The ideas of step 1 help to show other results.

Conjecture: $\forall$ even d: $p_{n+1} - p_n = d$ i.o. (inf. often)
(de Polignac) [1849]

<u>Def</u> $\mathcal{D}_W$ = de Polignac numbers (1849) in the weaker sense $p, p+d \in \mathcal{P}$ i.o.

<u>Def</u> $\mathcal{D}_S$ = strong de Polignac numbers: $p_{n+1} - p_n = d$ i.o.

(weak de Polignac numbers could be called Kronecker numbers (1901))

Thm G. (GPY) $\mathcal{D}_W$ has a pos. lower density if $\vartheta > 1/2$

Thm H. (GPY). $|\mathcal{D}_S| \geq 1$ if $\vartheta > 1/2$

Thm 3. $\mathcal{D}_S$ has a pos. lower density if $\vartheta > 1/2$.

Remark: $\mathcal{D}_W \neq \{0\} \Leftrightarrow \underline{\lim}(p_{n+1} - p_n) < \infty$

<u>Problem</u>: Thm C asserts that

$\forall c > 0 \quad p_{n+1} - p_n < c \log p_n$ i.o.

However, is this true for $\forall c > 0$ for a set of primes with relative pos. lower density?

<u>Thm I</u> (GY, 200?) If $c > 1/4$ the answer is yes.

<u>Thm 4</u> (GPY, 201?) The answer is yes for $\forall c > 0$

<u>Remark</u>. The answer is no, if the fixed $c > 0$ is substituted by any $g(n) \to 0$ as $n \to \infty$.

The strongest hypothesis about the distribution level of primes, $\vartheta = 1$, the Elliott-Halberstam conjecture implies

(*) $p_{n+1} - p_n \leq 16$ inf. often (GPY) (E)

and $\exists$ m-term AP's with (*) $\forall m$ (Thm 1)

Problem: does there exists a plausible hypothesis $\Rightarrow \forall m\ \exists$ m-term AP of twin primes [Not. $\theta(n) = \begin{cases} \log p & n = p \\ 0 & \text{else} \end{cases}$

Thm 5. Suppose $\vartheta > 0.724$ is a distribution level for primes AND for $f(n) = \lambda(n), \lambda(n)\lambda(n+h), \overset{\theta(n)\lambda(n+h)}{\log p\, \lambda(p+h)}$ and $\lambda(p-h)\log p$, i.e. $\forall \varepsilon, A > 0$:

$$\left[\sum_{q \leq N^{\vartheta - \varepsilon}} \max_{a} \left| \sum_{n \equiv a\,(q)} f(n) \right| \ll_{\varepsilon, A} \frac{N}{\log^A N} \right]$$

Then $\forall m\ \exists$ m-term AP of primes p such that $p+h$ is prime too