# Limits of functions on abelian groups and higher order Fourier analysis

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## Szemerédi's regularity lemma

- Szemerédi: If we look at an astronomically big graph from a very big distance with somewhat weak eyes then we see an interesting bounded structure.

- Analytic interpretation (Lovász-Sz): If we look at larger and larger graphs from bigger and bigger distance with very slowly improving eyes then what we see is more and more similar to a two variable symmetric measurable function.

- Regularization $\iff$ limit objects

## Addressing the main questions

- **We investigate subsets of abelian groups or more generally functions on abelian groups.**

- Is there a limit theory of such structures?

- What does regularization mean?

- If $S \subseteq A$ then we can create a $k$ uniform hypergraph $H_k$ with edges $(a_1, a_2, \ldots, a_k)$ where $a_1 + a_2 + \cdots + a_k \in S$. What is the regularization of $H_k$? What are limits of such hypergraphs?

- What does the Gowers norm $\|.\|_{U_k}$ measure?

## Comments

- Good news: There is a quite good understanding of all the above questions.

- Some of the answers in this topic are rather surprising.

- There is a hierarchy of questions and answers depending on a natural number $k$. The simplest case is when $k = 2$. We will use the convention that "degree" $= k - 1$.

- The linear case is much simpler than the higher order cases. It is closely tied to Fourier analysis.

Gowers's norms

Definition: $\Delta_t(f)(x) = f(x)\overline{f(x+t)}$

$$\|f\|_{U_k} = \left(\mathbb{E}_{x,t_1,t_2,\ldots,t_k}\Delta_{t_1,t_2,\ldots,t_k}f(x)\right)^{2^{-k}}$$

$\|f\|_{U_2}^4$ is the average of

$$f(x)\overline{f(x+t_1)f(x+t_2)}f(x+t_1+t_2).$$

$$\|f\|_{U_2}^4 = \sum_{\chi\in\hat{A}} |(f,\chi)|^4$$

$$\|f\|_\infty \leq 1 \Rightarrow |\lambda_{\max}| \leq \|f\|_{U_2} \leq \sqrt{|\lambda_{\max}|}$$

What do Gowers's norms measure?

- $\|f\|_{U_2} \sim 0 \iff f$ is Fourier noise

- $\|f\|_{U_k} \sim 0 \Rightarrow \|f\|_{U_{k-1}} \sim 0$.

-There are functions with $\|f\|_{U_2} \sim 0$ and $\|f\|_{U_3} >> 0$

-Such functions are Noise in Fourier theory but they have structure in a higher order sense! What kind of structure?

-Green-Tao: Quadratic structure

## Regularity (first approximation)

- We want to understand the meaning of the Gowers norms through decomposition theorems.

- Scheme of the theorem: Every measurable function $f$ with $|f| \leq 1$ on a compact abelian group $A$ is decomposable in the form

$$f = f_s + f_e + f_r$$

where $f_s$ has a bounded complexity structure, $|f_s| \leq 1$, $\|f_e\|_2 \leq \epsilon$ and $\|f_r\|_{U_k}$ is very small in terms of $\epsilon$ and the complexity of $f_s$.

Nilmanifolds

-Definition Let $L$ be a $k$-nilpotent Lie group and $\Gamma \subset L$ be a co-compact subgroup. Then the coset space $L/\Gamma$ is called a $k$-step **nil-manifold**

-Bounded complexity $k$-degree structure:

$$A \to N \to \mathbb{C}$$

where $N$ is a $k$-step nilmanifold of bounded dimension, the first map is an appropriate morphism and the second map is a Lipschitz function with bounded Lipschitz constant.

## Problems with nilmanifolds

- When we want to prove the regularity lemma we have to find these curious structures somewhere. They are defined indirectly through a nilpotent Lie group which is not even compact (it is locally compact).

- If $A$ has many small degree elements then things go crazy. (Cyclic group is fine, vectorspace over $\mathbb{F}_2$ is very bad.)

## Solution

- Very good news: There is a nice simple set of axioms defining structures that are exactly what we need.

- A similar axiom system was defined by Host-Kra in a beautiful paper. They used it to find two step nilpotent groups.

- Jointly with O.A.Camarena we introduced a variant of this axiom system and then we proved that they can characterize nilmanifolds.

# Nilspace axioms

- We say that $\{0,1\}^n$ is an $n$-dimensional cube.

- A function $\phi : \{0,1\}^n \to \{0,1\}^k$ is called a morphism if it extends to an affin homomorphism $\phi : \mathbb{Z}^n \to \mathbb{Z}^k$. Note that there is a nice combinatorial description of morphisms.

- Let $N$ be a set and assume that for every $n$ there is a collection of morphisms $f : \{0,1\}^n \to N$ that we denote by $\hom(\{0,1\}^n, N)$.

- Axiom 1. (pre-sheaf) Morphisms are closed under composition.

- Axiom 2. (ergodicity) For every two points $x, y \in N$ the map $\phi : \{0,1\} \to N$ with $\phi(0) = x, \phi(1) = y$ is a morphism.

- Axiom 3. (closing) If $\phi : \{0,1\}^n \setminus \{1^n\} \to N$ is a morphism restricted to every $n-1$ dimensional face containing $0^n$ then $\phi$ extends to a morphism of the full $n$ dimensional cube.

- If the closing is unique for some dimension $n = k+1$ then we say that $N$ is a $k$-step nilspace.

# Remarks

- If $\phi : \{0,1\}^n \to \{0,1\}^k$ is a morphism then it induces a function
$$\phi' : \mathrm{hom}(\{0,1\}^k, N) \to \mathrm{hom}(\{0,1\}^n, N).$$

- We can define topological, compact, differentiable etc... versions of nilspaces.

- 1-step nilspaces are abelian groups.

- $k$-step nilspaces are $k$-fold abelian bundles

- Nilspaces are forming a category. Morphisms are the cube preserving maps.

# Structure of nilspaces

- Theorem (Camarena-Sz.): A compact $k$-step nilspace is the inverse limit of finite dimensional $k$-step nilspaces.

- Theorem (Camarena-Sz.): (rigidity) A continuous almost morphism from a compact $k$-step nilspace to a bounded dimensional one can be corrected to a proper morphism

- Theorem (Camarena-Sz.): Finite dimensional compact connected $k$-step nilspaces are $k$-step nilmanifolds.

## Regularization of functions on abelian groups

- Theorem (Sz.) For an arbitrary decreasing function $F : \mathbb{R}^+ \times \mathbb{N} \to \mathbb{R}^+$ and $\epsilon > 0$ there a constant $c = c(F, \epsilon)$ such that every function $f : A \to \mathbb{C}$ with $|f| \leq 1$ has a decomposition

$$f = f_s + f_e + f_r$$

  such that $f_s$ has complexity $c' \leq c$, $|f_s| \leq 1$, $\|f_e\|_2 \leq \epsilon$, $\|f_r\|_{U_k} \leq F(\epsilon, c')$.

- $f_s = h \circ g$ where $h : A \to N$ is a morphism to a $k-1$ step nilspace of complexity at most $c'$ and $g$ is a Lipschitz function with constant at most $c'$.

## Remarks

several strengthenings follow easily from the proof.

- One can assume that the image of the nil-morphism is close to equi-distributed

- Restrictions on the structure of $A$ impose strong restrictions on the structure of $N$

- Green-Tao-Ziegler and Green-Tao recent results are closely connected

## Limits of functions on Abelian groups

For every degree $d = k-1$ there is an interesting limit concept for functions on compact abelian groups.

- If $d = 1$ then the limit object is again a function on an abelian group
- It is related to Fourier analysis and the limits of the related Cayley graphs.
- For a general $d$ the limit object is a measurable function on a $d$-step compact nilspace.

## Some surprising facts

- $d = 1$: Limit of functions on the circle is not necessarily a function on the circle.

- $d = 1$ (non-commutative case): Limit of functions on the orthogonal group in fix dimension $> 1$ is again a function on the same orthogonal group.

- The limit of simple quadratic functions such as $t \mapsto \lambda^{t^2}$ where $t = 1, 2, \ldots, n$ and $A = \mathbb{Z}_n$ can have a limit object which is a measurable function on the Heisenberg nilmanifold.

## If time allows

- What is higher order Fourier analyis?
- Spectral algorithm
- About Proofs
- More details on limits (sampling)