Combinatorial and Topological Aspects of Helly Type Theorems

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Happy Birthday, dear Endre Szemeredi! Budapest 2010.

Some basic principles

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- **Helly type theorems and other theorems in convexity have strong topological flavour.**
- **Helly-type theorems and other theorems in combinatorial geometry often have very general combinatorial underlying explanation.**

In the lecture I mentioned Trotter Szemeredi theorem as a quick motivating example.

Prologue: a problem about Families of sets

Let $\mathcal{F}_1, \mathcal{F}_2, \dots, \mathcal{F}_t$ be disjoint nonempty families of subsets of $[n] = \{1, 2, \dots, n\}$
Suppose that the following condition holds:
for every $i < j < k$ and every $R \in \mathcal{F}_i$ and $T \in \mathcal{F}_k$ there is $S \in \mathcal{F}_j$ such that $R \cap T \subset S$.

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Theorem: $f(n) \leq n^{\log n + 1}$.
Big question: Is there a polynomial upper bound?

Quasi Polynomial upper bound, remark

Suppose we start with such families and
a) Consider only sets containing an element $'m'$,
b) Remove $'m'$ from all these sets.

We will obtain a new such sequence of families, this time the ground set will have size $n-1$.
Some families in the beginning or at the and will vanish.

Quasi Polynomial upper bound, proof

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There is an element $'m'$ common to a set in the $s+1$ families and to a set in the last $r+1$ families. Therefore, if we eliminate $'m'$ the families $\mathcal{F}_{s+1}, \ldots \mathcal{F}_{t-r}$ survive.

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$t - s - r \leq f(n-1)$ and therefore $f(n) \leq f(n-1) + 2f(n/2)$.

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This would follow from a polynomial upper bound for $f(n)$ Friedrich Eisenbrand, Nicolai Hahnle, Sasha Razborov, and Thomas Rothvoss proved and almost quadratic lower bound for $f(n)$.

Helly's theorem

Helly's theorem: The family of compact convex sets in R^d has Helly number $d+1$.

A family $\mathcal{F}$ of sets has *Helly number k* if for every finite subfamily $\mathcal{G} \subset \mathcal{F}$, $|\mathcal{G}| \geq k$, if every k members of $\mathcal{G}$ have a point in common, then all members of $\mathcal{G}$ have a point in common.
And, moreover, k is the smallest integer with this property.

Topological Helly theorem

Topological Helly's theorem (proved by Helly himself!) The class of compact sets homehomorphic to a ball in R^d or empty has Helly order $d+1$.

A family $\mathcal{F}$ has *Helly order k* if for every finite subfamily $\mathcal{G}$, $|\mathcal{G}| \geq k$, with the property that all intersections of sets in $\mathcal{G}$ is in $\mathcal{F}$, if every k members of $\mathcal{G}$ have a point in common, then all members of $\mathcal{G}$ have a point in common.
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Lemma: Let $\mathcal{F}$ be a family of sets with Helly number $h(\mathcal{F})$ and Radon number $r(\mathcal{F})$, then $1 + h(\mathcal{F}) \leq r(\mathcal{F})$,

Tverberg's theorem

Let $\mathcal{F}$ be a family of subsets of X. Define $t_r(\mathcal{G})$ to be the smallest integer with the following property: Every set of $t_r(\mathcal{G})$ points from X can be divided into r parts, $X_1, X_2, \ldots, X_r$ such that for every $S_1, S_2, \ldots, S_k \in \mathcal{G}$ with $X_i \subset S_i$ there is a point in common to all the $S_i's$.

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Tverberg's theorem: The family $\mathcal{F}$ of convex sets in R^d has $t_r(\mathcal{F}) = (d+1)(r-1)+1$.

History: Birch (conjectured), Rado (proved a weaker result), Tverberg (proved), Tverberg (reproved), Tverberg & Vrecica (reproved), Sarkaria (reproved), Roundeff (reproved)

Topological Tverberg's theorem

Topological Helly theorem Let $f : \Delta^{(d+1)(r-1)} \to R^d$ be a continuous function from the $(d+1)(r-1)$ dimensional simplex to R^d. Then there are r disjoint faces of the simplex whose images have a point in common.

History: Bárány and Bajmóczy , Bárány, Shlosman and Szücs... Zivaljevic and Vrecica, Blagojecic, Matschke, and Ziegler

Eckoff's partition conjecture

Conjecture: (Eckhoff) Let $\mathcal{F}$ is a family of subsets of X closed under intersection. Suppose that $X \in \mathcal{F}$. Then

$$t_r(\mathcal{F}) - 1 \leq (r-1)(t_2(\mathcal{F}) - 1).$$

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Boris Bukh told me after the lecture that he disproved Eckhoff's conjecture!

Amenta's theorem

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Theorem (Alon-Kalai and Matousek): Let $\mathcal{F}$ be the family of union of r compact convex sets in R^d. Then the Helly order of $\mathcal{F}$ is finite.

Topological Amenta

Theorem: (Kalai and Meshulam, 2008): Let $\mathcal{F}$ be the family of union of r disjoint contractible sets in R^d. Then the Helly order of $\mathcal{F}$ is $(d+1)r$.

Combinatorial Amenta

Theorem(Eckhoff and Nischke 2008) Let $\mathcal{F}$ be a family with Helly order k, let $\mathcal{G}$ consists of unions of at most r disjoint members of $\mathcal{F}$, then $\mathcal{G}$ has Helly order kr.

The fractional Helly property

Let $\mathcal{F}$ be a family of sets. $\mathcal{F}$ satisfies **The weak fractional Helly property (WFHP)** with index k, if For every α there is β such that for every subfamily $\mathcal{G}$ of n sets if a fraction α of all k-subfamilies are intersecting then a fraction β of all members of $\mathcal{G}$ have nonempty intersection.

The strong FHP with index k: Also $\alpha \to 1$ when $\beta \to 1$.

Piercing property with index k: For every $p > k$ there is $f(p)$ such that if from every p sets k have a point in common there are $f(p)$ points such that every set contains one of them.

Theorem (Katchalski and Liu, Eckhoff, Kalai) Convex sets in R^d have the strong fractional Helly property with index $d+1$.

Theorem (Alon and Kleitman): Convex sets in R^d have the piercing property with index $d+1$.

Theorem (Alon, Kalai, Matousek, Meshulam): Weak fractional Helly implies piercing property with the same parameter.

The Barany-Matousek theorem

Integral Helly theorem (Scarf and others): Let $\mathcal{F}$ be a collection of n convex sets in R^d. If every 2^d sets in $\mathcal{F}$ have an integer point in common then there is an integer point common to all of the sets.

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Barany-Matousek Theorem
Sets of integer points in convex sets in R^d satisfy the weak fractional Helly property with parameter d+1.

In particular:
There is a positive constant $\alpha(d)$ such that the following statement holds:
Let $\mathcal{F}$ be a collection of n convex sets in R^d. If every $d+1$ sets in $\mathcal{F}$ have an integer point in common then there is an integer point common to $\alpha(d)n$ of the sets.

The Leray property

A simplicial complex is called *d-Leray* if all homology groups of dimension d or more of all induced subcomplexes vanish.

Examples: 0-Leray = complete complexes
1-Leray = chordal graphs

"Forbidden induced subcomplexes".

(immediate) d-Leray implies Helly number $\leq d+1$
(hard) d-Leray implies (strong) fractional helly

What type of properties implies (weak) fractional Helly?

Theorem: (Matousek) Bounded VC-dimension implies the weak fractional Helly property.

Conjecture (Kalai and Meshulam): Weak fractional Helly of parameter k follows from polynomial growth (like n^k) of the total Betti numbers of the nerve.

The case $k = 0$ and Gyárfás type questions!

For a graph G, $I(G)$ is the independent complex of G and $\beta(I(G))$ is the sum of (reduced) Betti numbers of $I(H)$.

Conjecture: Let G be a graph. If $\beta I(H) < K$ for every induced subgraph then $\chi(G)$ is bounded.

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This leads to very interesting **Gyárfás type** questions about uniform upper bound for the chromatic number of all graphs G with certain conditions on induced subgraphs.