On percolation critical probabilities and unimodular random graphs

Dorottya Beringer Gábor Pete Ádám Timár

Abstract

We investigate generalisations of the classical percolation critical probabilities p_c, p_T and the critical probability $\tilde{p}_c$ defined by Duminil-Copin and Tassion [12] to bounded degree unimodular random graphs. We further examine Schramm's conjecture in the case of unimodular random graphs: does $p_c(G_n)$ converge to $p_c(G)$ if $G_n \to G$ in the local weak sense? Among our results are the following:

- $p_c = \tilde{p}_c$ holds for bounded degree unimodular graphs. However, there are unimodular graphs with sub-exponential volume growth and $p_T < p_c$; i.e., the classical sharpness of phase transition does not hold.
- We give conditions which imply $\lim p_c(G_n) = p_c(\lim G_n)$.
- There are sequences of unimodular graphs such that $G_n \to G$ but $p_c(G) > \lim p_c(G_n)$ or $p_c(G) < \lim p_c(G_n) < 1$.

As a corollary to our positive results, we show that for any transitive graph with sub-exponential volume growth there is a sequence $\mathcal{T}_n$ of large girth bi-Lipschitz invariant subgraphs such that $p_c(\mathcal{T}_n) \to 1$. It remains open whether this holds whenever the transitive graph has cost 1.

1 Introduction

1.1 Motivation and results

There are several definitions of the critical probability for percolation on the lattices $\mathbb{Z}^d$, which have turned out to be equivalent not only on $\mathbb{Z}^d$, but also in the more general context of arbitrary transitive graphs [27, 1, 16, 4, 11]. One of our goals is to investigate the relationship between these different definitions when the graph G is an ergodic unimodular random graph [9, 2], which is the natural extension of transitivity to the disordered setting. We examine the generalisations of $p_c = \sup\{p : \mathbb{P}_p(\text{there is an infinite cluster}) = 0\}$, $p_T = \sup\{p : \mathbb{E}_p(|\mathcal{C}_o|) < \infty\}$ and $\tilde{p}_c$ defined by Duminil-Copin and Tassion in [11]. The last quantity was in fact designed to give a simple new proof of $p_c = p_T$ for transitive graphs, and to address the question of locality of critical percolation: whether the value of p_c depends only on the local structure of the graph.

More precisely, Schramm's "locality conjecture", stated first explicitly in [8], says that $p_c(G_n) \to p_c(G)$ holds whenever G_n is a sequence of vertex-transitive infinite graphs such that G_n converges locally to G (i.e., for every radius r, the r-ball in G_n, for n large enough, is isomorphic to the r-ball in G) and $\sup_n p_c(G_n) < 1$. Typically, however, the natural setting for such locality statements is not the class of transitive graphs, but the class of unimodular random graphs. Indeed, there are several interesting probabilistic quantities, most often related in some way to random walks, which have turned out to possess locality, mostly in the generality of unimodular random graphs: see [9, 22, 24, 10, 6, 17] for specific examples, and [29, Chapter 14] for a partial overview. Therefore, it is natural to investigate Schramm's conjecture in the setup of unimodular random graphs and see what the proper notion of critical probability may be from the point of view of locality.

The conjecture has been proved for some special transitive graphs: Grimmett and Marstrand [18] proved that $p_c\left(\mathbb{Z}^2 \times \{-n,\ldots,n\}^{d-2}\right) \xrightarrow{n\to\infty} p_c(\mathbb{Z}^d)$. Benjamini, Nachmias and Peres [8] verified that the convergence holds if (G_n) is a sequence of d-regular graphs with large girth and Cheeger constants uniformly bounded away from 0. Martineau and Tassion [26] proved that the convergence holds if (G_n) is a sequence of Cayley graphs of Abelian groups converging to a Cayley graph G of an Abelian group, and $p_c(G_n) < 1$ for all n. The inequality

$$\liminf_{n\to\infty} p_c(G_n) \geq p_c(G)$$

is known for any convergent sequence of transitive graphs; see [29, Section 14.2], and [11].

In Subsection 1.3 we define the generalized critical probabilities p_c, p_T, $\tilde{p}_c$, $\hat{p}_T$, and $\hat{p}_c$ for unimodular random graphs; somewhat simplistically saying, the first three will be quenched versions of the quantities mentioned above, while the last two will be annealed versions. In Section 2 we examine the relationship between these different generalizations. Our results are summarized in Table 1. The one sentence summary is that $p_c = \tilde{p}_c$ always holds, but otherwise almost anything can happen, unless the random graph satisfies some very strong uniformity conditions; one that we call "uniformly good" suffices for most purposes.

In Section 3 we investigate the extension of Schramm's conjecture for unimodular random graphs: does $p_c(G_n)$ converge to $p_c(G)$ if $G_n \to G$ in the local weak sense (i.e., the laws of the r-balls in G_n converge weakly, for every r) and $\sup p_c(G_n) < 1$? First we note (Example 3.2) that locality holds for unimodular Galton-Watson trees with bounded degrees, but not in general; this shows that it is natural to restrict one's attention to bounded degree unimodular random graphs. In Subsection 3.2, we give conditions which imply $\lim p_c(G_n) = p_c(G)$. In Subsection 3.3, we show by examples that there are sequences of unimodular random graphs such that $G_n \to G$ but $p_c(G) > \lim p_c(G_n)$ or $p_c(G) < \lim p_c(G_n) < 1$.

A corollary to our positive results is that if G is a transitive graph of subexponential volume growth, then there exists a sequence of invariant bi-Lipschitz spanning subgraphs G_n such that $p_c(G_n) \to 1$. As we will explain in Section 4, this is a strengthening of the simple fact that groups of subexponential growth have cost 1, as defined in [19], studied further in [14, 15]. We do not know if this strengthening holds for all groups of cost 1, which class includes, besides all amenable groups, direct products $\mathbb{G} \times \mathbb{Z}$ for any group $\mathbb{G}$, and $SL(d,\mathbb{Z})$ with $d \geq 3$. A related question is whether every amenable transitive graph has an invariant random Hamiltonian path. This is the invariant infinite version of what is known as Lovász' conjecture, namely, that every finite transitive graph has a Hamiltonian path, even though he has not conjectured a positive answer. The best general results seem to be [5] and [28].

1.2 Notation

Graphs. We always consider locally finite and rooted graphs. The root is denoted by o. We denote by e^- and e^+ the endpoints of the (directed) edge e. When a subgraph S is given (maybe implicitly) and it contains exactly one endpoint of e, then we denote that endpoint by e^-. We write $x \sim y$ if x and y are adjacent vertices in G. We will use $\mathrm{dist}_G(x,y)$ for the graph distance between the vertices x and y in the graph G. We denote by $B_G(o,r)$ the ball around o of radius r in G, i.e., the subgraph induced by the vertex set $\{x \in V(G) : \mathrm{dist}_G(o,x) \leq r\}$. For any subset S of the vertices, let $\partial_E S := \{e \in E(G) : e^- \in S, e^+ \notin S\}$ be the *edge boundary* of S, let $\partial_V^{\mathrm{in}} S := \{x \in S : \exists y \sim x, y \notin S\}$ be the *internal vertex boundary* of S, and let $\partial_V^{\text{out}} S := \{x \notin S : \exists y \sim x, y \in S\}$ be the *outer vertex boundary* of S.

Several of our examples will use percolation on $\mathbb{Z}^2$. The subgraph spanned by the box $[-n, n]^2$ will be denoted by Q_n. We will also use the standard dual percolation on the dual lattice $(\mathbb{Z}+\frac{1}{2})^2$.

When we talk about invariant random subgraphs of a Cayley graph Γ of a group $\mathbb{G}$, we will always mean that the measure on subgraphs is invariant under the natural action of $\mathbb{G}$.

Unimodular random graphs. Let $\mathcal{G}_\star$ be the space of isomorphism classes of locally finite labeled rooted graphs, and let $\mathcal{G}_{\star\star}$ be the space of isomorphism classes of locally finite labeled graphs with an ordered pair of distinguished vertices, each equipped with the natural local topology: two (doubly) rooted graphs are "close" if they agree in "large" neighborhoods of the root(s). If (G, o) is a random rooted graph, then denote by μ_G the distribution of it on $\mathcal{G}_\star$, and let $\overline{\mathbb{E}}_G$ be the expectation with respect to μ_G. We omit the index G from this notation if it is clear what the measure is.

Definition 1.1 ([2], Definition 2.1)**.** *We say that a random rooted graph (G, o) is* unimodular *if it obeys the Mass Transport Principle:*

$$\overline{\mathbb{E}}_G \left(\sum_{x \in V(\omega)} f(\omega, o, x) \right) = \overline{\mathbb{E}}_G \left(\sum_{x \in V(\omega)} f(\omega, x, o) \right)$$

for each Borel function $f : \mathcal{G}_{\star\star} \to [0, \infty]$.

There are several other equivalent definitions; see [29, Definition 14.1]. Also, it is an open question if this class is strictly larger than the class of sofic measures: the closure of the set of finite graphs under local weak convergence.

An important class of unimodular graphs consists of Cayley graphs of finitely generated groups and of invariant random subgraphs of a Cayley graph:

Proposition 1.2 ([2], Remark 3.3)**.** *Let Γ be a Cayley graph of a finitely generated group and let o be a vertex of Γ. If G is a random subgraph of Γ that is invariant under the action of the group, then (G, o) is unimodular.*

The class of unimodular probability measures is convex. A unimodular probability measure is called *extremal* if it cannot be written as a convex combination of other unimodular probability measures.

Percolation. For simplicity, we will consider only *bond* percolation processes on unimodular random graphs. For a fixed configuration ω of the random graph G let $\mathbb{P}_p^\omega$ be the probability measure obtained by the Bernoulli(p) bond percolation on ω and let $\mathbb{E}_p^\omega$ be the expectation with respect to $\mathbb{P}_p^\omega$. The percolation *cluster* (i.e., the connected component) of the root o will be $\mathcal{C}_o$.

1.3 Critical probabilities

The long studied critical probabilities $p_c = \sup\{p : \mathbb{P}_p(|\mathcal{C}_o| = \infty) = 0\}$ first defined by Hammersley and $p_T = \sup\{p : \mathbb{E}_p(|\mathcal{C}_o|) < \infty\}$ introduced by Temperley have natural generalizations to extremal unimodular random graphs. Let G be an extremal unimodular random graph. In this case the critical probability $p_c(\omega)$ of a configuration of G is almost surely a constant and the same holds for p_T (see [2], Section 6.). Hence one can define

$$\begin{aligned} p_c &= \sup\left\{p : \mu\left(\mathbb{P}_p^\omega\left(|\mathcal{C}_o| = \infty\right) > 0\right) = 1\right\} \\ &= \inf\left\{p : \mu\left(\mathbb{P}_p^\omega\left(|\mathcal{C}_o| = \infty\right) = 0\right) = 1\right\} \end{aligned}$$

and

$$\begin{aligned} p_T &= \sup\left\{p : \mu\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) < \infty\right) = 1\right\} \\ &= \inf\left\{p : \mu\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) = \infty\right) = 1\right\}. \end{aligned}$$

It may happen that although $\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) < \infty$ for μ-almost every ω, the expectation of these quantities with respect to μ is infinite. This provides a second natural extension of p_T to unimodular random graphs:

$$\begin{aligned} \hat{p}_T &= \sup\left\{p : \overline{\mathbb{E}}\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right)\right) < \infty\right\} \\ &= \inf\left\{p : \overline{\mathbb{E}}\left(\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right)\right) = \infty\right\}. \end{aligned}$$

It follows from the definitions that $p_c \geq p_T \geq \hat{p}_T$. It is known that $p_c = p_T$ in the case of transitive graphs; see [27, 1, 4, 11]. For unimodular random graphs (even with sub-exponential volume growth), the three critical probabilities can differ; we will present such graphs in Examples 2.7 and 2.9.

Duminil-Copin and Tassion [11] introduced the following local quantity for transitive graphs: let G be a rooted graph, S be a finite subgraph containing the root, and define

$$\phi_p(S) := \sum_{e\in\partial_E S} p\,\mathbb{P}_p(o \overset{S}{\longleftrightarrow} e^-)\,,$$

the expected number of open edges on the boundary of S such that there is an open path from o to e^- in S. Then, they defined the critical probability

$$\begin{aligned} \tilde{p}_c &:= \sup\{p : \text{there is a finite } S \text{ containing } o \text{ s.t. } \phi_p(S) < 1\} \\ &= \inf\{p : \phi_p(S) \geq 1 \text{ for all finite } S \text{ containing } o\}. \end{aligned} \tag{1.1}$$

They proved that transitive graphs satisfy $p_c = \tilde{p}_c$.

How to generalize this definition to unimodular random graphs is not a priori clear. The simplest way to define a similar critical probability seems to be a quenched version: find a suitable S_ω for almost every configuration ω. For a subgraph $S \subset \omega$ containing the root, denote by

$$\phi_p^\omega(S) := \sum_{e\in\partial_E S} p\,\mathbb{P}_p^\omega\left(o \overset{\omega,p}{\underset{S}{\longleftrightarrow}} e^-\right) \tag{1.2}$$

the expected number of open edges on the boundary of S in ω such that there is an open path from o to e^- in the percolation on ω with parameter p. Then let

$$\tilde{p}_c := \sup\left\{p : \mu\left(\left\{\omega : \exists S_\omega \text{ finite containing the root s.t. } \phi_p^\omega(S_\omega) < 1\right\}\right) = 1\right\}. \tag{1.3}$$

Remark 1.3. *Suppose p satisfies the following: for almost every ω there is a finite set S_ω with $\phi_p^\omega(S_\omega) < c$. Then unimodularity implies [2, Lemma 2.3.] that for almost every ω and every vertex x there is some finite connected set $S_{\omega,x} \ni x$ such that*

$$\phi_p^{\omega,x}(S_{\omega,x}) := p \sum_{e\in\partial_E S_{\omega,x}} \mathbb{P}\left(x \overset{\omega,p}{\underset{S_{\omega,x}}{\longleftrightarrow}} e^-\right) < c.$$

In the original definition (1.1) of $\tilde{p}_c$, there is no control on what the set S could be, which makes the definition rather ineffective. This becomes particularly problematic in the random graph case (1.3), where a bad neighborhood of o may force S_ω to be huge and hard to find. However, it will follow from our Lemma 2.3 that, for transitive graphs, the existence of an S with $\phi_p(S) < 1$ is equivalent to the existence of a positive integer r with $\phi_p(B(o,r)) < 1$. This provides a second natural extension of the definition of $\tilde{p}_c$ to the random case: we consider the ball of radius r in the random graph ω and we take the expectation of $\phi_p^\omega(B_\omega(o,r))$ with respect to μ. Then the following critical probability is another extension of the definition of $\tilde{p}_c$:

$$\hat{p}_c := \sup\{p : \exists r \text{ such that } \overline{\mathbb{E}}\left(\phi_p^\omega(B_\omega(o,r))\right) < 1\}. \tag{1.4}$$

1.4 Operations preserving unimodularity

Some of our examples arise from Cayley graphs using operations from $\mathcal{G}_\star$ to $\mathcal{G}_\star$. One of this operations is the *edge replacement* defined in [2], Example 9.8: we replace each edge of a unimodular graph G by a finite graph with two distinguished vertices corresponding to the endpoints of the edge, then we find the correct new distribution for the root that makes the measure unimodular. If the finite graphs are random, each must have finite expected vertex size. In this section, we define further operations, called *vertex replacement* and *contraction*, and we prove that if the initial graph is a unimodular labeled graph with appropriate labels, then the resulting graph by such an operation is also unimodular.

Vertex replacement. Let (Γ, o) be a unimodular random labeled graph with distribution μ, where the labels are in the form (G_x, φ_x), where G_x is a finite graph and φ_x is a map from $\{(x,y) \in E(\omega) : y \sim x\}$ to $V(G_x)$. If the labeling satisfies $\overline{\mathbb{E}}_\mu|V(G_o)| < \infty$, then we can define the following rooted random graph $H(\Gamma)$: we choose $(\Gamma, o, \{(G_x,\varphi_x) : x \in V(\Gamma)\})$ with respect to the probability measure μ biased by $|V(G_o)|$, and replace each vertex x of Γ by the graph G_x and each edge e of Γ by the edge $\{\varphi_{e^-}(e), \varphi_{e^+}(e)\}$. Let the root o' of $H(\Gamma)$ be a uniform random vertex of $V(G_o)$. Denote the law of $(H(\Gamma), o')$ by μ'.

We claim that if μ is unimodular with $\overline{\mathbb{E}}_\mu|V(G_o)| < \infty$, then μ' is also unimodular. Let $f(\omega,u,v)$ be a Borel function from $\mathcal{G}_{\star\star}$ to $[0,\infty]$ and let

$$\bar{f}(\bar{\omega},x,y) := \frac{1}{\overline{\mathbb{E}}_\mu|V(G_o)|} \sum_{u\in V(G_x), v\in V(G_y)} f(H(\bar{\omega}),u,v)$$

which is an isomorphism-invariant Borel function on the subspace of $\mathcal{G}_{\star\star}$ that consists of graphs with labels of the above form. We show that μ' obeys the Mass Transport Principle:

$$\begin{aligned}
\int \sum_{v\in V(\omega)} f(\omega,o',v)d\mu'(\omega,o') &= \int \sum_{o'\in V(G_o), v\in V(H(\bar{\omega}))} \frac{1}{|V(G_o)|} f(H(\bar{\omega}),o',v) \frac{|V(G_o)|}{\overline{\mathbb{E}}_\mu|V(G_o)|} d\mu(\bar{\omega},o) \\
&= \int \sum_{x\in V(\bar{\omega})} \sum_{o'\in V(G_o), v\in V(G_x)} \frac{1}{\overline{\mathbb{E}}_\mu|V(G_o)|} f(H(\bar{\omega}),o',v)d\mu(\bar{\omega},o) \\
&= \int \sum_{x\in V(\bar{\omega})} \bar{f}(\bar{\omega},o,x)d\mu(\bar{\omega},o) \\
&= \int \sum_{x\in V(\bar{\omega})} \bar{f}(\bar{\omega},x,o)d\mu(\bar{\omega},o) \\
&= \int \sum_{v\in V(\omega)} f(\omega,v,o')d\mu'(\omega,o').
\end{aligned}$$

Contraction. Let (Γ, o) be a unimodular random edge-labeled graph with distribution μ, where the labels of the edges are 0 or 1. We denote by G the random subgraph of Γ spanned by all the vertices and the edges with label 1. For a vertex x of Γ let $\mathcal{C}_x$ be the connected component of x in G. We define the contracted graph $H(\Gamma)$: in practice, this is what we get by identifying every vertices in the same component of G. More formally, first we choose (Γ, o, G) with respect to the distribution μ biased by $\frac{1}{|\mathcal{C}_o|}$. The vertices of $H(\Gamma)$ are the connected components of G and we join two vertices by an edge iff there is an edge in Γ which connects the two components. Let the root o' of $H(\Gamma)$ be the connected component $\mathcal{C}_o$. Denote the law of $(H(\Gamma), o')$ by μ'.

We claim that if μ is unimodular then μ' is also unimodular. Let $f(\omega, u, v)$ be a Borel function from $\mathcal{G}_{\star\star}$ to $[0, \infty]$ and let

$$\bar{f}(\bar{\omega}, x, y) := \frac{1}{|\mathcal{C}_x||\mathcal{C}_y|} f(H(\bar{\omega}), \mathcal{C}_x, \mathcal{C}_y)$$

which is an isomorphism-invariant Borel function on the subspace of $\mathcal{G}_{\star\star}$ that consists of graphs with edges labeled by 0 or 1, such that the subgraph defined by the edges with label 1 consists of finite components. We show that μ' obeys the Mass Transport Principle:

$$\begin{aligned}
\int \sum_{v \in V(\omega)} f(\omega, o', v) d\mu'(\omega, o') &= \int \sum_{x \in V(\bar{\omega})} \frac{1}{|\mathcal{C}_x|} f(H(\bar{\omega}), \mathcal{C}_o, \mathcal{C}_x) \frac{1}{|\mathcal{C}_o| \overline{\mathbb{E}}_\mu \left(\frac{1}{|\mathcal{C}_o|} \right)} d\mu(\bar{\omega}, o) \\
&= \frac{1}{\overline{\mathbb{E}}_\mu \left(\frac{1}{|\mathcal{C}_o|} \right)} \int \sum_{x \in V(\bar{\omega})} \bar{f}(\bar{\omega}, o, x) d\mu(\bar{\omega}, o) \\
&= \frac{1}{\overline{\mathbb{E}}_\mu \left(\frac{1}{|\mathcal{C}_o|} \right)} \int \sum_{x \in V(\bar{\omega})} \bar{f}(\bar{\omega}, x, o) d\mu(\bar{\omega}, o) \\
&= \int \sum_{v \in V(\omega)} f(\omega, v, o') d\mu'(\omega, o').
\end{aligned}$$

2 Relationship of the critical probabilities

We will start by proving in Theorem 2.1 that all bounded degree unimodular graphs satisfy $p_c = \tilde{p}_c$. This will be useful in many of our later results.

In the transitive case, the quantity $\phi_p(S)$ in the definition of $\tilde{p}_c$ can be used to give a short proof (see [11]) of Menshikov's theorem [27]: if Γ is a transitive graph and $p < p_c(\Gamma)$, then there exist a $\varphi(p)$ such that

$$\mathbb{P}_p \left(o \leftrightarrow B(o, r)^c \right) \leq e^{-\varphi(p) r}. \tag{2.1}$$

If a graph satisfies this exponential decay for each $p < p_c$ and has sub-exponential volume growth, then it is easy to see that $p_T = p_c$. In Lemma 2.3, we give a condition for unimodular random graphs that implies (2.1), and we prove in Corollary 2.5 that this condition implies $p_c = p_T = \hat{p}_T$ if the graph has uniform sub-exponential volume growth. However, in Examples 2.7 and 2.9 we present unimodular random graphs with uniform polynomial volume growth and $p_T < p_c$ and $\hat{p}_T < p_T$, respectively. This shows that Menshikov's theorem is not true in the generality of unimodular graphs.

The results of this section are summarized in the following table:

$\tilde{p}_c = p_c$	bounded degree
$p_c \geq p_T \geq \hat{p}_T$	always
$p_c = \hat{p}_T$	bounded degree uniformly good with sub-exp. growth
$p_c > p_T$	Example 2.7, with polynomial growth
$p_T > \hat{p}_T$	Example 2.9, with polynomial growth
$p_c \leq \hat{p}_c$	bounded degree uniformly good
$p_c > \hat{p}_c$	Example 2.8, not uniformly good
$p_c < \hat{p}_c$	Example 2.10, uniformly good

Table 1: Relationship of the critical probabilities

2.1 Positive results

Our first result is indispensable to the rest of the paper. The first part of the proof depends on new ideas, the second is a slight modification of the proof in [11] for our settings.

Theorem 2.1. *If G is a bounded degree unimodular random rooted graph, then $p_c(G) = \tilde{p}_c(G)$.*

Proof. We prove first that $\tilde{p}_c \leq p_c$. Fixing $p < \tilde{p}_c$, we will show that $p \leq p_c$. We claim that there exists a constant $c = c(p) < 1$ such that we can find for almost every ω a set S_ω that contains the root and satisfies $\phi_p^\omega(S_\omega) \leq c$. Let $p' := \frac{p+\tilde{p}_c}{2} < \tilde{p}_c$. Let S_ω be a finite subset of ω such that $o \in S_\omega$ and $\phi_{p'}^\omega(S_\omega) < 1$. The sets S_ω satisfy

$$\begin{aligned}\phi_p^\omega(S_\omega) = \sum_{e \in \partial_E S_\omega} p\mathbb{P}_p^\omega(o \leftrightarrow e^-) &\leq \frac{p}{p'} \sum_{e \in \partial_E S_\omega} p'\mathbb{P}_{p'}^\omega(o \leftrightarrow e^-) \\ &= \frac{p}{p'}\, \phi_{p'}^\omega(S_\omega) \leq \frac{p}{p'} =: c\,.\end{aligned}$$

Recall the definition of $\phi_p^{\omega,x}(S_{\omega,x})$ from Remark 1.3. Unimodularity implies that almost every ω satisfies the following: for each $x \in \omega$ there is a set $S_{\omega,x}$ containing x such that $\phi_p^{\omega,x}(S_{\omega,x}) \leq c$. Fix an $S_{\omega,x}$ as above for every ω and x.

Fix ω and denote by T^ω the following recursively defined tree: the vertices of the tree are finite sequences of vertices of ω. The root of the tree is (o). If $(x_0, x_1, \ldots, x_k)$ is a vertex of T^ω then its children are the sequences $(x_0, x_1, \ldots, x_k, x_{k+1})$ such that for all $j = 1, \ldots, k+1$, we have $x_j \in \partial_V^{\text{out}} S_{\omega,x_{j-1}}$, and there exist vertices $x'_j \in \partial_V^{\text{in}} S_{\omega,x_{j-1}}$ such that $x'_j \sim x_j$ and there are paths from x_{j-1} to x'_j in $S_{\omega,x_{j-1}}$ which are disjoint from each other and from the edges $\{x'_j, x_j\}$, as $j = 1, \ldots, k+1$. We say that the union of the above paths and edges is a *good path* through $x_0, x_1, \ldots, x_k, x_{k+1}$. Denote by $L_n := \{(x_0, x_1, \ldots, x_n) \in T^\omega\}$ the vertex set of T^ω on the nth level.

Let $T^\omega(p)$ be the random subtree of T^ω defined in a similar way but using the random subset of ω obtained by the Bernoulli(p) percolation instead of ω. It is easy to check that in fact $T^\omega(p) \subseteq T^\omega$. Denote by $L_n(p)$ the set of vertices of $T^\omega(p)$ in the nth level. The event that the cluster of the origin in the p-percolation on ω is infinite coincides with the event that there exists an infinite path in the tree $T^\omega(p)$ (that is, $T^\omega(p)$ survives).

We claim that for almost every ω the expected number of vertices in $L_n(p)$ converges to 0 as $n \to \infty$. More precisely, the expectation of the number of vertices in $L_n(p)$ decreases exponentially in n. In the first two inequalities we use the notation $\square$ for the occurence of events on disjoint edge sets and we apply the BK inequality ([16], Theorem 2.12). We denote the event

$\left\{ x_0 \overset{\omega,p}{\underset{B}{\longleftrightarrow}} x_k \text{ by a good path through } x_0, x_1, \dots, x_k \right\}$ by $\left\{ x_0 \overset{\omega,p}{\underset{B,(x_0,x_1,\dots,x_k)}{\longleftrightarrow}} x_k \right\}$.

$$
\begin{aligned}
\mathbb{E}^\omega\big(|L_n(p)|\big) &= \sum_{(x_0,\dots,x_n)\in L_n} \mathbb{P}^\omega\left(x_0 \overset{\omega,p}{\underset{S,(x_0,\dots,x_n)}{\longleftrightarrow}} x_n \right) \\
&\le \sum_{\substack{(x_0,\dots,x_{n-1})\in L_{n-1} \\ e\in\partial_E S_{\omega,x_{n-1}}}} \mathbb{P}^\omega\left(\left\{ x_0 \overset{\omega,p}{\underset{S,(x_0,\dots,x_{n-1})}{\longleftrightarrow}} x_{n-1} \right\} \square \{e \text{ is open}\} \square \left\{ x_{n-1} \overset{\omega,p}{\underset{S_{\omega,x_{n-1}}}{\longleftrightarrow}} e^- \right\} \right) \\
&\le \sum_{\substack{(x_0,\dots,x_{n-1})\in L_{n-1} \\ e\in\partial_E S_{\omega,x_{n-1}}}} \mathbb{P}^\omega\left(x_0 \overset{\omega,p}{\underset{S,(x_0,\dots,x_{n-1})}{\longleftrightarrow}} x_{n-1} \right) p\, \mathbb{P}^\omega\left(x_{n-1} \overset{\omega,p}{\underset{S_{\omega,x_{n-1}}}{\longleftrightarrow}} e^- \right) \\
&= \sum_{(x_0,\dots,x_{n-1})\in L_{n-1}} \mathbb{P}^\omega\left(x_0 \overset{\omega,p}{\underset{S,(x_0,\dots,x_{n-1})}{\longleftrightarrow}} x_{n-1} \right) \phi_p^{\omega,x_{n-1}}(S_{\omega,x_{n-1}}) \le \mathbb{E}^\omega\left(|L_{n-1}(p)|\right) c\,.
\end{aligned}
$$

It follows by induction that $\mathbb{E}^\omega\big(|L_n(p)|\big) \le c^n$. Therefore,

$$
\mathbb{P}^\omega(|\mathcal{C}_o| = \infty) = \mathbb{P}^\omega(T^\omega(p) \text{ survives}) = \lim_{n\to\infty} \mathbb{P}^\omega(|L_n(p)| \ge 1) \le \lim_{n\to\infty} \mathbb{E}^\omega |L_n(p)| = 0\,,
$$

hence $p \le p_c$.

Next we prove that $\tilde{p}_c \ge p_c$. Let

$$
q(p) := \mu\big(\{\omega : \phi_p^\omega(S) \ge 1 \text{ for all finite } S \subset \omega \text{ that contains } o\}\big),
$$

$$
q_r(p) := \mu\big(\{\omega : \phi_p^\omega(S) \ge 1 \text{ for all finite } S \subset B_\omega(o,r) \text{ that contains } o\}\big).
$$

Note that for any fixed p the sequence $(q_r(p))_{r\ge1}$ converges decreasingly to $q(p)$ as $r \to \infty$, and $q(p) > 0$ for every $p > \tilde{p}_c$ by the definition of $\tilde{p}_c$.

Fix ω and let $H \subseteq \omega$ be any fixed finite subgraph that contains the root. We will use Lemma 1.4. of [11]:

$$
\frac{d}{dp}\mathbb{P}_p^\omega\left(o \overset{\omega,p}{\longleftrightarrow} H^c \right) \ge \left(1 - \mathbb{P}_p^\omega\left(o \overset{H,p}{\longleftrightarrow} H^c \right) \right) \inf_{S: o\in S\subseteq H} \phi_p^H(S) \ge C(p) \inf_{S: o\in S\subseteq H} \phi_p^H(S)\,,
$$

where $C(p) = (1-p)^D \le 1 - \mathbb{P}^\omega\left(o \overset{\omega,p}{\longleftrightarrow} H^c \right)$ for every ω and H, with D being the almost sure bound on the degree of the graph G. The probabilities above depend only on the structure of ω in $K = H \cup \partial_V^{\text{out}} H$, hence we can use the above inequality to estimate the derivative of the probability $\mu\left(o \overset{\omega,p}{\longleftrightarrow} B^\omega(o,r)^c \right)$, as follows. Consider the following sets of finite rooted graphs:

$$
\begin{aligned}
\mathcal{H}_r &:= \{(K,o) : \text{dist}_K(o,x) \le r+1 \text{ and } \deg_K(x) \le D\,, \text{ for all } x \in V(K)\}\,, \\
\mathcal{H}_r(p) &:= \big\{(K,o) \in \mathcal{H}_r : \phi_p^K(S) \ge 1\,, \text{ for all } S \subseteq B_K(o,r)\big\}\,.
\end{aligned}
$$

Note that $\sum_{K\in\mathcal{H}_r(p)} \mu\left(\{\omega : B_\omega(o,r+1) = K\}\right) = q_r(p) \ge q(p)$.

$$
\begin{aligned}
\frac{d}{dp}\mu\left(o \overset{\omega,p}{\longleftrightarrow} B(o,r)^c \right) &= \sum_{(K,o)\in\mathcal{H}_r} \mu\big(B_\omega(o,r+1) = K\big) \frac{d}{dp}\mathbb{P}_p\left(o \overset{K,p}{\longleftrightarrow} B_K(o,r)^c \right) \\
&\ge \sum_{(K,o)\in\mathcal{H}_r(p)} \mu\big(B_\omega(o,r+1) = K\big) C(p) \inf_{S: o\in S\subseteq B_K(o,r)} \phi_p^K(S) \\
&\ge q(p)\, C(p).
\end{aligned}
$$

Integrate the above inequality on the interval $\left[\frac{p+\tilde{p}_c}{2}, p\right]$. Using the monotonicity of $q(p)$ and $C(p)$, we get

$$\mu\left(o \overset{\omega,p}{\longleftrightarrow} B(o,r)^c\right) \geq \frac{p-\tilde{p}_c}{2}\, q\left(\frac{p+\tilde{p}_c}{2}\right) C(p).$$

This gives a positive lower bound that is uniform in r. Thus $\mu\left(o \overset{\omega,p}{\longleftrightarrow} \infty\right) > 0$, and $p \geq p_c$. ■

One advantage of the definition of $\tilde{p}_c$ for transitive graphs is that it enables one to check whether a certain p is under $\tilde{p}_c$ using a finite witness. This characteristic makes the next definition natural.

Definition 2.2. *We say that a bounded degree unimodular random graph G is* uniformly good *if for any $p < p_c$ there exists a positive integer $r(p)$ such that $\mu_G(\{\omega : \exists S_\omega \subseteq B_\omega(o, r(p)), o \in S_\omega \text{ s.t. } \phi_p^\omega(S_\omega) < 1\}) = 1$.*

Uniformly good unimodular graphs satisfy the following exponential decay of $\phi_p(B_\omega(o,r))$ in r, which will imply the coincidence of p_c and $\hat{p}_c$.

Lemma 2.3. *Let G be a bounded degree unimodular random graph. G is uniformly good if and only if for all $p < p_c$ there are constants $c = c(p) < 1$ and $R(p)$ such that if $r \geq R(p)$, then $\phi_p^\omega(B) \leq c^r$ for almost every ω and every finite $B \supseteq B_\omega(o,r)$.*

Proof. If the constants $c(p)$ and $R(p)$ exist, then the sets $S_\omega := B_\omega(o, R(p))$ indicate that G is uniformly good.

To prove the other direction, assume that G is uniformly good, and fix $p < p_c$. We can show as in the proof of Theorem 2.1 that there exists a constant $c_0 < 1$ and a positive integer r_0 such that for almost every ω and every $x \in \omega$ there exists a finite connected set $S_{\omega,x} \subseteq B_\omega(x, r_0)$ containing x which satisfies $\phi_p^{\omega,x}(S_{\omega,x}) \leq c_0$. Fix an ω and the sets $S_{\omega,x}$ as above, a positive integer r and a finite set $B \supseteq B_\omega(o,r)$. We define the trees T^ω and $T^\omega(p)$ as in the proof of Theorem 2.1. On every directed path in T^ω from o to infinity there is a first vertex $(x_0, \ldots, x_k)$ such that $x_k \notin B$. Let π be the set of these vertices, which is a minimal set in T^ω that separates o from infinity, and let $\pi(p) := \pi \cap T^\omega(p)$. A similar argument as in the first part of the proof of Theorem 2.1 shows that

$$\begin{aligned}
\mathbb{E}^\omega\left(|\pi(p)|\right) &= \sum_{(x_0,\ldots,x_k)\in\pi} \mathbb{P}^\omega\left(x_0 \xleftrightarrow[B,(x_0,\ldots,x_k)]{\omega,p} x_k\right) \\
&\leq \sum_{(x_0,\ldots,x_k)\in\pi} \sum_{(x_1',\ldots,x_k')} \prod_{j=1}^{k} \mathbb{P}^\omega\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x_j'\right) p =: F(\pi,p),
\end{aligned}$$

where $(x_1', \ldots, x_k')$ denotes a sequence of vertices in ω such that $x_j' \in S_{\omega,x_{j-1}}$ and $x_j' \sim x_j$ for any $j = 1, \ldots, k$. To estimate $F(\pi,p)$ let $\pi_n := \bigcup_{m\leq n} (\pi \cap L_m) \cup \{v \in L_n : v \text{ has a descendant in } \pi\}$, which is a minimal vertex set that separates the root from infinity. Let $R := \max\{n : L_n \cap \pi \neq \emptyset\} < \infty$, thus $\pi = \pi_R$. Note that each π_n is the disjoint union of $\pi_{n+1} \setminus L_{n+1} \subseteq \pi$ and $\pi_n \setminus \pi_{n+1} \subseteq L_n$. We estimate $F(\pi,p)$ by summing over a larger set: the union of $\pi_R \setminus L_R$ and $\{(x_0, \ldots, x_R) : (x_0, \ldots, x_{R-1}) \in \pi_{R-1} \setminus \pi_R, x_R \in \partial_V^{\text{out}} S_{\omega,x_{R-1}}\} \supseteq \pi_R \cap L_R$. That is, using the bound

$$\sum_{e\in\partial_E S_{\omega,x_{R-1}}} \mathbb{P}^\omega\left(x_{R-1} \xleftrightarrow[S_{\omega,x_{R-1}}]{\omega,p} e^-\right) p = \phi_p^{\omega,x_{R-1}}(S_{\omega,x_{R-1}}) \leq 1$$

for the second term in the following estimation, we have

$$
\begin{aligned}
F(\pi,p) \le & \sum_{(x_0,\dots,x_k)\in\pi_R\setminus L_R} \sum_{(x_1',\dots,x_k')} \prod_{j=1}^{k} \mathbb{P}^\omega\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x_j'\right) p \\
& + \sum_{\substack{(x_0,\dots,x_{R-1})\in\pi_{R-1}\setminus\pi_R \\ (x_1',\dots,x_{R-1}')}} \left(\prod_{j=1}^{R-1} \mathbb{P}^\omega\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x_j'\right) p\right) \sum_{e\in\partial_E S_{\omega,x_{R-1}}} \mathbb{P}^\omega\left(x_{R-1} \xleftrightarrow[S_{\omega,x_{R-1}}]{\omega,p} e^-\right) p \\
\le & \sum_{(x_0,\dots,x_k)\in\pi_{R-1}} \sum_{(x_1',\dots,x_k')} \prod_{j=1}^{k} \mathbb{P}^\omega\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x_j'\right) p = F(\pi_{R-1},p).
\end{aligned}
$$

A similar argument shows that $F(\pi,p) \le F(\pi_n,p)$ for any $n \le R$. If $(x_0,\dots,x_k) \in \pi$, then $\mathrm{dist}_\omega(o,x_k) \ge r$, hence the distance between o and π in T^ω is at least $\frac{r}{r_0}$, thus $\pi_n = L_n$ for any $n \le \frac{r}{r_0}$. If we apply the above argument for $F(\pi_n,p)$ whith $n \le \frac{r}{r_0}$, then the first term disappear, and the inequality $\phi_p^{\omega,x_{n-1}}(S_{\omega,x_{n-1}}) \le c_0$ gives

$$
F(\pi,p) \le F(\pi_{\frac{r}{r_0}},p) \le F(\pi_{\frac{r}{r_0}-1},p)c_0 \le \cdots \le c_0^{\frac{r}{r_0}}.
$$

Denote by $\bar\pi$ the set of the parents of the vertices in π. If for some $e \in \partial_E B$ the event $\left\{o \xleftrightarrow[B]{\omega,p} e^-\right\}$ occurs, then there is some $(x_0,\dots,x_k) \in \bar\pi$ such that there is a good path through $x_0,\dots,x_k$ in the percolation and a disjoint path from x_k to e^- in S_{ω,x_k}. For any fixed $(x_0,\dots,x_k)$ the number of edges in $\partial_E B \cap (E(S_{\omega,x_k}) \cup \partial_E S_{\omega,x_k})$ is bounded above by $|E(S_{\omega,x_k}) \cup \partial_E S_{\omega,x_k}| \le D^{r_0+1}$ where D is the almost sure bound on the degree of the graph G. We have

$$
\begin{aligned}
\phi_p^\omega(B) &= p \sum_{e\in\partial_E B} \mathbb{P}^\omega(o \xleftrightarrow[B]{\omega,p} e^-) \\
&\le \sum_{e\in\partial_E B} \sum_{\substack{(x_0,\dots,x_k)\in\bar\pi \\ (x_0',\dots,x_k')}} \mathbb{P}_p^\omega\left(\left\{x_0 \xleftrightarrow[B,(x_0,\dots,x_k)]{\omega,p} x_k\right\} \square \{x_k \xleftrightarrow[S_{\omega,x_k}]{\omega,p} e^-\}\right) \\
&\le \sum_{\substack{(x_0,\dots,x_k)\in\bar\pi \\ (x_0',\dots,x_k')}} \left(\prod_{j=1}^{k} \mathbb{P}^\omega\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x_j'\right) p\right) \sum_{e\in\partial_E B\cap\left(E(S_{\omega,x_k})\cup\partial_E S_{\omega,x_k}\right)} \mathbb{P}_p^\omega(x_k \xleftrightarrow[S_{\omega,x_k}]{\omega,p} e^-) \\
&\le \sum_{\substack{(x_0,\dots,x_k)\in\bar\pi \\ (x_0',\dots,x_k')}} \left(\prod_{j=1}^{k} \mathbb{P}^\omega\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x_j'\right) p\right) D^{r_0+1} = F(\bar\pi,p)D^{r_0+1}
\end{aligned}
$$

To estimate this inequality, note that

$$
\begin{aligned}
F(\pi,p) &= \sum_{\substack{(x_0,\dots,x_k)\in\bar\pi \\ (x_0',\dots,x_k')}} \left(\prod_{j=1}^{k} \mathbb{P}^\omega\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x_j'\right) p\right) \sum_{\substack{x_{k+1}:(x_0,\dots,x_{k+1})\in\pi \\ x_{k+1}'\in S_{\omega,x_k}, x_{k+1}'\sim x_{k+1}}} \mathbb{P}^\omega\left(x_k \xleftrightarrow[S_{\omega,x_k}]{\omega,p} x_{k+1}'\right) p \\
&\ge \sum_{\substack{(x_0,\dots,x_k)\in\bar\pi \\ (x_0',\dots,x_k')}} \left(\prod_{j=1}^{k} \mathbb{P}^\omega\left(x_{j-1} \xleftrightarrow[S_{\omega,x_{j-1}}]{\omega,p} x_j'\right) p\right) p^{r_0+1} = F(\bar\pi,p)p^{r_0+1}
\end{aligned}
$$

by the assumption that the graph is uniformly good. We have

$$\phi_p^\omega(B) \leq \frac{D^{r_0+1}}{p^{r_0+1}} F(\pi, p) \leq \frac{D^{r_0+1}}{p^{r_0+1}} c_0^{\frac{r}{r_0}} \leq c^r,$$

with some $c < 1$ if r is large enough. ■

Corollary 2.4. *If G is a uniformly good unimodular graph, then $p_c \leq \hat{p}_c$.*

Proof. Let $p < p_c$, and let c and $R(p)$ be as in Lemma 2.3. We have $\overline{\mathbb{E}}\left(\phi_p^\omega\left(B_\omega(o, R(p))\right)\right) \leq c^R < 1$, thus $p \leq \hat{p}_c$. ■

We will see in Remark 2.8 that, without the assumption of uniform goodness, the inequality $p_c \leq \hat{p}_c$ does not necessarily hold. Also, we will show in Example 2.10 that there are uniformly good graphs with $p_c < \hat{p}_c$.

Corollary 2.5. *If G is a uniformly good unimodular graph with uniform sub-exponential volume growth (i.e., for any $c < 1$ and $\varepsilon > 0$ there is an R such that $\mu\left(\omega : |B_\omega(o, r)|c^r < \varepsilon\right) = 1$ for any $r > R$), then $p_c = p_T = \hat{p}_T$.*

Proof. Let $p < p_c = \tilde{p}_c$ and let c and $R(p)$ be as in Lemma 2.3. Denote by D the maximum degree of G. Let $R > R(p)$ such that $\mu\left(\{\omega : |B_\omega(o, r)|c^{r/2} < 1\}\right) = 1$ for any $r > R$ and let ω satisfy this event for all $r > R$ simultaneously. Then we have

$$\begin{aligned}
\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right) &= \sum_{n=1}^{\infty} \mathbb{P}_p^\omega\left(|\mathcal{C}_o| \geq n\right) = \sum_{r=1}^{\infty} \sum_{n=|B_\omega(o,r)|+1}^{|B_\omega(o,r+1)|} \mathbb{P}_p^\omega\left(|\mathcal{C}_o| \geq n\right) \\
&\leq \sum_{r=1}^{\infty} \sum_{n=|B_\omega(o,r)|+1}^{|B_\omega(o,r+1)|} \mathbb{P}_p^\omega\left(o \overset{p,\omega}{\longleftrightarrow} B_\omega(o, r)^c\right) \\
&\leq \sum_{r=1}^{\infty} |B_\omega(o, r+1)| \min\{\phi_p^\omega\left(B_\omega(o, r)\right), 1\} \\
&\leq \sum_{r=2}^{R+1} |B_\omega(o, r)| + \sum_{r=R+1}^{\infty} |B_\omega(o, r+1)| c^r \\
&\leq \sum_{r=2}^{R+1} D^r + \sum_{r=R+1}^{\infty} c^{r/2} < \infty
\end{aligned}$$

This gives a uniform upper bound on $\mathbb{E}_p^\omega\left(|\mathcal{C}_o|\right)$ thus $\overline{\mathbb{E}}_p\left(|\mathcal{C}_o|\right) < \infty$. It follows that $p \leq \hat{p}_T$, hence $\hat{p}_T \geq p_c$. The other direction follows from the definition of $\hat{p}_T$. ■

Subexponential volume growth will also appear in Example 3.6 and Corollary 4.1.

2.2 Counterexamples

We show in Examples 2.7 and 2.9 that there are unimodular random graphs of uniform subexponential (in fact, quadratic) volume growth, but $p_T < p_c$ and $\hat{p}_T < p_T$. Both constructions will use Bernoulli percolation on $\mathbb{Z}^2$ as an ingredient; moreover, although we define the graph in the second example as a vertex replacement of $\mathbb{Z}^2$, it could be defined even as an invariant random subgraph of $\mathbb{Z}^2$. We further give examples of graphs with $\hat{p}_c < p_c$ and $\hat{p}_c > p_c$; see Examples 2.8 and 2.10, respectively. First we need a lemma that will be useful in our examples.

Lemma 2.6. *For any $\varepsilon > 0$ there is a probability $p_1 < 1$ such that for n large enough, the vertices $(0,-n)$, $(0,n)$, $(-n,0)$, $(n,0)$ are in the same cluster in Bernoulli(p_1) percolation on Q_n with probability at least $1-\varepsilon$.*

Proof. The occurrence of the events in the following two claims implies the occurrence of the event in the statement of the lemma, hence we will be done by a union bound.

Claim 1: For any $p > 1/2$ and $n > n_0(p,\varepsilon)$ large enough, in Bernoulli(p) percolation on Q_n, with probability at least $1-\varepsilon/2$, there is a *giant cluster* with the following properties: it joins all the sides of Q_n, while every other cluster in Q_n has diameter at most $n/5$. This was proved in [3, Proposition 2.1].

Claim 2: There exists $p_1 < 1$ such that for all n and all $p > p_1$,

$$\mathbb{P}_p\left(\mathrm{diam}(\mathcal{C}_{(0,n)}) \geq n\right) \geq 1-\varepsilon/8\,.$$

Similarly for $(0,-n)$, $(-n,0)$, and $(n,0)$, instead of $(0,n)$.

Proof of Claim 2: If there is no open path in the dual percolation joining a dual vertex in $[-n+\frac{1}{2},-\frac{1}{2}]\times\{n+\frac{1}{2}\}$ to a dual vertex in $[\frac{1}{2},n-\frac{1}{2}]\times\{n+\frac{1}{2}\}$, then there is a primal open path from $(0,n)$ to $(\{-n\}\times[0,n])\cup([-n,n]\times\{0\})\cup(\{n\}\times[0,n])$, and hence $\mathrm{diam}(\mathcal{C}_{(0,n)})\geq n$.

On the other hand, for any pair of dual vertices, $x\in[-n+\frac{1}{2},-\frac{1}{2}]\times\{n+\frac{1}{2}\}$ and $y\in[\frac{1}{2},n-\frac{1}{2}]\times\{n+\frac{1}{2}\}$, we have

$$\mathbb{P}_p\left(x \overset{Q_n}{\longleftrightarrow} y \text{ by a dual-open path of length } k\right) \leq (3(1-p))^k\,.$$

Moreover, if the distance of x and y is larger than k, then this probability is of course 0, hence for each k there are at most k^2 relevant pairs (x,y). Therefore, for every n,

$$\mathbb{P}_p\left(\mathrm{diam}(\mathcal{C}_{(0,n)}) < n\right) \leq \sum_{k=1}^{\infty} k^2(3(1-p))^k =: f(p) < \infty\,,$$

where $f(p)$ converges to 0 as $p\to 1$. ∎

Example 2.7. *There is a unimodular graph with uniform polynomial volume growth and $p_T < p_c$. In particular, the exponential decay of two-point connection probabilities fails for $p\in(p_T,p_c)$ on this graph.*

Proof. Let $\mathbb{T}$ be the 3-regular infinite rooted tree with a distinguished end ξ and a *Busemann function* (see [30]) $\mathfrak{h}:\mathbb{T}\to\mathbb{Z}$ that gives the levels w.r.t. to ξ. More precisely, to define $\mathfrak{h}$, fix a root $o\in\mathbb{T}$. For any vertex x, let (ξ,x) be the unique infinite simple path from x which is in the equivalence class ξ. Denote by $o\wedge x$ the unique vertex in $\mathbb{T}$ such that $(\xi,x\wedge o)=(\xi,x)\cap(\xi,o)$. Finally, let $\mathfrak{h}(x):=\mathrm{dist}(o,x\wedge o)-\mathrm{dist}(x,x\wedge o)$.

Let $\Lambda\subset\mathbb{T}$ be the subgraph spanned by the vertices x with $\mathfrak{h}(x)\geq 0$. This tree Λ is called the *canopy tree.* Denote by $L(n):=\{x\in V(\mathbb{T}):\mathfrak{h}(x)=n\}$ the n^{th} vertex level and by $L_E(n):=\{e\in E(\mathbb{T}): e^-\in L(n), e^+\in L(n+1)\}$ the n^{th} edge level of $\mathbb{T}$, or, for $n\geq 0$, of Λ. If we choose the root o of Λ such that $\mathbb{P}(o\in L(n))=2^{-n-1}$, we get a unimodular random graph.

We define the graph G as an edge replacement (see [2], Example 9.8) of the canopy tree: each $e\in L_E(n)$ is replaced by $(Q_{2^n}(e),(0,-2^n),(0,2^n))$, where $Q_{2^n}(e)$ is isomorphic to Q_{2^n}. It is clear that the volume of $B_G(o,r)$, for any root o and radius r, is at most Cr^2, for some $C<\infty$. We will now show that $p_T(G) < p_c(G) = 1$.

Consider Bernoulli(p) percolation ω on G and, as a deterministic function of it, define the following percolation λ on Λ: an edge $e \in L_E(n)$ is open in λ if and only if the vertices $(0,-n)$ and $(0,n) \in Q_n(e)$ are connected by an open path in ω. Clearly, there exists an infinite cluster in ω if and only if there is an infinite cluster in λ. The law of λ is stochastically dominated by a Bernoulli($1-(1-p)^3$) percolation on Λ, because if $e \in L_E(n)$ is open, then at least one of the edges in $Q_n(e)$ adjacent to $(0,n)$ is open. The tree Λ has one end, hence, for any $p < 1$,

$$\mathbb{P}_p^G(\exists \text{ an infinite cluster}) \leq \mathbb{P}^{\Lambda}_{1-(1-p)^3}(\exists \text{ an infinite cluster}) = 0\,.$$

That is, $p_c(G) = 1$.

An easy first moment computation (that we omit) shows that $p_T(\Lambda) = 1/\sqrt{2}$. Now let $0 < \varepsilon < 1 - 1/\sqrt{2}$. It follows from Lemma 2.6 that there exists $p_1 < 1$ and some large N such that $\mathbb{P}_{p_1}(e \in \lambda) \geq 1 - \varepsilon$ for all $e \in L_E(n)$ with $n \geq N$. Thus, for $o \in L(N)$, the cluster $\mathcal{C}_o$ in λ, restricted to the levels $n \geq N$, stochastically dominates Bernoulli($1-\varepsilon$) percolation on Λ. The latter has infinite expected size, hence the expected size of the cluster in ω of $(0,-N) \in Q_N(e)$ for $e \in L_E(N)$ is also infinite. That is, $p_T(G) \leq p_1 < 1$. ∎

Example 2.8. *The canopy tree Λ defined in the previous example satisfies $\hat{p}_c = \frac{1}{\sqrt{2}}$, thus this is an example of a not uniformly good unimodular graph with $p_c > \hat{p}_c$.*

Proof. It is easy to check that $\overline{\mathbb{E}}(\phi_p(B(o,r)))$ equals $2p(\sqrt{2}p)^r$ if r is even, and equals $3(\sqrt{2}p)^{r+1}/2$ if r is odd. Thus it converges to 0 for $p < 1/\sqrt{2}$, while remains above 1 for $p < 1/\sqrt{2}$, which implies the claim. ∎

Example 2.9. *There is a unimodular graph with polynomial volume growth and $\hat{p}_T < p_T$.*

Proof. Let X be a positive integer valued random variable such that $\mathbb{P}(X = k) = ck^{-5/2}$ for all $k \geq 1$. Then $\mathbb{E}X < \infty$ and $\mathbb{E}(X^2) = \infty$. We define the graph G as a vertex replacement (see Subsection 1.4) of $\mathbb{Z}^2$ with respect to the following labels as follow. Let $\{X_n, X'_n : n \in \mathbb{Z}\}$ be iid copies of X, and for each vertex $(m,n) \in \mathbb{Z}^2$, let $G_{(m,n)}$ be isomorphic to the subgraph of $\mathbb{Z}^2$ spanned by the vertices in $[0, 2X_m] \times [0, 2X'_n]$, and for the edges going from (m,n) to North, East, South, and West, let the image of $\varphi_{(m,n)}$ be the corresponding midpoint of the box $G_{(m,n)}$. We can also think of the resulting graph as an invariant random subgraph of $\mathbb{Z}^2$.

Denote by Y and Y' half the length of the sides of the box of o in G, i.e., the law of X_0 and X'_0 biased by $X_0X'_0$. Then

$$\mathbb{P}(Y = k, Y' = l) = \frac{kl}{(\mathbb{E}X)^2}\mathbb{P}(X = k, X' = l),$$

hence Y and Y' are independent with distribution $\mathbb{P}(Y = k) = \frac{ck^{-3/2}}{\mathbb{E}X}$.

First we show that $\hat{p}_T = \frac{1}{2}$. G is a subgraph of $\mathbb{Z}^2$, hence $\hat{p}_T(G) \geq \frac{1}{2}$. Fix $p > \frac{1}{2}$ and let $\varepsilon > 0$. Denote by $M(Q_n)$ the largest cluster in percolation with parameter p in the box Q_n, and let

$$\mathcal{A}(Q_n) := \big\{|M(Q_n)| \geq (1-\varepsilon)\theta(p)|Q_n|, \quad \operatorname{diam}(C) < \nu \log n \;\forall \text{ open cluster } C \neq M(Q_n)\big\},$$

where $\theta(p) = \mathbb{P}_p(|\mathcal{C}_o(\mathbb{Z}^2)| = \infty)$, and ν is chosen as follows: by [16, Theorem 7.61], there is an $N = N(p)$ and $\nu = \nu(p)$ such that, for any $n \geq N$,

$$\mathbb{P}_p(\mathcal{A}(Q_n)) > 1 - \varepsilon\,.$$

Let $Z := \min\{Y, Y'\}$, and consider the event $\mathcal{D}(G_{0,0}) := \{\operatorname{dist}(o, \partial_V^{\mathrm{in}} G_{0,0}) \geq \nu \log Z\}$. If Z is large enough, then $\mathbb{P}(\mathcal{D}(G_{0,0}) \mid Z) \geq 1 - \varepsilon$, since o is uniform in $G_{0,0}$. Assuming that $\mathcal{D}(G_{0,0})$ occurs, choose a box $Q_Z \subseteq G_{0,0}$ that contains o such that $\operatorname{dist}(o, \partial_V^{\mathrm{in}} Q_Z) \geq \nu \log Z$. Consider percolation on $\mathbb{Z}^2 \supset Q_Z$. If o is in the unique infinite cluster of this percolation on $\mathbb{Z}^2$, then the diameter of $\mathcal{C}_o(Q_Z)$ is at least $\nu \log Z$, hence

$$\mathbb{P}_p\Big(o \in M(Q_Z), \mathcal{A}(Q_Z) \,\Big|\, Z = n, \mathcal{D}(G_{0,0})\Big) > \theta(p) - \varepsilon$$

for n large enough. It follows that there is an N' such that

$$\begin{aligned}\overline{\mathbb{E}}_p\left(|\mathcal{C}_o|\right) &\geq \sum_{n=N'}^{\infty} \mathbb{P}_p\Big(o \in M(Q_Z), \mathcal{A}(Q_Z), \mathcal{D}(G_{0,0}) \Big| Z = n\Big)\, \mathbb{P}(Z = n)\,(1-\varepsilon)\theta(p) n^2 \\ &\geq \sum_{n=N'}^{\infty} (\theta(p) - \varepsilon)(1-\varepsilon)\, \mathbb{P}(Z = n)\,(1-\varepsilon)\theta(p) n^2 = \infty\,,\end{aligned}$$

as desired.

To show that $p_T > \frac{1}{2}$ let e be an edge in $\mathbb{Z}^2$, and let G_{e^-} and G_{e^+} be the subgraphs of G that correspond to the endpoints of the edge. Let $x := \varphi_{e^-}(e)$ and $y := \varphi_{e^+}(e)$, i.e. let $\{x, y\}$ be the edge in G that joins G_{e^-} and G_{e^+}. If there is an open path in $G(p)$ through the edge $\{x, y\}$, that joins two vertices in $G_{e^-} \setminus \{x\}$ and in $G_{e^+} \setminus \{y\}$, then the event $J(\{x, y\}) := \{\exists e' \in E(G_{e^-}) : e' \sim x, e' \text{ open}\} \cap \{\exists e' \in E(G_{e^+}) : e' \sim y, e' \text{ open}\} \cap \{\{x, y\} \text{ open}\}$ occurs. For a fixed configuration of G the events $J(\{\varphi_{e^-}(e), \varphi_{e^+}(e)\})$ are independent for different edges, and $\mathbb{P}_p(J(\{\varphi_{e^-}(e), \varphi_{e^+}(e)\})) = p(1-(1-p)^3)^2$. This probability is strictly increasing in p and there is a $p_0 > \frac{1}{2}$ such that $p(1-(1-p)^3)^2 > \frac{1}{2}$ iff $p > p_0$. We consider a random subset $H = H(G(p)) \subseteq E(\mathbb{Z}^2)$ obtained from the percolation $G(p)$: let $e \in H$ if and only if the event $J(\{\varphi_{e^-}(e), \varphi_{e^+}(e)\})$ occurs in $G(p)$. The law of H is the same as the law of Bernoulli($p(1-(1-p)^3)^2$) bond percolation. We want to estimate the expected size of $\mathcal{C}_o(G)$ conditioned on the size of $G_{0,0}$. If $\mathcal{C}_o(G)$ intersects a box G_v, then the connected component of o in H contains v. Therefore

$$\begin{aligned}\overline{\mathbb{E}}_p\Big(|\mathcal{C}_o| \,\Big|\, Y, Y'\Big) &\leq \overline{\mathbb{E}}_p\left(\sum_{v \in \mathbb{Z}^2 : v \in \mathcal{C}_o(H)} |G_v| \,\middle|\, Y, Y'\right) \\ &\leq \mathbb{E}(|\mathcal{C}_o(H)|) \max\left\{Y^2, (Y')^2, (\mathbb{E}X)^2\right\},\end{aligned}$$

which is finite if $p < p_0$. It follows that for almost every configuration of (G, o) the expected size $\mathbb{E}_p^G(\mathcal{C}_o)$ is finite if $p < p_0$, hence $p_T \geq p_0$. ∎

Example 2.10. *There is a quasi-transitive graph with $\hat{p}_c > p_c$.*

Proof. Let $H_{k,l}$ be the following finite directed multigraph: the vertex set is $\{x_0, x_1, \ldots, x_k\}$, and we have l loops at x_0, then one edge from x_0 to each x_j, $j = 1, \ldots, k$, and one from each x_j back to x_0. Let $T_{k,l}$ be the directed cover of $H_{k,l}$ based at x_0. Consider two copies of $T_{k,l}$ and connect the roots of them by an edge to get the infinite quasi-transitive graph $G_{k,l}$, which has vertices of degree 2 and $k + l + 1$. One can easily compute that to get a unimodular random graph one has to choose the root according to $\mu(\deg o = 2) = 1 - \mu(\deg o = k + l + 1) = \frac{k}{k+2}$. Hence $\overline{\mathbb{E}}(\deg o) = \frac{4k+2l+2}{k+2}$. The equality $\overline{\mathbb{E}}\left(\phi_p^\omega(B_\omega(o,0))\right) = p\,\overline{\mathbb{E}}(\deg o)$ implies that $\hat{p}_c \geq \left(\overline{\mathbb{E}}(\deg o)\right)^{-1} = \frac{k+2}{4k+2l+2}$. On the other hand, the critical probability of a directed cover of a finite graph is $p_c(T_{k,l}) = (\operatorname{br}(T_{k,l}))^{-1} = (\operatorname{growth}(T_{k,l}))^{-1} = (\lambda_*(H_{k,l}))^{-1}$, where $\lambda_*(H)$ is the largest positive eigenvalue of the directed adjacency matrix of $H_{k,l}$; see [25], Section 3.3 and [21]. One can thus compute that $p_c(G_{k,l}) = p_c(T_{k,l}) = \frac{2}{l+\sqrt{l^2+4k}}$. If we set, e.g., $k = 3, l = 5$, then we have $p_c(G_{3,5}) = \frac{2}{5+\sqrt{37}} < \frac{5}{24} = \left(\overline{\mathbb{E}}_{G_{3,5}}(\deg o)\right)^{-1} \le \hat{p}_c(G_{3,5})$. ■

3 Locality of the critical probability

In this section we examine the question of Schramm's locality conjecture: does $p_c(G_n)$ converge to $p_c(G)$ if $G_n \to G$ in the local weak sense? The original question in [8] was phrased for sequences of transitive graphs that converge to a transitive graph in the local sense and satisfy $\sup p_c(G_n) < 1$. First we provide some simple examples of unimodular graphs where the conjecture holds. In Example 3.1, we note that if G_n and G are infinite clusters of an independent percolation with appropriate parameters, then the convergence holds. In Example 3.2, we discuss unimodular Galton-Watson trees, and give sufficient and necessary conditions on the offspring distribution to satisfy locality of p_c. Then we investigate the inequality $\liminf p_c(G_n) \ge p_c(G)$, which is known for transitive graphs; see [11] for a simple proof. In Proposition 3.3 we show by a similar argument that the critical probability $\hat{p}_c$ satisfies this inequality for unimodular random graphs. We show in Proposition 3.4 that under certain restrictions on the graphs G and G_n the convergence $\lim p_c(G_n) = p_c(G)$ is true for unimodular random graphs. Examples 3.5 and 3.6 provide graph sequences with $\lim p_c(G_n) < p_c(G)$. These indicate that unimodular graphs do not satisfy Schramm's conjecture in general and show that both of the conditions in Proposition 3.4 are necessary. We show in Example 3.7 a sequence with $p_c(G) < \lim p_c(G_n) < 1$. In this example G and each G_n satisfy the conditions of Corollaries 2.4 and 2.5, thus $p_c = p_T = \hat{p}_T$ and also $\hat{p}_c(G) < \lim \hat{p}_c(G_n) < 1$. This shows that none of the generalisations of the critical probabilities satisfies the extension of Schramm's conjecture for unimodular graphs in general.

3.1 Basic examples

We present now two natural classes of unimodular random graphs that satisfy Schramm's conjecture. The first example is very easy; the proof is left as an exercise.

Example 3.1. *Let Γ be a transitive unimodular graph and let $p_n \to p \in (p_c(\Gamma), 1]$. Let G_n be the connected component of the root in the Bernoulli(p_n) percolation on Γ conditioned to be infinite. Then $p_c(G_n) \to p_c(G) < 1$.*

Our second class of examples, unimodular Galton–Watson trees, is less trivial. Let X be a non-negative integer valued random variable, the *offspring distribution* of the tree, and let $UGW(X)$ be the unimodular Galton–Watson tree measure on rooted trees: the probability that the root o has k children is

$$\mathbb{P}^{UGW(X)}(\deg o = k) = \frac{\mathbb{P}(X = k-1)}{k\,\mathbb{E}(\frac{1}{X+1})} \tag{3.1}$$

for $k \ge 1$, while the number of children of each descendant is according to X, independently of the other vertices. This measure is unimodular (see [2], Example 1.1), and if $\mathbb{E}X > 1$, then $\mathbb{P}(|UGW(X)| = \infty) > 0$, thus we can consider the measure $UGW_\infty(X)$ which is $UGW(X)$ conditioned on the event $\{|UGW(X)| = \infty\}$. The measure UGW_∞ is also unimodular, being an ergodic component of a unimodular measure.

Example 3.2. *Let $UGW_\infty(X)$ be the unimodular Galton–Watson tree with offspring distribution X, conditioned to be infinite. If X_n and X are non-negative integer valued random variables s.t. the X_n satisfy $\mathbb{E}X_n > 1$, while X satisfies $\mathbb{E}X > 1$ or $\mathbb{P}(X = 1) = 1$, then*

(1) *$UGW_\infty(X_n) \to UGW_\infty(X)$ in the local weak sense iff $X_n \to X$ in distribution;*

(2) *$p_c(UGW_\infty(X_n)) \to p_c(UGW_\infty(X))$ iff $\mathbb{E}X_n \to \mathbb{E}X$.*

Before the proof, note that this example shows that p_c is a continuous function of $UGW_\infty(X)$ when the trees have a uniform bound on their degrees (by the Dominated Convergence Theorem), but not necessary otherwise: if $X_n \to X$ in distribution, with $\mathbb{E}X_n > 1$ and $\mathbb{E}X > 1$, but $\mathbb{E}X_n \nrightarrow \mathbb{E}X$, then the critical probabilities $p_c(UGW_\infty(X_n))$ do not converge to $p_c(UGW_\infty(X))$. Nevertheless, Fatou's lemma implies that the inequality $\limsup p_c(UGW_\infty(X_n)) \le p_c(UGW_\infty(X))$ does hold without further assumptions. That is, if the trees do not satisfy the locality of p_c, then they also fail to satisfy the lower semicontinuity discussed in the next subsection, proved to hold in many cases, including transitive graphs. This suggests that a uniform bound on the degrees is a natural condition when we investigate the locality of p_c for unimodular graphs.

Proof. The critical probability $p_c(UGW_\infty(X))$ equals $\frac{1}{\mathbb{E}X}$ (see [25], Proposition 5.9), therefore $p_c(UGW_\infty(X_n)) \to p_c(UGW_\infty(X))$ iff $\mathbb{E}X_n \to \mathbb{E}X$. This shows part (2).

For part (1), for any nonnegative integer random variable X, let $p_k(X) := \mathbb{P}(X = k)$, let $f_X(t) := \sum_{k=0}^\infty p_k(X)t^k$ be the probability generating function of X, and let $q = q(X) := \mathbb{P}(|GW(X)| < \infty)$, which is the smallest non-negative number that satisfies $f_X(q) = q$.

Assume that $X_n \to X$ in distribution, first with $\mathbb{E}(X) > 1$. From $X_n \to X$ it follows easily that $UGW(X_n) \to UGW(X)$, while, from the uniform convergence of the convex functions f_{X_n} to the strictly convex function f_X on $[0, 1]$, we also get $q_n = q(X_n) \to q(X) < 1$. Thus $UGW_\infty(X_n) \to UGW_\infty(X)$.

Now assume that $\mathbb{P}(X = 1) = 1$ and $\mathbb{P}(X_n = 1) \to 1$ with $\mathbb{E}(X_n) > 1$. Using Bayes' rule and (3.1),

$$\begin{aligned}
\mathbb{P}^{UGW_\infty(X_n)}(\deg o = 2) &= \frac{\mathbb{P}^{UGW(X_n)}\left(|UGW(X_n)| = \infty \,|\deg o = 2\right)\mathbb{P}^{UGW(X_n)}(\deg o = 2)}{\mathbb{P}\left(|UGW(X_n)| = \infty\right)} \\
&= \frac{1 - q_n^2}{\mathbb{P}(|UGW(X_n)| = \infty)} \, \frac{\mathbb{P}(X_n = 1)}{2\,\mathbb{E}(\frac{1}{X_n+1})} \\
&= \frac{(1 - q_n^2)\mathbb{P}(X_n = 1)}{2\sum_{j=1}^\infty \mathbb{P}(X_n = j - 1)(1 - q_n^j)/j}\,. \qquad (3.2)
\end{aligned}$$

We claim that $\mathbb{P}^{UGW_\infty(X_n)}(\deg o = 2) \to 1$. If q_n converges to some $q_\infty < 1$, then plugging $\mathbb{P}(X_n = 1) \to 1$ into (3.2) yields the claim immediately. If $q_n \to 1$, then, simplifying the numerator and the denominator of (3.2) by $1 - q_n$, it becomes

$$\frac{(1 + q_n)\,\mathbb{P}(X_n = 1)}{2\sum_{j=1}^\infty \mathbb{P}(X_n = j - 1)(1 + q_n + \cdots + q_n^{j-1})/j} \ge \frac{(1 + q_n)\mathbb{P}(X_n = 1)}{2} \to 1\,. \qquad (3.3)$$

Finally, if q_n does not converge, we can still apply one of these two arguments to any convergent subsequence, and obtain the claim. Therefore, in the local weak limit, the root has degree 2 almost surely. By unimodularity, this limit must be $\mathbb{Z}$. This is also $UGW_\infty(X)$, thus we have $UGW_\infty(X_n) \to UGW_\infty(X)$.

For the other direction of part (1), suppose that there are X_n and X such that $UGW_\infty(X_n) \to UGW_\infty(X)$, but $X_n \nrightarrow X$. The set $\{X_n\}$ of probability distributions must be tight: otherwise, a uniform random neighbour of o in $UGW_\infty(X_n)$, whose offspring distribution stochastically dominates X because of the conditioning on $\{|UGW(X_n)| = \infty\}$, would have arbitrarily large degrees with a uniform positive probability, and thus $UGW_\infty(X_n)$ could not converge to the locally finite graph $UGW_\infty(X)$. It follows from this tightness that there is a subsequence $\{X_{k(n)}\}$ that converges in distribution to a random variable $Y \neq X$.

First we show that $\mathbb{E}Y \geq 1$. Suppose $\mathbb{E}Y < 1$, then $\lim q_n = q(Y) = 1$, hence

$$\mathbb{P}^{UGW_\infty(X_n)}(\deg o = k) = \frac{\mathbb{P}(X_n = k-1)(1 + \cdots + q_n^{k-1})}{k\sum_{j=1}^{\infty}\mathbb{P}(X_n = j-1)(1 + \cdots + q_n^{j-1})/j} \to \mathbb{P}(Y = k-1).$$

It follows, that the expected degree of the root in the limit graph is $\mathbb{E}Y + 1 < 2$. The local weak limit of the graphs $UGW_\infty(X_n)$ is almost surely infinite, hence the expected degree of the root is at least 2 (see [2], Theorem 6.1), a contradiction.

If we have $\mathbb{P}(Y = 1) = 1$, then the first direction of part (1) implies that $UGW_\infty(X_{k(n)}) \to UGW_\infty(Y) = \mathbb{Z}$. But we also have $UGW_\infty(X_{k(n)}) \to UGW_\infty(X)$, and it is obvious that $UGW_\infty(X) = \mathbb{Z}$ implies that $\mathbb{P}(X = 1) = 1$. That is, X_n would in fact converge in distribution to X, a contradiction.

If $\mathbb{E}Y = 1$, but $\mathbb{P}(Y = 1) \neq 1$, then the generating function $f_Y(t)$ is strictly convex, hence $q(X_{k(n)}) \to q(Y) = 1$. A computation similar to (3.2) and (3.3) gives that the degree distribution of o in $UGW_\infty(X_{k(n)})$ converges to that of $Y+1$. This must be the degree distribution of o in the local limit $UGW_\infty(X)$. Since $\mathbb{P}(Y+1=2) \neq 1$, we must be in the case $\mathbb{E}X > 1$. However, then we would have $p_c(UGW_\infty(X)) = 1/\mathbb{E}X < 1$, while $\mathbb{E}(\deg o) = \mathbb{E}(Y+1) = 2$ implies that $UGW_\infty(X)$ is a tree with at most two ends (see [2], Theorem 6.2) hence $p_c = 1$, again a contradiction.

The final case is that $\mathbb{E}Y > 1$, for which we can again use the first direction of part (1), saying that $UGW_\infty(X_{k(n)}) \to UGW_\infty(Y)$. If we prove that the distribution of $UGW_\infty(X)$ determines X, then we must have $X = Y$, and we are done, as before.

This invertibility follows from the construction in [25], Theorem 5.28, as follows. Let $T^* := GW(X^*)$, where the probability generating function of the positive integer valued random variable X^* is $f^*(t) := \frac{f_X(q+(1-q)t)}{1-q}$, and let $\bar{T} := GW(\bar{X})$, where $\bar{f}(t) = f_{\bar{X}}(t) := \frac{f(qs)}{q}$, and hence $\bar{T}$ is almost surely finite. The law of $GW(X)$ conditioned to be infinite equals the law of the tree T constructed as follows: consider the rooted tree T^*, and attach to each vertex of T^* an appropriate number of independent copies of $\bar{T}$. We get the law of $UGW_\infty(X)$ if we attach to the root an appropriate random number of independent copies of T and $\bar{T}$. It follows that the law of $UGW_\infty(X)$ determines $(f^*, \bar{f})$. We get the function f from $(f^*, \bar{f})$ by the transform $f(s) = q\bar{f}\left(\frac{s}{q}\right)$, if $0 \leq s \leq q$ and $f(s) = (1-q)f^*\left(\frac{s-q}{1-q}\right)$, if $q \leq s \leq 1$. There is a unique q for which the resulting $f(s)$ has the same second derivative from the left and from the right at $s = q$. Since $f(s)$ has to be analytic, we see that $(f^*, \bar{f})$ uniquely determines f and hence X. ■

3.2 Lower semicontinuity

The quantity $\phi_p(S)$ can be used to give a short proof that $p_c(G)$ is lower semicontinuous in the local topology of transitive graphs: that is, $\liminf p_c(G_n) \geq p_c(G)$ holds; see [11, Section 1.2]. One can show in a similar way that this inequality is also true for $\hat{p}_c$ and unimodular graphs.

Proposition 3.3. *Let G_n and G be unimodular random graphs with uniformly bounded degrees. If G_n converges to G then* $\liminf_{n\to\infty} \hat{p}_c(G_n) \geq \hat{p}_c(G)$.

Proof. Let $p < \hat{p}_c(G)$ and let r be such that $\overline{\mathbb{E}}_G\left(\phi_p^\omega(B_\omega(o,r))\right) < 1-\varepsilon$ with some $\varepsilon > 0$. Let n be large enough to satisfy

$$\sum_{H \in \mathcal{H}_{r+1}} \left|\mu_{G_n}\left(B_\omega(o,r+1)=H\right) - \mu_G\left(B_\omega(o,r+1)=H\right)\right| < \frac{\varepsilon}{2D^{r+1}},$$

where D is a uniform bound on the degrees of G_n and G and $\mathcal{H}_r$ is the set of possible r-neighbourhoods of the root in graphs with maximum degree D. Any $H \in \mathcal{H}_{r+1}$ satisfies $\phi_p^H(B_\omega(o,r)) \leq D^{r+1}$. We obtain

$$\begin{aligned}
\overline{\mathbb{E}}_{G_n}\left(\phi_p^\omega(B_\omega(o,r))\right) &= \sum_{H \in \mathcal{H}_{r+1}} \mu_{G_n}\left(B_\omega(o,r+1)=H\right)\phi_p^H(B_\omega(o,r)) \\
&\leq \sum_{H \in \mathcal{H}_{r+1}} \big[\mu_G\left(B_\omega(o,r+1)=H\right)\phi_p^H(B_\omega(o,r)) \\
&+ \left|\mu_{G_n}\left(B_\omega(o,r+1)=H\right) - \mu_G\left(B_\omega(o,r+1)=H\right)\right| |\partial_E B_H(o,r)|\big] \\
&\leq \overline{\mathbb{E}}_G\left(\phi_p^\omega(S)\right) + \frac{\varepsilon}{2} < 1.
\end{aligned}$$

It follows that $\hat{p}_c(G_n) \geq p$ thus $\liminf \hat{p}_c(G_n) \geq \hat{p}_c(G)$. ■

Proposition 3.4. *Let G be a uniformly good unimodular random graph. Furthermore, let G_n be unimodular random graphs converging to G in the local weak sense, in a* uniformly sparse *way: there is a positive integer k such that for each n there is a coupling ν_n of μ_G and μ_{G_n} such that $G \subseteq G_n$ and there is a sequence of positive integers $r_n \to \infty$ that satisfies $|(E(G_n) \setminus E(G)) \cap B_{G_n}(o,r_n)| \leq k$ ν_n-almost surely. Then*

$$\lim_{n\to\infty} p_c(G_n) = p_c(G).$$

Proof. First, $G \subseteq G_n$ implies that $p_c(G) \geq p_c(G_n)$ for all n. For the sake of simplicity, we prove the inequality $\lim p_c(G_n) \geq p_c(G)$ for $k=1$. It can be proved for general k in a similar way. Let $p < p_c(G)$ fixed and let c and $R(p)$ be as in Lemma 2.3. Let n be sufficiently large to satisfy $r_n/2 > R(p)$ and $c^{r_n/2} < \frac{1}{3}$. Fix a pair (ω,ω_n) that satisfies the sparseness condition for r_n. Then, in the smaller ball $B_{\omega_n}(o,r_n/2)$, there is at most one edge $\{x,y\} \in \omega_n \setminus \omega$. If this edge exists, let $B_n := B_{\omega_n}(o,r_n/2) \cup B_{\omega_n}(x,r_n/2) \cup B_{\omega_n}(y,r_n/2)$; otherwise, just let $B_n := B_{\omega_n}(o,r_n/2)$. Note that $B_n \subset B_{\omega_n}(o,r_n)$. Similarly, let $B := B_\omega(o,r_n/2) \cup B_\omega(x,r_n/2) \cup B_\omega(y,r_n/2)$, omitting those terms in the union that do not exists in ω. (Note that it may happen that x or y does not exist in

ω, but not both, since $B_{\omega_n}(o, r_n/2)$ is connected). We claim that we have $\phi_p^{\omega_n}(B_n) < 1$. Indeed:

$$\begin{aligned}
\phi_p^{\omega_n}(B_n) &= p \sum_{e \in \partial_E B_n} \mathbb{P}^{\omega_n}\left(o \overset{\omega_n,p}{\underset{B_n}{\longleftrightarrow}} e^-\right) \\
&= p \sum_{e \in \partial_E B_n} \Bigg[\mathbb{P}^{\omega_n}\left(o \overset{\omega_n\setminus\{x,y\},p}{\underset{B_n}{\longleftrightarrow}} e^-\right) + \mathbb{P}^{\omega_n}\left(\{o \overset{\omega_n,p}{\underset{B_n}{\longleftrightarrow}} x\}\Box\{\{x,y\} \text{ open}\}\Box\{y \overset{\omega_n,p}{\underset{B_n}{\longleftrightarrow}} e^-\}\right) \\
&\qquad + \mathbb{P}^{\omega_n}\left(\{o \overset{\omega_n,p}{\underset{B_n}{\longleftrightarrow}} y\}\Box\{\{x,y\} \text{ open}\}\Box\{x \overset{\omega_n,p}{\underset{B_n}{\longleftrightarrow}} e^-\}\right)\Bigg] \\
&\leq p \sum_{e \in \partial_E B} \Bigg[\mathbb{P}^{\omega}\left(o \overset{\omega,p}{\underset{B}{\longleftrightarrow}} e^-\right) + p\,\mathbb{P}^{\omega}\left(\{o \overset{\omega,p}{\underset{B}{\longleftrightarrow}} x\}\Box\{y \overset{\omega,p}{\underset{B}{\longleftrightarrow}} e^-\}\right) \\
&\qquad + p\,\mathbb{P}^{\omega}\left(\{o \overset{\omega,p}{\underset{B}{\longleftrightarrow}} y\}\Box\{x \overset{\omega,p}{\underset{B}{\longleftrightarrow}} e^-\}\right)\Bigg] \\
&\leq p \sum_{e \in \partial_E B} \Bigg[\mathbb{P}^{\omega}\left(o \overset{\omega,p}{\underset{B}{\longleftrightarrow}} e^-\right) + \mathbb{P}^{\omega}\left(y \overset{\omega,p}{\underset{B}{\longleftrightarrow}} e^-\right) + \mathbb{P}^{\omega}\left(x \overset{\omega,p}{\underset{B}{\longleftrightarrow}} e^-\right)\Bigg] \\
&= \phi_p^{\omega}(B) + \phi_p^{\omega,y}(B) + \phi_p^{\omega,x}(B) \ < \ 1\,;
\end{aligned}$$

if x or y does not exist in ω, all its appearances in the above formulas involving ω can be replaced by the other vertex, and the inequalities remain true. It follows that $p \leq \tilde{p}_c(G_n) = p_c(G_n)$. ■

3.3 Counterexamples

Our first example will show that even if we keep the condition of uniformly sparse convergence of G_n to G of Proposition 3.4, without G being uniformly good, the conclusion may not hold. Next, Example 3.6 will show that keeping the limit uniformly good but removing the condition of uniform sparseness will make the conclusion false. Finally, Example 3.7 will show that the inequality of the lower semicontinuity may be strict even when invariant subgraphs G_n of $\mathbb{Z}^2$ converge to $\mathbb{Z}^2$.

Example 3.5. *There exists a sequence (G_n) of invariant random subgraphs of a Cayley graph, converging to an invariant subgraph G in a uniformly sparse way, such that* $\lim p_c(G_n) < p_c(\lim G_n)$.

Proof. The first step is to construct an invariant percolation on a Cayley graph of the lamplighter group all whose clusters are isomorphic to the canopy tree Λ. In more detail:

Consider the generators $\{Rs, R, sL, L\}$ of the lampligher group $\mathbb{Z}_2 \wr \mathbb{Z} = \oplus_{\mathbb{Z}}\mathbb{Z}_2 \rtimes \mathbb{Z}$, where $R := (0,1), L := (0,-1)$, and $s := (e_0, 0) \in \mathbb{Z}_2 \wr \mathbb{Z}$ with $e_0 \in \{0,1\}^{\mathbb{Z}}$, $(e_0)_j = \delta_{0,j}$. It is well-known (see, e.g., [30]) that the Cayley graph with respect to these generators is the Diestel-Leader graph DL(2,2). This graph can be defined using two trees $\mathbb{T}^1$ and $\mathbb{T}^2$ which both are 3-regular infinite rooted trees with a distinguished end and Busemann functions $\mathfrak{h}_i : \mathbb{T}^i \to \mathbb{Z}, i = 1, 2$, as in Example 2.7. Each vertex $x \in \mathbb{T}^i$ has exactly one neighbour $\bar{x}$ with $\mathfrak{h}_i(\bar{x}) = \mathfrak{h}_i(x) - 1$, called the parent of x. We call the other two neighbours the children of x. Now consider the following percolation on $\mathbb{T}^1$: for each vertex x we delete the edge connecting x to one of its two children, independently with equal probabilities. We get a random subgraph of $\mathbb{T}^1$ consisting of infinite simple paths. We then delete the edges in the graph DL(2,2) whose first coordinate is a deleted edge in $\mathbb{T}^1$. The resulting random subgraph $\mathcal{F} \subset$ DL(2,2) is invariant under the action of the lamplighter group and it consists of infinitely many components which are all isomorphic to the canopy tree $\Lambda \subset \mathbb{T}$. The probability that the root is in the n^{th} level of its component in $\mathcal{F}$ is clearly 2^{-n-1}. The canopy tree with a random root chosen according to this distribution is a unimodular random graph, as it also must be the case by Proposition 1.2.

The significance of the canopy tree for this construction (as in Example 2.7) will be that it has one end, thus $p_c(\Lambda) = 1$, while one can easily compute that $p_T(\Lambda) = 1/\sqrt{2}$.

Now let $\mathbb{G}$ be the free product of $\mathbb{Z}_2 := \mathbb{Z}/2\mathbb{Z}$ and the lamplighter group $\mathbb{Z}_2 \wr \mathbb{Z}$. Let Γ be the left Cayley graph of $\mathbb{G}$ with respect to the generators $\{a, Rs, R, sL, L\}$ where a is the generator of the free factor $\mathbb{Z}_2$. Let $\beta : \mathbb{G} \longrightarrow \mathbb{Z}_2 \wr \mathbb{Z}$ be the natural projection homomorphism: if $w = a_1 b_1 \dots a_k b_k$ is a word in $\mathbb{G}$ such that $a_j \in \mathbb{Z}_2, b_j \in \mathbb{Z}_2 \wr \mathbb{Z}, j = 1, \dots, k$, then $\beta(w) := b_1 \dots b_k \in \mathbb{Z}_2 \wr \mathbb{Z}$. We now define G to be the following random spanning subgraph of Γ: let e be in $E(G)$ iff $\beta(e^-)$ and $\beta(e^+)$ are connected by an edge in $\mathcal{F}$. The distribution of G is invariant under the action of $\mathbb{G}$ and each component of G is a canopy tree, hence $p_c(G) = 1$.

We define a sequence (G_n) of random subgraphs of Γ converging to G. We choose an element $b \in \{0, 1, \dots n-1\}$ uniformly at random. For each vertex in $L_{\mathbb{T}^1}(b+kn), k \in \mathbb{Z}$ we choose one of its descendants in $L_{\mathbb{T}^1}(b + (k+1)n)$ uniformly at random and we choose all vertices in $L_{\mathbb{T}^2}(-b + kn)$. Let S_n be the set of edges $e \in E(\Gamma)$ such that e is labelled by the generator a and both coordinates of $\beta(e^-) = \beta(e^+)$ are chosen vertices in the above procedure. Let $G_n := G \cup S_n$.

We show that $p_c(G_n) \le \frac{1}{\sqrt{2}}$ for all n. Let $p > \frac{1}{\sqrt{2}} = p_T(\Lambda)$, let n be a positive integer and consider Bernoulli(p) percolation on G_n. Denote by $T(v)$ the component of the vertex v in G and by $\mathcal{C}_v$ the component of the vertex v in the percolation on G_n. Let $s(v) := \min\{l : L_{T(v)}(l) \cap S_n \neq \emptyset\}$. We define a branching process depending on the percolation on G_n. For each vertex v of Γ let $N_v := \{ax : x \in T(v) \cap \mathcal{C}_v \cap S_n \setminus \{v\}, \{x, ax\} \text{ is open}\}$. Let $Z_1 := N_o$ and let $Z_{k+1} := \bigcup_{v \in Z_k} N_v$. Note that $Z_i \neq Z_j, i \neq j$ and $Z_j \subset \mathcal{C}_o$. The distribution of $|N_v|$ depends only on the level of v in $T(v)$ and on $s(v)$. The distribution of $|N_v|$ conditioned on $\{o \in L_{T(o)}(l), s(v) = s\}$ with any l and s stochastically dominates the distribution of $|N_v|$ conditioned on the event $\{v \in L_{T(v)}(0), s(v) = n-1\}$. Therefore the distribution of $|Z_k|$ stochastically dominates the distribution of the k^{th} generation of the Galton-Watson process with offspring distribution $|N_v|$ conditioned on $\{v \in L_{T(v)}(0), s(v) = n-1\}$, which has infinite expectation. Hence $\mu(\liminf |Z_k| > 0) > 0$ which implies $\mu(|\mathcal{C}_o| = \infty) > 0$. ■

Example 3.6. *There exists a sequence (G_n) of invariant random subgraphs of a Cayley graph such that $\lim p_c(G_n) < p_c(\lim G_n)$ and $\lim G_n$ is uniformly good.*

Proof. Let Γ be a Cayley graph of a finitely generated group such that there exists a random subgraph $\bar{G}$ which satisfies the following: the distribution of $\bar{G}$ is invariant under the action of the group, it consists of infinitely many infinite components and each component has critical percolation probability $\bar{p} < 1$. Let G' be an invariant random connected subgraph of Γ such that $p_c(G') > \bar{p}$. For example, if Γ is amenable, then one can choose G' to be an invariant spanning tree of Γ, which always exists and has at most two ends; see [7], Theorem 5.3. Moreover, if Γ has sub-exponential volume growth, then for each $p < 1$ there is a positive integer $r = r(p)$ such that

$$\phi_p^{\omega}(B_{\omega}(o, r)) \le |B_{\Gamma}(o, r)| p^r < 1 \tag{3.4}$$

for almost every configuration ω of the invariant spanning tree G', because $B_{\omega}(o, r)$ is a tree in $B_{\Gamma}(o, r)$. It follows that G' is uniformly good. Let $\varepsilon_n \to 0$ be a sequence of positive numbers and let G_n be the following random subgraph of Γ: we remove each component of $\bar{G}$ with probability $1 - \varepsilon_n$ and keep it with probability ε_n independently for each component. Let G_n be the union of G' and the remaining components of $\bar{G}$. It follows from Proposition 1.2 that G_n is unimodular. The sequence (G_n) converges to G', but $p_c(G_n) \le \bar{p} < p_c(G')$ for each n. The sequence $p_c(G_n)$ has a convergent subsequence, hence we can choose the corresponding subsequence $\varepsilon_{k(n)}$, and get $\lim p_c(G_{k(n)}) \le \bar{p} < p_c(G')$.

We get a similar example that is uniformly good if we set $\Gamma := \mathbb{Z}^5$, $\bar{G} := \bigcup_{y\in\mathbb{Z}^2}\{y\} \times \mathbb{Z}^3$ and $G' := \bigcup_{x\in\mathbb{Z}^3} \mathbb{Z}^2 \times \{x\}$. In this example G' is not connected, but each G_n is connected almost surely, and $p_c(G_n) \le p_c(\mathbb{Z}^3) < p_c(\lim G_n) = p_c(\mathbb{Z}^2) < 1$ for each n. ■

Example 3.7. *There exists a sequence* (G_n) *of invariant random subgraphs of a Cayley graph such that* $1 > \lim p_c(G_n) > p_c(\lim G_n)$.

Proof. We define G_n as a vertex and edge replacement (see Subsection 1.4 and [2], Example 9.8) of $\mathbb{Z}^2$ where we replace each vertex x by the graph Q_x isomorphic to Q_n and we replace each edge by a path of length two that joins the middle points of the neighbouring sides of the boxes corresponding to the endpoints of the edge. The graphs G_n can be considered as deterministic subgraphs of $\mathbb{Z}^2$ with a randomly chosen root. The sequence G_n converges to $\mathbb{Z}^2$.

We show that $\frac{1}{2} < \lim p_c(G_n) < 1$. Denote by $G_n(p)$ the subgraph obtained by the Bernoulli(p) percolation on G_n, and let $H_n(p)$ be the following percolation on $\mathbb{Z}^2$: let an edge $\{x, y\}$ open, iff both edges are open in the path that joins the boxes Q_x and Q_y in G_n. The existence of an infinite cluster in $G_n(p)$ implies the existence of an infinite cluster in $H_n(p)$. The law of H_n equals the law of the Bernoulli(p^2) percolation on $\mathbb{Z}^2$, hence $p_c(G_n) \ge \frac{1}{\sqrt{2}}$ for each n.

To show that $\limsup p_c(G_n) < 1$, we define the percolation $\bar{H}_n(p)$ on $\mathbb{Z}^2$. Denote by $\mathcal{A}_x(n)$ the event that the vertices $(0, -n)$, $(0, n)$, $(-n, 0)$, $(n, 0)$ are in the same cluster in Bernoulli(p) percolation on the box $Q_x \subset G_n$. Let an edge $\{x, y\} \in \bar{H}_n(p)$, iff $\{x, y\} \in H_n(p)$, and both of the events $\mathcal{A}_x(n)$ and $\mathcal{A}_y(n)$ occurs. The existence of an infinite cluster in $\bar{H}_n(p)$ implies the existence of an infinite cluster in $G_n(p)$. Let $1 > p_0 > \frac{1}{2}$ be arbitrary. There is an $\varepsilon > 0$ such that if the marginals of a 2-dependent percolation on $\mathbb{Z}^2$ are at least $(1 - \varepsilon)^4$, then this percolation stochastically dominates Bernoulli(p_0) percolation; see [20, Theorem 0.0]. Lemma 2.6 implies, that we can find constants $1 - \varepsilon < p_1 < 1$ and N such that for any $p > p_1$, $n \ge N$ and for any vertex $x \in V(\mathbb{Z}^2)$ the event $\mathcal{A}_x(n)$ occurs whith probability at least $1 - \varepsilon$, thus $\mathbb{P}(e \in \bar{H}_n(p)) \ge p_1^2(1 - \varepsilon)^2 \ge (1 - \varepsilon)^4$ for any edge $e \in E(\mathbb{Z}^2)$. The events $\{e_1 \in \bar{H}_n\}$ and $\{e_2 \in \bar{H}_n\}$ are independent if the distance of e_1 and e_2 is at least 2, hence $\bar{H}_n(p)$ stochastically dominates Bernoulli(p_0) percolation. It follows that $\limsup p_c(G_n) \le p_1 < 1$. ■

4 On transitive graphs of cost 1

The *cost of a group* $\mathbb{G}$ is defined as half of the infimum of the expected degrees of its invariant connected spanning graphs. (See Subsection 1.1 for references.) The *cost of a transitive graph* Γ may be defined similarly, over $\mathbb{G}$-invariant random connected spanning *sub*graphs, where $\mathbb{G} \le Aut(\Gamma)$ is a vertex-transitive subgroup of graph-automorphisms that will usually be fixed implicitly. It is not known in general that if we first fix a Cayley graph Γ of $\mathbb{G}$, then the $\mathbb{G}$-cost of Γ is always as small as the cost of $\mathbb{G}$. Nevertheless, cost 1 can be achieved inside any Cayley graph of any amenable group: as proved in [7, Theorem 5.3], an infinite transitive graph Γ has an invariant spanning tree $\mathcal{T}$ with at most two ends (hence with expected degree 2 and $p_c(\mathcal{T}) = 1$) iff it is amenable.

The main point of our next proposition is that, under the stronger condition of subexponential decay, we get a spanning tree with the stronger property $\hat{p}_c(\mathcal{T}) = 1$, and using this, we can achieve approximately 1-dimensional percolation behaviour $p_c(G_k) \to 1$ via connected spanning subgraphs that have the same large-scale geometry as Γ. The bi-Lipschitz condition is also natural from the point of view of Elek's combinatorial cost for sequences of finite graphs [13].

Proposition 4.1. *If G is a transitive amenable graph, then there is a sequence of invariant random subgraphs G_k which satisfies the following: each G_k is a bi-Lipschitz (in particular, connected) spanning subgraph of G, the girth of G_k tends to infinity and G_k locally converges to an invariant random spanning tree $\mathcal{T}$ with at most two ends.*

If G is a unimodular transitive graph with sub-exponential volume growth then $\lim p_c(G_k) = 1$.

Proof. We construct $\mathcal{T}$ as in [7], Theorem 5.3: let F_n be a sequence of Følner sets such that $\sum_{n=1}^{\infty} \frac{|\partial_E F_n|}{|F_n|} < 1$. For each n and $x \in V(G)$ choose a random $g_{x,n} \in Aut(G)$ that takes o to x, and a random bit $Z_{x,n}$ that equals 1 with probability $\frac{1}{|F_n|}$. Choose all $g_{x,n}$ and $Z_{x,n}$ independently. Let $\omega_n := E(G) \setminus \bigcup_{x \in V(G), Z_{x,n}=1} \partial_E(g_{x,n} F_n)$; i.e., we remove all edges in the boundaries of the translates of F_n with $Z_{x,n} = 1$. Let $\bar{\omega}_n = \bigcap_{k \geq n} \omega_k$. Each $\bar{\omega}_n$ has only finite components.

To construct $\mathcal{T}$ and G_k, choose uniform labels L_e in [0,1] independently for each $e \in E(G)$. For each finite component of $\bar{\omega}_1$ take the minimal spanning tree of the component with respect to the labels. Denote by T_1 the union of these trees. Let T_2 be the union of T_1 and the edges in $\bar{\omega}_2 \setminus \bar{\omega}_1$ with minimal labels such that the components of T_2 are spanning trees of the components of $\bar{\omega}_2$. Continue inductively, and let $\mathcal{T} := \bigcup T_n$. This is an invariant random spanning tree, which has at most 2 ends (otherwise it would have infinitely many ends, which is impossible, since G is amenable).

To construct G_k we define a color for each edge. Let all edges in $\mathcal{T}$ be green. In each component of $\bar{\omega}_1$ do the following: consider the edge with the smallest label which has no color. If there is a path of length at most k between its endpoints consisting of green edges, then color it red, otherwise color it green. Continue inductively for the edges in the component. This procedure defines a color for each edge of $\bar{\omega}_1$. If all edges in $\bar{\omega}_n$ have a color, then continue coloring the edges of $\bar{\omega}_{n+1} \setminus \bar{\omega}_n$ in the same way. Let G_k be the union of the green edges. It follows from the construction that G_k is invariant, its girth is at least $k+2$ and for each edge of G there is a path in G_k between its endpoints with length at most k. The sequence G_k converges to $\mathcal{T}$.

If G has sub-exponential volume growth, then for each $p < 1$ there is a positive integer $r = r(p)$ that satisfies the inequality (3.4) (with G in the place of Γ) for almost every configuration ω that is a forest in $B_G(o, r)$. It follows that $\hat{p}_c(\mathcal{T}) = 1$. Moreover, for any $p < 1$ there is a positive integer k_0 such that G_k is almost surely a forest in $B_G(o, r(p))$ for any $k \geq k_0$. The random subsets $S_\omega := G_k \cap B_G(o, r(p))$ attest $\tilde{p}_c(G_k) \geq p$. Since $p_c(G_k) = \tilde{p}_c(G_k)$ holds by Theorem 2.1, we get $p_c(G_k) \to 1$. ∎

Our proposition may be thought of as a strengthening of having cost 1:

Lemma 4.2. *If Γ is a Cayley graph of $\mathbb{G}$, and there exists a sequence of $\mathbb{G}$-invariant connected spanning subgraphs $G_k \subset \Gamma$ with $p_c(G_k) \to 1$, then the cost of Γ, hence of $\mathbb{G}$, is 1.*

Proof. Take $\varepsilon_k \to 0$ such that $p_c(G_k) > 1 - \varepsilon_k$. Then, all clusters of Bernoulli$(1 - \varepsilon_k)$ percolation on G_k are finite almost surely. Let the set of closed edges be denoted by $\eta_k \subset G_k \subset \Gamma$, an invariant percolation itself. In each finite cluster, take a uniform random spanning tree, a subtree of G_k. The union of all these finite spanning trees and η_k will be ω_k. One the one hand, it is clear that ω_k is a connected spanning subgraph of G_k, hence of Γ. On the other hand, the expected degree of o in ω_k is at most $\mathbb{E} \deg_{\eta_k}(o) + 2 \leq d\varepsilon_k + 2$, where $\deg_\Gamma(o) = d$. As $k \to \infty$, we obtain that the cost of Γ is 1. ∎

As we mentioned above, having a $\mathbb{G}$-invariant connected spanning graph $\mathcal{T}$ with $p_c(\mathcal{T}) = 1$ implies that $\mathbb{G}$ is amenable, because it is not hard to construct an invariant mean [7, Theorem 5.3]. However, the sequence $p_c(G_k) \to 1$ does not imply amenability:

Example 4.3. $\mathbb{T}_3 \times \mathbb{Z}$ *has a sequence of invariant bi-Lipschitz subgraphs* G_k *with* $p_c(G_k) \to 1$.

Proof. One can partition the edges of $\mathbb{T}_3$ into 3 disjoint perfect matchings M_1, M_2 and M_3 in an invariant way. (See, for instance, [23], around Proposition 2.4.) Then, consider the following subgraphs $G_k \subseteq \mathbb{T}_3 \times \mathbb{Z}$: we keep all the edges in the subgraphs $\{v\} \times \mathbb{Z}$ and the edges $\{e\} \times \{3jk+ik\}$ where $e \in M_i$, $j \in \mathbb{Z}$. We choose a uniform random integer $b \in \{0, \dots k-1\}$ and translate this subgraph by (id, b) to get the invariant subgraph G_k of $\mathbb{T}_3 \times \mathbb{Z}$. Each G_k is clearly bi-Lipschitz equivalent to $\mathbb{T}_3 \times \mathbb{Z}$. On the other hand, we have $p_c(G_k) \to 1$: either from Proposition 3.4, or more directly, by observing that the universal cover T_k of G_k can be obtained from $\mathbb{T}_3$ by replacing "two thirds" of the edges by a path of length k, for which it is easy to see that $p_c(T_k) \to 1$, while $p_c(T_k) \leq p_c(G_k)$ holds by [25, Theorem 6.47]. ■

We do not know if the converse of Lemma 4.2 holds:

Question 4.4. *Does there exist, for any Cayley graph* Γ *of any group* $\mathbb{G}$ *of cost 1, a sequence of* $\mathbb{G}$*-invariant bi-Lipschitz spanning subgraphs* $G_k \subset \Gamma$ *with* $p_c(G_k) \to 1$*?*

For amenable Cayley graphs Γ, a first step of independent interest could be a positive answer to the following question, mentioned in Subsection 1.1:

Question 4.5. *For any amenable Cayley graph, is there an invariant random spanning subtree of subexponential growth? More boldly, does there always exist an invariant random Hamiltonian path?*

Acknowledgments

We are grateful to Sébastien Martineau for comments on the manuscript.

Our work was partially supported by the Hungarian National Research, Development and Innovation Office, NKFIH grant K109684. Timár's research was supported by a Marie Curie Intra European Fellowship within the 7th European Community Framework Programme.

References

[1] Aizenman, M. and Barsky, D. (1987) Sharpness of the phase transition in percolation models, *Comm. Math. Phys.* Vol. 108, 489–526.

[2] Aldous, D. and Lyons, R. (2007) Processes on unimodular random networks, *Electron. J. Probab.*, **12**, Paper 54, 1454–1508. `http://128.208.128.142/~ejpecp/viewarticle.php?id=1754`

[3] Antal, P. and Pisztora, A. (1996) On the chemical distance for supercritical Bernoulli percolation, *The Annals of Probability*, Vol. 24, No. 2, 1036–1048

[4] Antunović, T. and Veselić, I. (2008) Sharpness of the phase transition and exponential decay of the subcritical cluster size for percolation on quasi-transitive graphs, *J. Statist. Phys.* Vol. 130, 983–1009. `arXiv:0707.1089 [math.PR]`

[5] Babai, L. (1979) Long cycles in vertex-transitive graphs, *Journal of Graph Theory* Vol. 3, 301–304.

[6] Backhausz, Á., Szegedy, B., and Virág, B. (2015) Ramanujan graphings and correlation decay in local algorithms. *Random Structures Algorithms* Vol. 47, 424–435. arXiv:1305.6784 [math.PR]

[7] Benjamini, I., Lyons, R., Peres, Y., and Schramm, O. (1999) Group-invariant percolation on graphs, *Geom. Funct. Anal.* Vol. 9, 29–66.

[8] Benjamini, I., Nachmias, A., and Peres, Y. (2011) Is the critical percolation probability local? *Probab. Theory Relat. Fields* Vol. 149, 261–269 arXiv:0901.4616 [math.PR]

[9] Benjamini, I. and Schramm, O. (2001) Recurrence of distributional limits of finite planar graphs. *Electron. J. Probab.* Vol. 6, paper no. 23, 13 pp. (electronic). [arXiv:math.PR/0011019]

[10] Csikvári, P. and Frenkel, P. E. (2016) Benjamini–Schramm continuity of root moments of graph polynomials. *Europ. J. Combin.* Vol. 52, 302–320. arXiv:1204.0463 [math.CO]

[11] Duminil-Copin, H. and Tassion, V. (2016) A new proof of the sharpness of the phase transition for Bernoulli percolation and the Ising model, *Comm. Math. Phys.* Vol. 343, no. 2, 725–745. arXiv:1502.03050 [math.PR]

[12] Duminil-Copin, H. and Tassion, V. (2015) A new proof of the sharpness of the phase transition for Bernoulli percolation on $\mathbb{Z}^d$, *Enseign. Math.*, to appear. arXiv:1502.03051 [math.PR]

[13] Elek, G. (2007) The combinatorial cost, *l'Enseignement Math.* Vol. 53, 225–236. [arXiv:math.GR/0608474]

[14] Gaboriau, D. (2000). Coût des relations d'équivalence et des groupes, *Invent. Math.*, Vol. 139, 41–98.

[15] Gaboriau, D. (2010) Orbit equivalence and measured group theory. *In: Proceedings of the 2010 Hyderabad ICM*, Vol. 3, 1501–1527. arXiv:1009.0132v1 [math.GR]

[16] Grimmett, G. (1999) *Percolation*, 2nd edition, Springer-Verlag, Berlin

[17] Grimmett, G. R. and Li, Z. (2015) Locality of connective constants, I. Transitive graphs, arXiv:arXiv:1412.0150 [math.CO]

[18] Grimmett, G. R. and Marstrand, J. M. (1990) The supercritical phase of percolation is well behaved *Proc. Roy. Soc. London, Ser. A: Math. and Phys. Sci.* Vol. 430, 439–457.

[19] Levitt, G. (1995) On the cost of generating an equivalence relation. *Ergodic Theory Dynam. Systems*, Vol. 15, 1173–1181.

[20] Liggett, T. M., Schonmann, R. H., Stacey, A. M. (1997) Domination by product measures *The Annals of Probability*, Vol. 25, No. 1, 71–95.

[21] Lyons, R. (1990) Random walks and percolation on trees. *Ann. Probab.* Vol. 18, 931–958. [arXiv:math.CO/0212165]

[22] Lyons, R. (2005) Asymptotic enumeration of spanning trees. *Combin. Probab. & Comput.* Vol. 14, 491–522. [arXiv:math.CO/0212165]

[23] Lyons, R. Factors of IID on trees. *Combin. Probab. Comput.*, to appear. arXiv:1401.4197 [math.PR]

[24] Lyons, R. and Nazarov, F. (2011) Perfect matchings as IID factors on non-amenable groups, *Europ. J. Combin.* Vol. 32, 1115–1125. arXiv:0911.0092v2 [math.PR]

[25] Lyons, R. and Peres, Y. (2015) *Probability on trees and networks*, Version of 14 December 2015. http://mypage.iu.edu/~rdlyons

[26] Martineau, S. and Tassion, V. (2013) Locality of percolation for abelian Cayley graphs, *Ann. Probab.*, to appear. arXiv:1312.1946 [math.PR]

[27] Menshikov, M. V. (1986) Coincidence of critical points in percolation problems, *Soviet Mathematics Doklady* Vol. 33, 856–859.

[28] Pak, I. and Radoičić, R. (2009) Hamiltonian paths in Cayley graphs, *Discrete Mathematics* Vol. 309, 5501–5508. http://www.sciencedirect.com/science/article/pii/S0012365X09000776

[29] Pete, G. (2015) *Probability and geometry on groups.* Lecture notes for a graduate course, Version of 3 August 2015. http://www.math.bme.hu/~gabor/PGG.pdf

[30] Woess, W. (2005) Lamplighters, Diestel-Leader graphs, random walks, and harmonic functions *Combin. Probab. & Comput.* Vol. 14, 415–433.

Dorottya Beringer
Bolyai Institute, University of Szeged
Aradi vértanúk tere 1, Szeged 6720 Hungary
beringer[at]math.u-szeged.hu

Gábor Pete
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
Reáltanoda u. 13-15, Budapest 1053 Hungary
and Institute of Mathematics, Budapest University of Technology and Economics
Egry J. u. 1., Budapest 1111 Hungary
http://www.math.bme.hu/~gabor

Ádám Timár
Alfréd Rényi Institute of Mathematics, Hungarian Academy of Sciences
Reáltanoda u. 13-15, Budapest 1053 Hungary
madaramit[at]gmail.com