On-Line Linear Discrepancy

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Szemerédi and WTT

Fact We are very close and have been for more than 30 years. In fact, back to back!!!

E. Szemerédi, Regular partitions of graphs, Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976), Colloq. Internat. CNRS, 260, Paris: CNRS, pp. 399–401.

Just check out the very next article!!!

General Framework

Two players, BUILDER and ASSIGNER.

BUILDER constructs a graph one point at a time.

ASSIGNER colors the graph as it is built.

All moves are permanent and correct.

Coloring a Tree can be Hard!

Trees are 2-colorable and are trivial to color off-line. On-line the situation is more complicated and the number of colors needed goes to infinity with the number of vertices.

Theorem (Bean) The minimum number of colors required to color trees on n vertices on-line is $1 + \lfloor \lg n \rfloor$.

Remark First Fit is optimal.

Bipartite Graphs are About the Same

Remark First Fit can be forced to use n colors on a bipartite graph having $2n - 1$ vertices.

Theorem (Lovász, Saks and Trotter)
The minimum number $f(2,n)$ of colors required to color bipartite graphs on-line satisfies $f(2,n) = \Theta(\lg n)$

3-Colorable Graphs are Hard!

Theorem (Alon, Kierstead) The minimum number $f(3,n)$ of colors required to color 3-colorable graphs on-line satisfies:

$$\lg^2 n < f(3,n) < n^{2/3} / \lg^{2/3} n$$

Some On-Line Problems are Easier

Theorem (Kierstead and Trotter) The minimum number of colors required to color on-line interval graphs with maximum clique size k is $3k - 2$.

Remark First Fit is not optimal when k is large.

First Fit Coloring of Interval Graphs

Theorem (Woodall; Brightwell, Kierstead, Trotter; Pemmaraju, Raman and Varadajan; Babu and Narayansamy; Smith; Qin; Chrobak and Slusarek; Howard)

On interval graphs with maximum clique size k, First Fit can be forced to use at least $(5-o(1))\,k$ colors … but cannot be forced to use more than $8k - 4$ colors.

Posets Off-Line

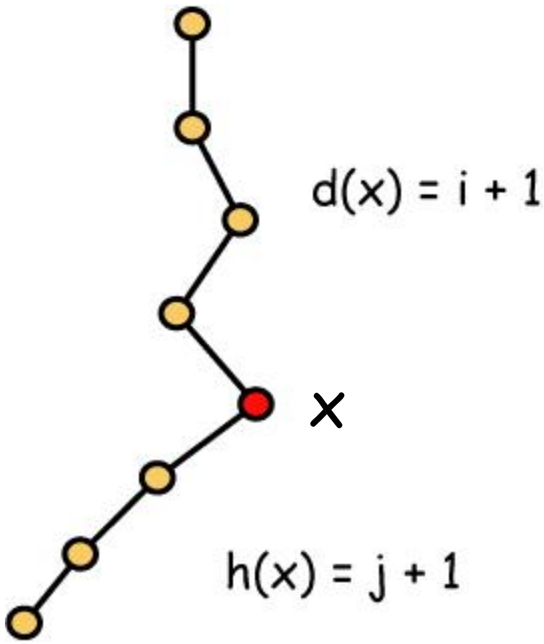
Theorem (Dilworth) The minimum number of chains required to cover a poset of width w is w.

Theorem The minimum number of antichains required to cover a poset of height h is h.

Question How do the corresponding on-line problems behave?

On-Line Antichain Partitioning

Theorem (Schmerl and Szemerédi) The minimum number of antichains required to cover posets of height h is $h(h+1)/2$.



Assign the new point x to antichain $A(i, j)$ when x is at depth $i + 1$ and height $j + 1$

Szemerédi's Observation

With height h, BUILDER can force ASSIGNER to use $h(h+1)/2$ antichains and use at least h on the set of maximal elements alone.

On-Line Chain Partitioning

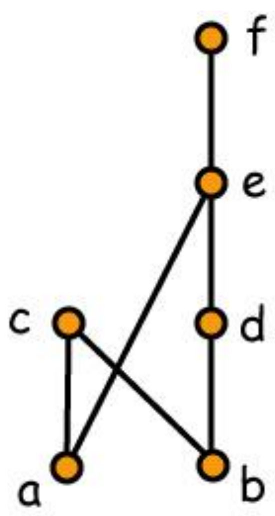
Remark First Fit can be forced to use arbitrarily many chains, even on posets of width 2.

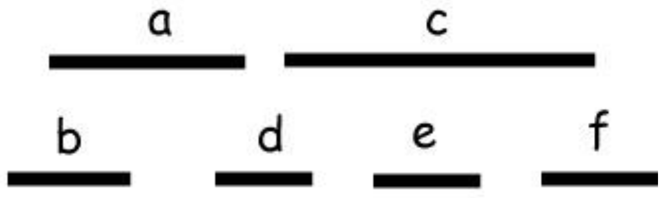
Theorem (Kierstead) Posets of width w can be partitioned on-line into $(5^w - 1)/4$ chains.

Theorem (Bosek and Krawczyk) Posets of width w can be partitioned on-line into $O(w^{16\log w})$ chains.

Interval Orders

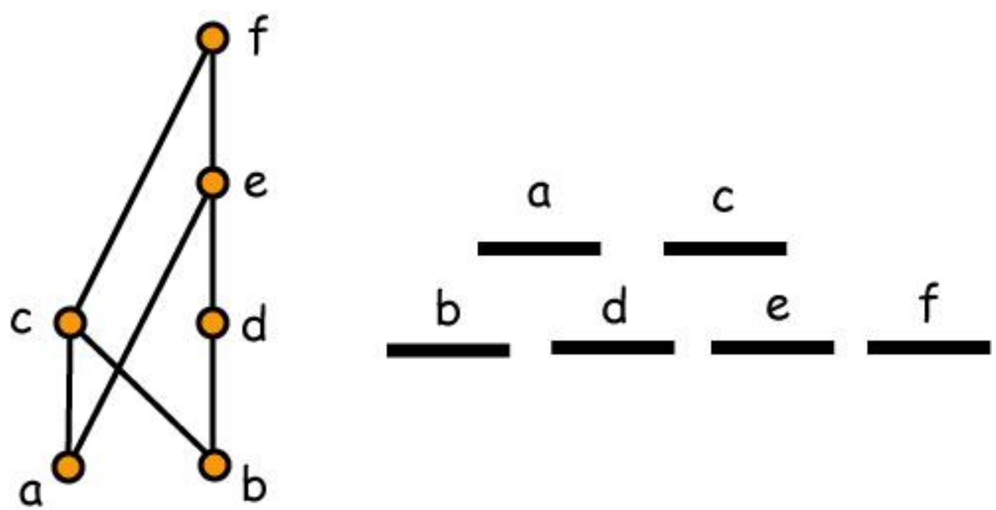
A poset P is an interval order if there exists a function I assigning to each x in P a closed interval $I(x) = [a_x, b_x]$ of the real line $\mathbf{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbf{R}$.





Semiorders – Unit Interval Orders

A poset P is a semiorder if there exists a function I assigning to each x in P a closed interval $I(x) = [a_x, b_x = a_x + 1]$ of the real line $\mathbf{R}$ so that $x < y$ in P if and only if $b_x < a_y$ in $\mathbf{R}$.



Recognizing Interval Orders

Theorem (Fishburn) Interval orders are just the posets excluding **2** + **2**.

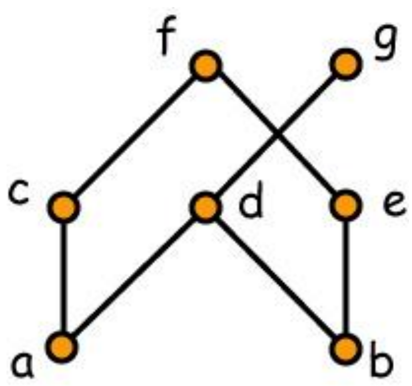
Theorem (Scott and Suppes) Semiorders are just the interval orders excluding **3** + **1**.

Interval Orders On-Line

Remark It is very easy to recognize interval orders and semiorders. So in on-line problems, BUILDER can present such a poset just by giving the order relation. On the other hand, BUILDER could actually present an interval representation. The second approach may be more restrictive.

Linear Discrepancy

Goal Find a "fair" linear extension of a poset, i.e., one that keeps incomparable points close together.



$L_1 = b < e < a < d < g < c < f$

$L_2 = a < b < c < d < e < f < g$

$L_3 = a < b < e < c < d < g < f$

Linear Discrepancy

Definition The linear discrepancy $ld(L)$ of a linear extension L of a poset P is the maximum value of $|h_L(x) - h_L(y)|$, taken over all pairs x, y of incomparable points in P.

Definition The linear discrepancy $ld(P)$ of a poset P is the minimum value of $ld(L)$, taken over all linear extensions of P.

Linear Discrepancy and Bandwidth

Theorem (Fishburn, Tannebaum and Trenk, Brightwell) The linear discrepancy of a poset P is the bandwidth of its incomparability graph.

Remark FTT reduced the problem to interval orders. Kleitman and Vohra gave an efficient algorithm for the bandwidth of an interval graph, which was order preserving on the complement.

Remark Brightwell has given (as yet unpublished) a direct argument, using local exchanges. Very clever!

Hard to Compute

Instance: Poset P

Integer d

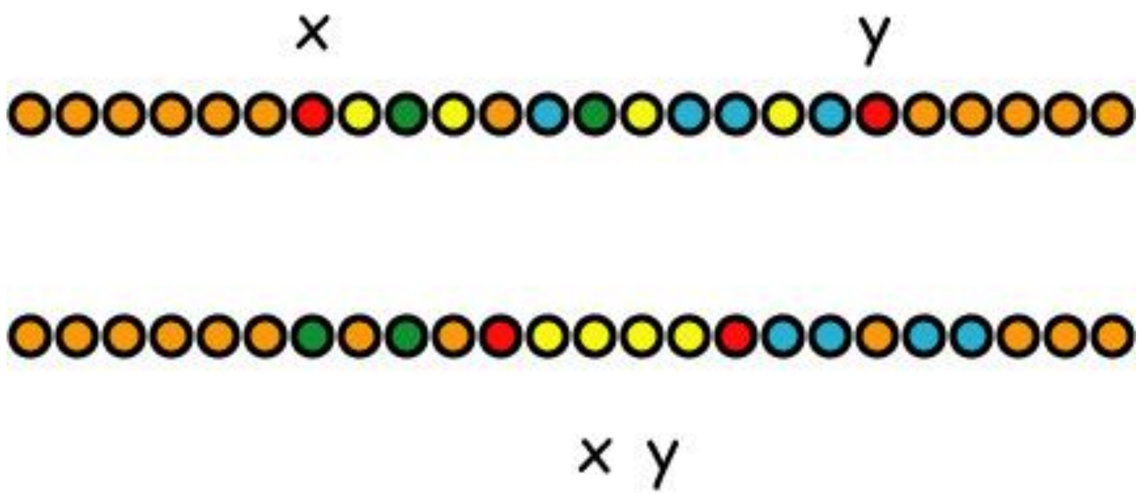
Question: Is $\mathrm{ld}(P) \leq d$?

Result: This is an NP-complete problem.

Easy to Approximate

Fact If L is any linear extension of a poset P, then

$$\mathrm{ld}\,(L) \leq 3\,\mathrm{ld}\,(P)$$



The On-Line Problem

BUILDER constructs a poset P and ASSIGNER assembles a linear extension L of P, both proceeding one point at a time. After a week or two, the game is halted and the referee determines that the linear discrepancy of P is k. The absolute worst that ASSIGNER can do is to produce an L with $ld(L) = 3k$. Is there a strategy for ASSIGNER that will do better?

On-Line Linear Discrepancy

Theorem (Keller, Streib and T) There is an on-line strategy **S** for ASSIGNER that will construct a linear extension L of P so that:

1. If $\mathrm{ld}(P) = k$, then $\mathrm{ld}(L) \leq 3k - 1$.
2. If P is a semiorder and $\mathrm{ld}(P) = k$, then $\mathrm{ld}(L) \leq 2k$.

Two Reasonable Strategies

Strategy **M** (Middle): Always inserting the new point as close to the middle.

Strategy **G** (Greedy): Insert the new point so that the linear discrepancy is minimized.

Reasonable Not Optimal

Fact For each $k \geq 1$, BUILDER can construct a semiorder P with $ld(P) = k$ while forcing an ASSIGNER using Strategy ***M***, to assemble a linear extension L with $ld(L) = 3k - 1$.

Fact For each $k \geq 1$, BUILDER can construct a semiorder P with $ld(P) = k$ while forcing an ASSIGNER using Strategy ***G***, to assemble a linear extension L with $ld(L) = 2k + \lceil (k-1)/2 \rceil$.

Reasonable Not Optimal (2)

Fact For each $k \geq 1$, BUILDER can construct a poset P with $\mathrm{ld}(P) = k$ while forcing an ASSIGNER using Strategy **M**, to assemble a linear extension L with $\mathrm{ld}(L) = 3k$.

Our Result is Best Possible

Fact For each $k \geq 1$, even when restricted to the class of interval orders excluding $\mathbf{5} + \mathbf{1}$, BUILDER can construct a poset P with $ld(P) = k$ while forcing ASSIGNER to assemble a linear extension L with $ld(L) \geq 3k - 1$.

Remark If BUILDER is restricted to interval orders excluding $\mathbf{4} + \mathbf{1}$, we can only show a lower bound of $2k + \lceil (k-1)/2 \rceil$.

Interval Representations

Theorem (Keller, Streib and T) There is an on-line strategy **L** for ASSIGNER that will construct a linear extension L of an interval order P presented in terms of an interval representation so that:

1. If ld (P) = k, then ld (L) ≤ 2k.
2. If P is a semiorder and ld (P) = k, then ld (L) = k.

This Result is Also Best Possible

Fact For each $k \geq 1$, even when restricted to the class of interval orders excluding $\mathbf{4} + \mathbf{1}$, BUILDER can construct a poset P with $ld(P) = k$ while forcing ASSIGNER to assemble a linear extension L with $ld(L) \geq 2k$.

Remark If BUILDER is restricted to interval orders excluding $\mathbf{3} + \mathbf{1}$, we can only show a lower bound of $k + \lceil (k - 1)/2 \rceil$.