SZEMERÉDI!

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Quasirandom Multitype Graphs

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B.B. King is still playing, even though he is sitting down now. But, hey, I'm sitting down already.

– Ringo Starr

Quasirandom Graphs

- Every H appears right number of times

- Every H with at most four vertices appears right number of times

- Good eigenvalues

- Right number of edges and 4-cycles

- Much much more!

Multitype Graphs

K types, $V_1, \ldots, V_k$

sizes $|V_i| = \alpha_i n$, $1 \leq i \leq K$

random p_{ij} on $V_i \times V_j$

random p_{ii} on V_i

(undirected, $p_{ij} = p_{ji}$)

`DATA`: α_i, p_{ij}, p_{ii}

Szemerédi Regularity:

All G look like this!

Finite Determination

$t(H, G) =$ normalized "copynumber" H in G.

Lovász, Sós (2008): For each choice of $\mathtt{DATA}$ there is a constant L such that the following are equivalent:

- $t(H, G) \to t(H, \mathtt{DATA})$ for every H with at most L vertices.

- $t(H, G) \to t(H, \mathtt{DATA})$ for every H.

Nice Data

$d_i := \sum_j \alpha_{ij} p_{ij}$ (degree of $v \in V_i$)

`DATA` nice if d_i distinct.

Assume nice.

Here: Explicit "short" list of sufficient H.

The Sufficient H

- $\texttt{STAR}(a_1)$, $0 \leq a_1 \leq 2K$

- $\texttt{EDGESTAR}(a_1, a_2)$, $0 \leq a_1, a_2 \leq 2K - 2$

- $\texttt{CYCLESTAR}(a_1, a_2, a_3, a_4)$, $0 \leq a_1, a_2, a_3, a_4 \leq 2K - 2$

The Degrees

$f(x) = \prod(x - d_i)^2 = \sum \beta_s x^j$

$t(\mathtt{STAR}(j), G) = \sum \mathsf{deg}(v)^j$

$A := \sum \beta_s t(\mathtt{STAR}(j), G)) = \sum f(\mathsf{deg}(v))$

$A = 0$ implies all $\mathsf{deg}(v) \in \{d_1, \ldots, d_k\}$.

Sieve $g_i(x) = \prod_{j \neq i}(x - d_j)^2/(d_i - d_j)^2 = \sum \gamma_s x^s$.

$B_i := \sum \gamma_s(\mathtt{STAR}(j), G)) = \sum g_i(\mathsf{deg}(v))$

$B_i = \alpha_i$ implies α_i with $\mathsf{deg}(v) = d_i$.

Counting Edges

Sieve $h(x,y) = g_i(x)g_j(y) = \sum \kappa_{rs} x^r y^s$

$h(\text{deg}(v), \text{deg}(w)) = 1$ iff $v \in V_i$, $w \in W_i$.

Else zero

$C := \sum \kappa_{r,s} t(\texttt{EDGESTAR}(r,s))$

$C = \sum h(\text{deg}(v), \text{deg}(w))$ over $\{v, w\} \in G$.

$C = p_{ij}\alpha_i\alpha_j$ implies p_{ij} edges $V_i \times V_j$.

($i = j$ similar)

Counting 4-cycles

Sieve for $V_i \times V_j$ 4-cycles

$h(x,y,z,w) = g_i(x)g_j(y)g_i(z)g_j(w) = \sum \tau_{rsr's'} x^r y^s z^{r'} w^{s'}$

$h(\text{deg}(v), \text{deg}(w), \text{deg}(v'), \text{deg}(w')) = 1$ if $v, v' \in$

V_i, $w, w' \in W_i$, Else zero

$D := \sum \tau_{r,s,r',s'} t(\texttt{CYCLESTAR}(r,s,r',s'))$

$D = \sum h(\text{deg}(v), \text{deg}(w), \text{deg}(v'), \text{deg}(w'))$ over

4-cycles v, w, v', w'

$D =$ number of 4-cycles, $v, v' \in V_i$, $w, w' \in W_i$.

$D = p_{ij}^4 \alpha_i^2 \alpha_j^2$ implies p_{ij}^4 4-cycles

($i = j$ similar)

Bipartite Quasirandom

$G \subset T \times B$

EDGE COUNT

$\sum \text{deg}(t) = \sum \text{deg}(b) = p|T| \cdot |B|$

VEE COUNT

$\sum \text{deg}^2(t) = |T|(p|B|)^2$, $\sum \text{deg}^2(b) = |B|(p|T|)^2$

THEN: All $\text{deg}(t) = p|B|$; All $\text{deg}(b) = p|T|$

ALSO: $\sum \text{cod}(t_1, t_2) = \sum \text{deg}(b)^2 = p^2|B| \cdot |T|^2$

4-CYCLE COUNT

$\sum \text{cod}^2(t_1, t_2) = \sum \text{deg}(b)^2 = p^4|B|^2 \cdot |T|^2 = \sum \text{cod}^2(b_1, b_2)$

$\Rightarrow$: All $\text{cod}(t_1, t_2) = p^2|B|$, $\text{cod}(b_1, b_2) = p^2|T|$

Counting Extensions

$$WIT := \{((t_1,\ldots,t_u),b) : b \sim t_1,\ldots,t_u\}$$

$WIT(t_1,\ldots,t_u) = \#b$.

$WIT(b) = \#(t_1,\ldots,t_u)$.

$WIT(b_1,b_2) = \#(t_1,\ldots,t_u)$

$WIT(b) = (p|T|)^u$, $WIT(b_1,b_2) = (p^2|T|)^u$

$\sum WIT(t_1,\ldots,t_u) = \sum WIT(b) = |T|^u p^u |B|$

$\sum WIT^2(t_1,\ldots,t_u) = \sum WIT(b_1,b_2) = |T|^u (p^u|B|)^2$

ALL $WIT(t_1,\ldots,t_u) = p^u|B|$

ALL H have right count!

Quasirandom Multitype Graphs

$t(H, G)$ splits: $j \in V_{i_j}$, $1 \leq j \leq |V(H)|$. Each $(v_1, \ldots, v_u)$, $v_j \in V_{i_j}$, adjacent to *same* number of $w \in V_i$.

Number of copies of H in given sets determined pointwise.

Total number of copies is finite sum.

Determined!

What is Minimal L??

For each choice of $\texttt{DATA}$ there is a constant L such that the following are equivalent:

- $t(H,G) \to t(H,\texttt{DATA})$ for every H with at most L vertices.

- $t(H,G) \to t(H,\texttt{DATA})$ for every H.

Maybe $L = 8K - 4$. $K = 1$: Yes.

A Strange Logic

For $\alpha \in [0, 1]$:

$$COUNT^{\alpha}(x_1, \ldots, x_r)P(x_1, \ldots, x_r)$$

means number of $(x_1, \ldots, x_r)$ is $\sim \alpha n^r$.

First Order Logic?

Complete Theories?

A great part of my discoveries arise out of pure speculation; one could undoubtedly think of them as manifestations of my dreaming. I am happy with that; for isn't the act of dreaming a virtual catastrophe that gives birth to knowledge? At a time when so many scientists are busy computing; isn't it desireable for some of them - if they can do it - to dream?

René Thom