ARE THERE ARBITRARILY LONG ARITHMETIC PROGRESSIONS IN THE SET OF TWIN PRIMES?

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OUR KNOWLEDGE AT THE BEGINNING OF 2004:

<u>Thm A</u> (Van der Corput, 1939): There are infinitely many 3-term AP's of primes ($\mathcal{P}$).

<u>Thm B</u>. (H. Maier 1988):

$$\Delta_1 := \liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} \le 0.2486\ldots < \frac{1}{4}$$

Definitions and notation

p, p_i primes, $\mathcal{P}$ = set of primes

$n = \prod_{i=1}^{k} p_i^{\alpha_i}$ $\quad \Omega(n) = \sum_{i=1}^{k} \alpha_i$, $\omega(n) = k$

$\mu(n) = \begin{cases} (-1)^{\Omega(n)} & \text{if } n \text{ is squarefree } \forall \alpha_i \leq 1 \\ 0 & \text{otherwise } (\exists \alpha_i \geq 2) \end{cases}$

$\lambda(n) = (-1)^{\Omega(n)} = \begin{cases} 1 & \text{if } \Omega(n) \text{ is even} \\ -1 & \text{if } \Omega(n) \text{ is odd} \end{cases}$

Def

$\mathcal{H} = \{h_i\}_{i=1}^{k}$ $\quad 0 \leq h_1 < h_2 < \dots h_k$ admissible

if # res. classes $\nu_p(\mathcal{H})$ covered by $\mathcal{H}$ mod p satisfies $\nu_p(\mathcal{H}) < p \ \forall p \iff$

$\mathfrak{S}(\mathcal{H}) := \prod \left(1 - \frac{\nu_p(\mathcal{H})}{p}\right)\left(1 - \frac{1}{p}\right)^{-k} > 0$

Def ϑ is a level of distribution for primes if $\forall \varepsilon > 0\ \forall A > 0$

$$\sum_{q \le N^{\vartheta - \varepsilon}} \max_{\substack{a \\ (a,q)=1}} \left| \sum_{\substack{p \equiv a\,(q) \\ p \le N}} \log p - \frac{N}{\varphi(q)} \right| \ll_{\varepsilon, A} \frac{N}{\log^A N} \qquad \Big(\varphi(q) = \sum_{\substack{a=1 \\ (a,q)=1}}^{q} 1 \Big)$$

Def Elliott-Halberstam Conjecture

$\vartheta = 1$ is a level of distribution for $\mathcal{P}$

Bombieri-Vinogradov Theorem:

$\vartheta = 1/2$ is a level of distribution for $\mathcal{P}$

End of 2004

Thm C (Green – Tao) $\forall m$ $(\exists \infty)$ m-term AP's in $\mathcal{P}$

Thm D $\Delta_1 = 0$ (GPY = D. Goldston, C. Yıldırım, J. Pintz)

Thm E. $\overbrace{\text{If}}^{\text{(GPY)}}$ the set of primes has a distribution level $\vartheta > 1/2$ $\overbrace{\exists d,}^{\text{then}}$ $0 < d \le C(\vartheta)$: $p_{n+1} - p_n = d$ inf. often

If $\vartheta = 1$ or even $\vartheta \ge 0.971$ then $\exists\, d \in [2, 16]$: $p_{n+1} - p_n = d$ inf. often (GPY)

Def ϑ is a level of distribution for $\mathcal{P}$ if

$$\sum_{q \le N^{\vartheta - \varepsilon}} \max_{\substack{a \\ (a,q)=1}} \Bigg| \sum_{\substack{p \equiv a\ (q) \\ p \le N}} \log p - \frac{N}{\varphi(q)} \Bigg| \ll_{\varepsilon, A} \frac{N}{\log^A N}$$

Thm (BV) $\vartheta = 1/2$ is a level of distr. for primes (1965)

<u>Thm 1</u>. If $\vartheta > 1/2$ then $\exists\, d > 0$
$d \leq C(\vartheta)$ and arbitrarily long AP's
of primes p such that $p+d \in \mathcal{P}$ too
(in fact $p+d$ is the prime following p)
for all elements of the progression

If the Elliott-Halberstam conj.
$\vartheta = 1$ is true (or $\vartheta \geq 0.971$) then $d \leq 16$.

Hope for combination of both
methods (GT + GPY): they both
use some sieve weights origina-
ting from Selberg's sieve and
used by Goldston and Yildirim
with the aim to show

$$\Delta_1 = \liminf_{n\to\infty} \frac{p_{n+1}-p_n}{\log p_n} = 0$$

GT weighted the primes with a measure

$$\rho(n) := \Big(\sum_{\substack{d \mid n \\ d \le R}} \mu(d) \log \frac{R}{d} \Big)^2 := \Lambda_R^2(n)$$

which appears in Selberg's sieve and which is a truncated version of

$$\Lambda^2(n) = \Big(\sum_{d \mid n} \mu(d) \log \frac{n}{d} \Big)^2 = \begin{cases} \log^2 p & \text{if } n = p^m \\ 0 & \text{otherwise} \end{cases}$$

$\mathcal{H} = \{h_i\}_{i=1}^{k}$

Properties of $\rho(n)$, more precisely of $\prod_{i=1}^{k} \rho(n+h_i)$ were used by GY.

Difficulties (differences in the two methods) $\mathcal{H} = \{h_i\}_{i=1}^{k}$

GT: $\prod_i \Lambda_R^2(n+h_i) = \prod_i \left(\sum_{d \le R} \mu(d) \log \frac{R}{d} \right)^2$

$n \in [N, 2N)$ $(n \sim N)$ $d \mid n+h_i$

Here $R = N^c$ c small

—o—o—o—

GPY: $\Lambda_R^2(n; \mathcal{H}, \ell) = \left(\sum_{d \le R} \mu(d) \log^{k+\ell} \frac{R}{d} \right)^2$

$\ell \to \infty$, $\ell = o(k)$ $d \mid \prod (n+h_i)$

Here $R = N^{\vartheta/2 - \varepsilon}$ $(R = N^{1/4 - \varepsilon})$

—o—o—o—

Def $\mathcal{H}$ admissible if # res. classes covered by $\mathcal{H}$ mod p, $\nu_p(\mathcal{H}) < p$ $\forall p$

Further difficulty:

GT embed the primes into the set of almost primes as a "subset" of positive relative density (in fact they embed the primes into $\mathbb{N}$, but elements of $\mathbb{N}$ are weighted, and

$$\left.\begin{array}{l}\mu(\mathbb{N})=1, \\ \mu(\mathcal{P}) = c^* > 0\end{array}\right\} \text{(subset} \Rightarrow \text{measure)}$$

and prove a relative Szemerédi thm.

<u>Szemerédi thm</u> (1975). Every subset of $\mathbb{N}$ with positive (lower) density contains arbitrarily long AP's

But (if $\vartheta > 1/2$) in GPY $\#\{p,\ p+d \in \mathcal{P},\ p \le N\}$

$$\gg N^{1-c/\log\log N},$$

much weaker than expected number

$$\mathfrak{S}(0,d)\ N/\log^2 N$$

Sketch of the idea of proof of Thm 1: Instead of combining just Thm C and E we combine Thm C, E and

<u>Thm F</u> (Halberstam – Richert 1974)
If $\mathcal{H}$ is admissible then there are ∞ many n's such that all $n+h_i$'s are ALMOST PRIMES and at least one of them is prime

<u>Def 1</u> n is a P_r almost prime if $\Omega(n) = \sum_{p \mid n} 1$ (with mult.) $\leq r$

<u>Def 2</u> n is almost prime of level c if $P^-(n) = \min_{p \mid n} p > \underset{n}{\cancel{x}}^{c} \Rightarrow P_r$ $r = \left[\frac{1}{c}\right]$

Thm 2: Let $\vartheta > 1/2$ If $k \geq k_0(\vartheta)$ then for any admissible set $\mathcal{H}_k$ $|\mathcal{H}_k| = k$ $\exists\, i, j \in \{1, 2, \ldots k\}$ $i \neq j$, $c > 0$ Such that there are arbitrarily long AP's of n's with $n + h_i \in \mathcal{P}$ $n + h_j \in \mathcal{P}$ for all elements of the AP, and for $\forall t \in [1, k]$ $P^-(n + h_t) > n^c$ for the elements of the AP. [$k_0(1) = 6$]

Step 1. Let $\vartheta > 1/2$ we obtain below N

$$\geq c_k \frac{\mathfrak{S}(\mathcal{H}) N}{\log^k N}$$ values n with

$n + h_i, n + h_j \in \mathcal{P}$, $\forall t\, P^-(n + h_t) > n^c$

(with modified GPY method and weights $\Lambda^*_R(n, \mathcal{H}, \ell)$ $R = N^{\vartheta/2 - \varepsilon}$

Step 2. Embed the pairs $p, p+d$ into the set of n's with $P^-\left(\prod (n + h_i)\right) > n^c$

(GT method $\prod_{i=1} \Lambda^2_{R_0}(n + h_i) \Leftarrow \Lambda^2_{R_0}(n)$) R_0 small

Crucial point:

The weight function of GT,

$$\varrho(n) = \Lambda_{R_0}^2(n) = \Big(\sum_{d \mid n,\, d \leq R_0} \mu(d) \log \frac{R}{d}\Big)^2$$

where $R_0 = m^c$, c small

can be substituted by

$$\varrho_{\mathcal{H}}(n) = \prod_{h_i \in \mathcal{H}} \varrho(n+h_i) = \prod_{h_i \in \mathcal{H}} \Lambda_{R_0}^2(n+h_i)$$

where $\mathcal{H}$ is a fixed admissible k-tuple $\mathcal{H} = \{h_1, \dots, h_k\}$

$$0 \leq h_i < h_{i+1} \quad (i = 1, \dots, k-1)$$

$\boxed{6B}$

$$\Lambda_R^*(n, \mathcal{H}, \ell) = \begin{cases} 0 & \text{if } (*) \\ \Lambda_R(n, \mathcal{H}, \ell) & \text{otherwise} \end{cases}$$

$$(*) \quad P^-\left(\prod_{i=1}^{k} (n+h_i)\right) < N^\delta \quad (n^\delta)$$

where $P^-(m) = \min\{p \,;\, p \mid m\}$