite axiom system, say $Ax2$, in L_2 and a first-order formula φ such that the answer to the question

($\star$) Is $Ax2 \models_2 \varphi$ true?

is unknowable. (We will soon explain what we mean by unknowable, but till then, we really mean unknowable.) Moreover, there is an infinity of different choices of $Ax2$ and φ making ($\star$) unknowable.

If we allow φ to be second-order too, then the situation becomes even worse, even $\models_2 \varphi$ becomes unknowable, where $\models_2 \varphi$ abbreviates $\emptyset \models_2 \varphi$, i.e. $\models_2 \varphi$ says that φ is logically valid in L_2.

Let us come back to ($\star$). It is possible to choose $Ax2$ and φ such that $Ax2 \models_2 \varphi$ will be true (according to the rules of the game called second-order logic) iff the

[1285] Standard second-order logic (usually called simply "second-order logic") should be sharply distinguished from Henkin's nonstandard version of second-order logic also known as "second-order logic with generalized models". Herein, by second-order logic we will understand standard second-order logic. Henkin's version of second-order logic is suitable for our purposes and is free of the anomalies and pathological properties of standard second-order logic L_2 discussed in this Appendix. Actually, in the present work we will make use of Henkin's nonstandard second-order logic. We will return to that logic in item (8) way below.

[1286] Of course, if we get interested in something new, it is useful to axiomatize it in L_2, as a first step. The only thing we claim is that, in most cases, sooner or later we will need to make a further step and that will be an axiomatization in FOL (or some of its variants).

continuum hypothesis[1287] CH is true in the meta theory floating above our heads. I.e. the statement $Ax2 \models_2 \varphi$ is equivalent with stating CH. But CH is independent of usual set theory, hence we cannot know whether it is true or false, and therefore we cannot know whether $Ax2 \models_2 \varphi$ is true, for our simple $Ax2$ and φ.

To clarify the situation for the nonlogician, let us recall that when we are writing down concrete formulas like $Ax2$ and φ and asking whether $Ax2 \models_2 \varphi$ holds, then we are assuming that the basic tools of mathematics are available for us. This means that a basic version of set theory, call it MetaZFC, is fixed and we are "living" inside a model (V, ϵ) of the MetaZFC, so to speak. We do not know which model we are living in, we only know that it satisfies the axioms of ZFC. Usually, people call this model (V, ϵ) our universe[1288], but it is important to keep in mind that we do not know more about our universe than that it satisfies the axioms of ZFC. The language of MetaZFC is called the meta language and (V, ϵ) could be called a meta model. What we will call usual models are all elements of V. Hence (V, ϵ) is *not* a usual model (or even a model), it is only a meta model. All the work we are doing, e.g. defining L_2, its formulas (e.g. $\varphi, \psi, Ax2$ above), its logical consequence relation $\models_2$ is done inside V. Hence the question $(\star)$ is also formulated inside V. Hence saying that $Ax2 \models_2 \varphi$ is true iff $(V, \epsilon) \models CH$ is completely useless information, since by the most basic rules of the game we cannot know whether CH is true in V (recall that (V, ϵ) was an arbitrary but fixed model of MetaZFC, and there are models of MetaZFC in which CH is true while in others it is false).

Saying that $Ax2 \models_2 \varphi$ is true iff CH is true in V is analogous to saying that $Ax2 \models_2 \varphi$ is true iff God wants it to be true. (The point in this analogy is that, here, a question asked on the level of our subject language is answered by pointing up to the meta language.)[1289]

In a simpler language, the above can be summarized by saying that if we use L_2, then it is very easy to run into natural formulas $Ax2$, φ such that the question whether $Ax2 \models_2 \varphi$ holds is independent from mathematics. This cannot happen in FOL![1290] Moreover, as we will see it later, if the above independence phenomenon can happen in our logic, then this can seriously interfere (negatively) with our planned analysis of the logical structure of the theory in question (in the present case, relativity). Further, it interferes with other goals formulated in the

[1287] To formulate CH, recall that $\aleph_0$ is the least infinite cardinal, $\aleph_1$ is its successor, and $\aleph_k$ is the k-th infinite cardinal. Now, CH is the assumption that $\aleph_1 = |2^{\aleph_0}|$, i.e. there is no cardinal between $\aleph_0$ and the continuum. By CH for $\aleph_k$ we will understand the assumption that $\aleph_{k+1} = |2^{\aleph_k}|$.

[1288] or "the real world"

[1289] In this analogy, the meta language is playing the role of God: the point is that it is one level higher up than the level is on which we are supposed to work.

[1290] We will soon say more about why this cannot happen in FOL.

Introduction (§1.1), too.

The expressibility (in L_2) of CH, CH for $\aleph_k$ ($k \in N = \omega$), and other statements independent from MetaZFC is showed e.g. in Sain [232], in Ebbinghaus et al. [77] and also in Barwise-Feferman [43, p.33, lines 14-20] where it writes about L_2: "... all useful model theoretic properties of first-order logic fail [for L_2]. Moreover ... we quickly run into set theoretic dependencies[1291] as well. ...".

In contrast with L_2, FOL is free from these pathological properties. This is shown by proving that the basic ingredients, like $\models \subseteq$ Formulas $\times$ Formulas and $\models \subseteq$ Models $\times$ Formulas, of FOL satisfy the condition which is called "absoluteness" in set theoretic investigations of metalogic. Such investigations are e.g. in Chapter XVII (by Väänänen) of [43, §XVII.2.1, pp.609-610], but also in [41] and in Manders [183]. Absoluteness of a logic L is defined and is treated as a desirable property (from the pont of view of L being a logic).[1292]

To illustrate the idea of absoluteness of logics, we quote from the basic book [43, p.609]. (They use $\models_{\mathcal{L}}$ to denote the validity relation of a logic $\mathcal{L}$.):

"The idea of absoluteness of a logic is that the truth or falsity of the predicate $\mathfrak{M} \models_{\mathcal{L}} \varphi$ ***should*** not depend on the entire set theoretic universe but rather should depend on the sets that are required to exist (in addition to $\mathfrak{M}$ and φ) by the axioms of a fixed set theory, say T, only." (Underlying and insertion of "say" from present authors.)

Summing up, FOL is an absolute logic, and therefore it is free of all the pathological properties discussed in the present Appendix (not only above, but also below).[1293]

(2) More concrete examples for ($\star$), i.e. for $Ax2$ and φ (and for the unknowability of $Ax2 \models_2 \varphi$).

Let $\mathfrak{N} = \langle N, 0, 1, +, \cdot \rangle$ be the standard structure of natural numbers. Then it is easy to write up a finite and complete axiomatization $Axnu$ of $\mathfrak{N}$ in L_2.

By Gödel's incompleteness theorem, there are infinitely many FOL-formulas $\varphi_1, \varphi_2, \ldots, \varphi_i, \ldots$ such that for each $i \in N$, it is independent from metamathematics (i.e. from MetaZFC) whether $Axnu \models_2 \varphi_i$ holds.

[1291] They mean by this that basic properties like $\varphi \models_2 \psi$, which should not be independent of set theory in case of any logic, turn out to be independent of set theory.

[1292] We mean to say that desirability of absoluteness is tied up with our ambition that we want to regard L as a logic. An arbitrary mathematical concept may be nice without being absolute. But it cannot be a nice ***logic***. Hence, it is the theory of logical systems, which led its specialists to conclude that *when* considering logics for logical purposes, then absoluteness is a desirable property.

[1293] Henkin's nonstandard higher-order logic is also absolute. As a contrast, as we have seen above, standard second-order logic L_2 is not absolute.

To be more concrete, we can choose e.g. $\varphi_1 \stackrel{\text{def}}{=} \mathrm{Con(ZFC)}$, $\varphi_2 \stackrel{\text{def}}{=} \mathrm{Con(ZFC}+\varphi_1)$, $\varphi_{n+1} \stackrel{\text{def}}{=} \mathrm{Con(ZFC}+\varphi_n)$. Here Con(ZFC) denotes the number theoretic formula expressing the consistency of ZFC constructed in the style of Gödel's Second Incompleteness Theorem.

It follows from Gödel's theorems that, for each $i \in N$, there are models of MetaZFC in which "$Axnu \models_2 \varphi_i$" holds while in other models of the same MetaZFC "$Axnu \models_2 \varphi_i$" fails, assuming MetaZFC is sufficiently consistent.[1294] This is what we mean when we say that the question whether $Axnu \models_2 \varphi_i$ holds is independent of mathematics (i.e. is unknowable, in some sense). Notice that by now we have infinitely many unanswerable (unknowable) questions about the logical consequence relation $\models_2$ of L_2.

It was shown in [16] that the same thing can happen with relativity theory (or almost any theory of dynamics allowing accelerated motion) in place of number theory, i.e. in place of $Axnu$ or in place of $\mathfrak{N}$. In passing we note that it was shown in Penrose [212, Chapter 5, pp.180-173 (section: "is life in the billiard-ball world computable?")] that even for a rather simple version of Newtonian Mechanics, Gödel's Incompleteness Theorems do apply. Therefore in the above construction of the independent questions $Axnu \models_2 \varphi_i$ we could replace number theory with Newtonian mechanics (even with a simplified version, if we wanted). Cf. also reference Fredkin-Toffoli (1982) "Conservative logic" in Int. J. Theor. Physics, listed in the bibliography of [212].

So the above construction provides us with an infinity of examples of formulas φ, ψ in L_2 where $\psi \models_2 \varphi$ is unknowable (i.e. independent of mathematics). Moreover, the references to Penrose, Fredkin-Toffoli and [16] indicate that the same anomalies can happen if we apply L_2 to the simplest kind of physical theories. In passing we note that Montague [198, (Deterministic theories)] is a classical application of model theory (and logic) to physical theories of motion and dynamics which is of high standards.

(3) One of the purposes of the present work is to elaborate logical analysis of some relativity theories. But the basic building blocks of any logical analysis are statements like ($\star$) way above or $\psi \models_2 \varphi$ where e.g. ψ is a (perhaps potential) axiom of our theory while φ is a prediction (or theorem) of the theory being analyzed. A completely analogous consideration applies to our sub-goal summarized in §1.1(X) "why"-type questions", p.12. Similarly for §1.1(III), p.7.

[1294] For $i = 1$ it is enough to assume that MetaZFC is consistent. For greater choices of i we assume (ZFC+Con(ZFC)) is consistent, [ZFC+Con(ZFC+Con(ZFC))] is consistent, etc.

In these investigations we stick with FOL, because the anomalies or pathologies outlined in items (1), (2) above cannot happen in FOL because, as we said, FOL is proved absolute and it is shown that absolute logics are free of such anomalies (almost any work on set theoretic properties of logics proves this).

(4) Below, we try to illustrate that standard higher-order logic has further disadvantages (besides the ones in (1)-(3)).

Assume, we are given a mathematical structure $\mathfrak{A}$, like e.g. $\mathfrak{N}$ way above. Assume $Ax2$ is a complete categorical axiomatization of $\mathfrak{A}$ in L_2. Then, we claim that it may happen that studying $Ax2$ (and L_2 of course) does not reveal, moreover cannot reveal, to us what the logical consequences of $Ax2$ are. Moreover, we can choose $Ax2$ and $\mathfrak{A}$ such that there are simple first order formulas $\varphi_1, \dots$ about $\mathfrak{A}$ such that any mathematician familiar with $\mathfrak{A}$ should easily answer the question whether φ_1 is true in $\mathfrak{A}$ while at the same time it is impossible to find out whether $Ax2 \models_2 \varphi_1$ in the sense that it is independent of MetaZFC whether $Ax2 \models_2 \varphi_1$.

Let $\mathfrak{N}_1$ be a structure similar to $\mathfrak{N}$ (in its language) but drastically different from $\mathfrak{N}$ (in its structure), e.g. assume $\mathfrak{N}_1$ has infinitely large elements with strange properties (like $x+x=x$). Further, choose $\mathfrak{N}_1$ such that it admits a finite categorical axiomatization in L_2.

Example 1 Let $Ax2$ be an L_2 formula expressing that

$$[\mathrm{CH} \Rightarrow \mathfrak{A} \cong \mathfrak{N}] \ \wedge\ [\neg\mathrm{CH} \Rightarrow (\mathfrak{A} \not\cong \mathfrak{N} \text{ moreover, } \mathfrak{A} \cong \mathfrak{N}_1)].$$

Since we are living in an arbitrary but fixed (V, ϵ), it is true in V that $Ax2$ is a categorical axiom system. Hence it is a theorem of mathematics that $Ax2$ is complete, categorical etc. At the same time, if somebody hands us $Ax2$ written on paper by pencil, we will never be able to figure out whether its models are like $\mathfrak{N}$ or like $\mathfrak{N}_1$.

Someone may argue that this is not so bad, since we still have only two models. However, the example can be refined to have many more non-isomorphic models, as follows.

Example 2 Let $Ax3$ be an L_2 formula expressing the following.

$$[\mathrm{CH \text{ for } } \aleph_0 \Rightarrow \mathfrak{A} \cong \mathfrak{N}] \wedge$$

$$[\neg(\mathrm{CH \text{ for } } \aleph_0) + (\mathrm{CH \text{ for } } \aleph_1) \Rightarrow \mathfrak{A} \cong \mathfrak{N}_1] \wedge$$

$$[\neg(\mathrm{CH \text{ for } } \aleph_0) + \neg(\mathrm{CH \text{ for } } \aleph_1) + (\mathrm{CH \text{ for } } \aleph_2) \Rightarrow \mathfrak{A} \cong \text{a 3-element field}] \wedge$$

$$[(\forall i \leq 2)\neg(\mathrm{CH \text{ for } } \aleph_i) \Rightarrow \mathfrak{A} \cong \text{a 5-element field}].$$

For any $k \in N$, we could continue the above outlined formula to include k-many choices (going up to $\aleph_{k-1}$) instead of only four choices as in $Ax3$ above. Let Ax_k denote the so obtained axiom system in L_2.

Again, similarly to Example 1, it is a mathematical theorem that Ax_k is complete and categorical. But now, if we are handed Ax_k printed on paper, we could not figure out whether the model Ax_k speaks about has 3 elements or 5 elements or infinitely many elements. The reason for this is that we cannot find out whether in the world (V, ϵ) we are sitting in CH for e.g. $\aleph_2$ is true or not. (The possibilities of CH holding for $\aleph_1$ but not for $\aleph_2$ or vice versa etc. are all realizable in models of ZFC. We mean that basically all combinations can occur.)

So, it may happen that a plain FOL axiomatization of $\mathfrak{N}$ (though not categorical) can tell us more about its models, e.g. $\mathfrak{N}$, than a complete and categorical axiomatization $Ax3$ of $\mathfrak{N}$ in L_2. The reader may argue that the axiomatization $Ax3$ is deliberately "ill minded" and artificial, but there is no guarantee that if we work hard in L_2 on axiomatizing some new and exciting structure, then by accident a situation like in the case of $Ax3$ will not happen to us spontaneously. Anyway, no such pathological axiomatizations are possible in FOL or in Henkin's nonstandard higher-order logic.[1295] What we understand by pathology here is that if we look at $Ax3$ in the perspective of (V, ϵ) in which we are living, it says something utterly different from what it says when studied from the perspective of the meta language MetaZFC.

(5) Many logicians maintain that logic is a science whose subject matter includes "rational reasoning", deduction, inference. The latter entails that if $\varphi \models \psi$ then we study the ways in which someone can prove ψ from φ and then use this proof to convince others. But if $\varphi \models_2 \psi$ is independent of mathematics, then we cannot even address the above outlined questions. This might point in the direction that despite of its name, standard second-order logic L_2 is not really a logic, after all.

The famous logician Quine [217] ("Philosophy of logic") argues that "standard second-order logic is not logic but set theory in disguise", cf. also [264, p.1022, line 8]. His arguments are different from ours but his conclusion is the same.

(6) We hope that the above considerations illustrate why we claim that if we chose standard higher-order logic as our language for axiomatizing and analyzing relativity theories, then this would render it practically impossible to carry through our aims outlined in the Introduction (§1.1).

[1295] This is so because of our earlier references to (set theoretic) absoluteness of FOL and the positive results about absolute logics, e.g. in [43, p.609] or in Sain [232].

(7) The explanations in items (1)-(6) above form only a strongly shortened outline of the methodological reasons coming from mathematical logic which say that we should stick with (a variant or reasonable extension) of FOL and avoid using standard higher-order logic. A full exposition of the already existing explanations for the subject matter of the present remark would take up a full technical paper (at least). All the same, we hope that the present shortened discussion is convincing enough (for the purposes of this work at least).

(8) Warning: To avoid misunderstanding we note the following. There is a version of higher-order logic called Henkin's nonstandard higher-order logic which is free of the above outlined anomalies, cf. e.g. the logic book Enderton [82, Chapter 4 ("Second-order logic"), pp.268-289], Monk [197, pp.495-498], and §4 of van Dalen [269, pp.147-156] or Henkin [128]. (The most useful reference in the present subject seems to be Enderton [82].) Actually, Henkin's higher-order logic is reducible to an axiomatisable theory of our many-sorted FOL which we use in the present work. So, we could say that besides axiomatizing our theories in FOL, we will also axiomatize some of them in Henkin's higher-order logic. Actually, some of our theories will be formalized in Henkin's nonstandard higher-order logic[1296], but we will de-emphasize this connection in order to avoid confusion with standard higher-order logic which we systematically avoid (for the above outlined methodological reasons). In passing we note that some confusion in applied logic is generated by the habit of some proof theorists of calling Henkin's non-standard higher-order logic simply "higher-order logic". An example of such confusion was tidied up in Makowski-Sain [182].

We note that a very useful translation of Henkin's second-order logic to FOL is given and explained in detail in van Dalen [269, pp.150-156] (cf. also Enderton [82, §§4.3-4.4]). Since this translation will occassionally play a role in the present work[1297], it is useful if the reader finds time for reading it (including the exercises on p.155-6) or reading [82, §§4.3-4.4], but this is not a prerequisite for understanding the present work. (There are useful such translations (from L_2 into FOL) elsewhere, too, e.g. in Monk [197] and of course in Henkin's original paper [128].)

(9) On terminology: We include the following to assist the reader with the literature. Henkin's nonstandard higher-order logic differs from standard higher-order logic only in that Henkin's version has more models. Therefore we call Henkin's extra models nonstandard models while we call the rest standard models. Several logic books, e.g. Enderton [82], agree with our usage, but not all books are such. A

[1296] Cf. e.g. the convention below the formulation of $\mathbf{Ax}_G$ on p.31 and footnote 42 there, footnote 651 on p.789, item 6 in Def.6.2.2(I) on p.789, Convention 6.2.3 on p.801, $\langle Points, Lines, Planes, \dots, \in, \in_{Pl}, \dots\rangle$ on p.998, etc.

[1297] Cf. e.g. footnote 42 on p.31, Convention 6.2.3 on p.801

large portion of the literature calls nonstandard models "general models" or weak models (of higher-order logic) while they call standard models "full models", such is van Dalen [269] except that he calls standard models simply "models". The branch of logic (beginning with e.g. Montague [198], Gallin [96]) applying logic to issues like the semantics of languages including natural languages often uses "general model", "generalized model" or "generalized structure" (g-structure for short) for Henkin's nonstandard models while they often write "standard structure" for standard models, cf. e.g. the important handbook [264, pp.1021-1022]. Similar terminology is used in works of van Benthem, Shapiro and Doets, cf. e.g. the references in the above handbook. Instead of higher-order logic many authors write "type theory" which is basically the same thing as such higher-order logic in which we allow not only second-order variables but also third-order ones and even ones of arbitrary order n where n is required to be finite.

In nonstandard analysis standard models are often called "full" ones (but sometimes simply "models"), while nonstandard ones are often called "weak" (cf. e.g. Csirmaz [64]). The Handbook of Philosophical Logic [93, Vol. II, pp.360, 364] uses "standard models" and "general models". Same holds for van Benthem [266, p.5(middle), p.12 (line 2)] and van Benthem [263, pp.211-∞]. The literature of Intensional Logic usually uses these terms.