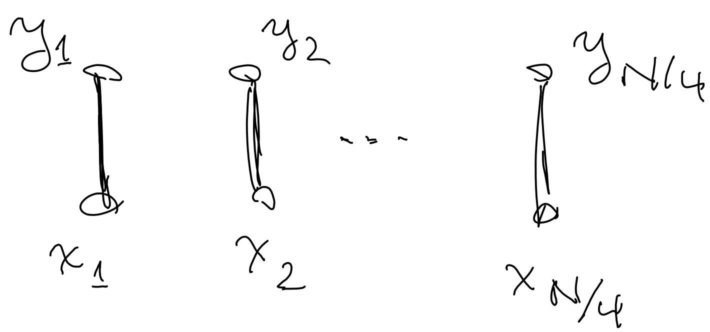
Q (Duffus–Frankl–Rödl):

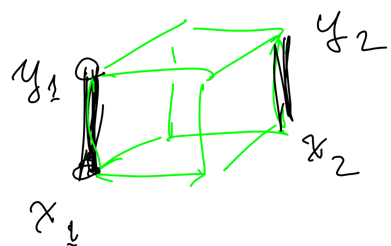
How many max'l ind. sets in $\{0,1\}^d$? ($=: Q^d$)

lower bd: $(2d)\, 2^{N/4}$ $\quad [N := 2^d]$:



← induced matching

(e.g.

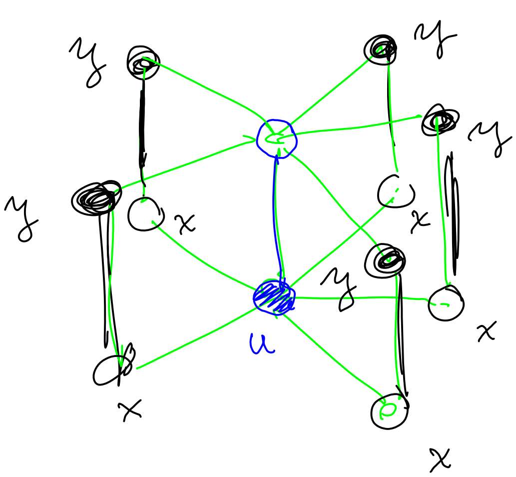


)

and do:

$|I \cap \{x_i, y_i\}| = 1 \quad \forall i$

& add still-legals



→ add u

$2^{N/4}$ I's

DFR:

Thm $N/4 < \log \text{m.i.s.}(Q^d) < 0.39N$

→ # ...

Q: asymptotics of log?

(we'll come back)

Themes:

- "sampling" (trivial?)
- entropy.

} "INFO"

E.g. of "sampling"

$\mathcal{P}_k$ proj. pl. $|\text{line}| = k$ (order $k-1$)

τ (= # pts to meet all lines) $= k$

(any line is a k-cover)

Q (Erdős & Lovász 73)

How many random lines force $\tau = k$?

ANS (K92; as conj'd) $\Theta(k \log k)$

[l.b.: HW; why?: never mind]

(V. ROUGH:) $P = \{\text{points}\}$

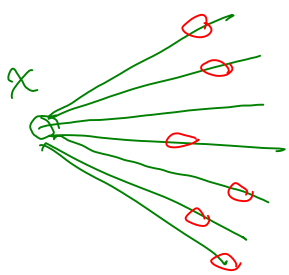
CRUX: bound

$\left|\left\{ X \in \binom{P}{k-1} : \text{few lines miss } X \right\}\right|$ (A)

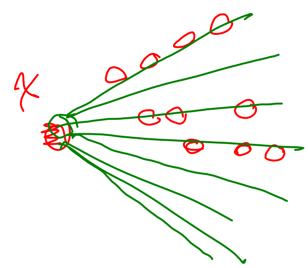
few $\approx k^2/\gamma$; bound depends on γ

X does (A) $\Longrightarrow$ most lines tangent to X

[X from x:



$x \notin X$



$x \in X$]

$\longrightarrow$

"SAMPLE": $\exists$ small $X_0 \subseteq X \ni$

$L(x, X_0)$ $\begin{cases} \text{small for most } x \in X \\ \text{large for most } x \notin X \end{cases}$

[lines through x meeting X_0]

→ choose (cheap):

- X_0
- $x \in X$ $\Rightarrow$ $L(x, X_0)$ large
- $x \notin X$ $\Rightarrow$ $L(x, X_0)$ small

(...)

"SKETCH" of X: $\{x : L(x, X_0) \text{ small}\}$

More e.g.'s

1 $i(G) :=$ # of indep't sets in G

Thm G d-regular, N vertices

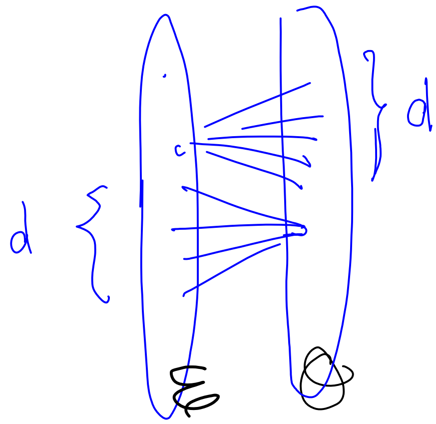
$$\Rightarrow \quad i(G) \le (2^{d+1} - 1)^{N/2d}$$

- sharp for disj't $K_{d,d}$'s
- suggested by Alon 91
  ($\leftrightarrow$ counting sum-free sets...)
- G bipartite: K 01; entropy
- gen'l: [illegible]
  (hints: (1) reduce to bipartite
  (2) a little embarrassing)

$$H(X) = \sum p(x) \log \frac{1}{p(x)}$$

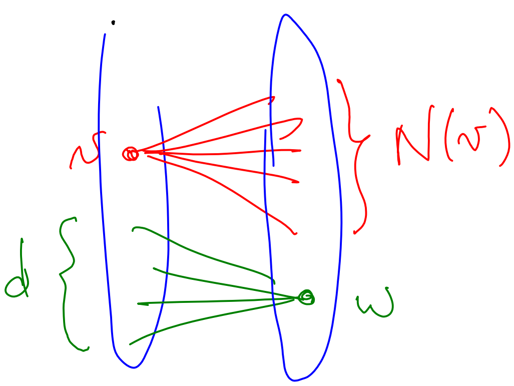
$$\left(p(x) = \Pr(X = x) \right)$$

Entropy $\longleftrightarrow$ information $\left[\log_2(\#) = \underline{\underline{\text{info}}}\right]$



$\mathbb{I}$ unif. ind.:

$$H(\mathbb{I}) = H(\mathbb{I} \cap \mathcal{O}) + H(\mathbb{I} \cap \mathcal{E} \mid \mathbb{I} \cap \mathcal{O})$$



IDEA: info in one of

$\mathbb{1}_{\{v \in \mathbb{I}\}}$, $\mathbb{I} \cap N(v)$

MAIN: Shearer's Lemma $\longrightarrow$

$$H(\mathbb{I} \cap \mathcal{O}) \leq d^{-1} \sum_{v \in \mathcal{E}} H(\mathbb{I} \cap N(v))$$

$w \in \mathcal{O}$: "$\sum$" gives info on w d times

2 **thm** (Korshunov-Sapozhenko 83
⇝ Sapozhenko 89 ("sampling"))*

$$i(Q^d) \sim 2\sqrt{e}\, 2^{2^{d-1}}$$

even/odd

"flaws"

all-even ind sets

⇝

* exposition: D. Galvin http...

3 $G = \mathbb{Z}_{2N}^d$ — bipartite

d large, fixed; N arb. [OR $\mathbb{Z}^d$ ...]

x, y even vertices; z odd vertex [think: far apart]

Thm (Galvin–K 04)

I unif. ind set:

$$\Pr(x \in I \mid y \in I) \approx 1/2$$

$$\Pr(x \in I \mid z \in I) \approx 0$$

← pub date

← we never got around to writing it date.

Thm (GKRS ~03, Peled 10)

σ unif 3-coloring (R, B, G):

$$\Pr(\sigma(x) = R \mid \sigma(y) = R) \approx 2/3$$

Conj: Sim. for k colors (& $d > d(k)$)

[4] φ unif $\in \{$homo's$: Q^d \to \mathbb{Z}$ w. $\underline{0} \to 0\}$

Thm (K 01) — entropy (& ...)

$$\exists\, b: \Pr(R(\varphi) > b) < e^{-\Omega(d)}$$

$R(\varphi) := |\text{range}(\varphi)|$

[**Conj** (Benjamini–Häggström–Mossel)

$$\Pr(R(\varphi) > \varepsilon n) \to 0 \quad \forall \varepsilon$$]

Thm (Galvin 03, conj'd K) — entropy + "Sapozhenko"

$$\Pr(R(\varphi) > 5) < e^{-\Omega(d)}$$

(5 → $\geq$)

$$(\Pr(R(\varphi) = 5) > \Omega(1))$$

Conj (BHM) ~~Q^d~~ n-vertex, bip. G

→ $\mathbb{E}\, R(\varphi)$ (e.g.) max for path

Back to maximal indept sets

[Nay, this is too much, to remember at night all the foolish things that were said in the morning.

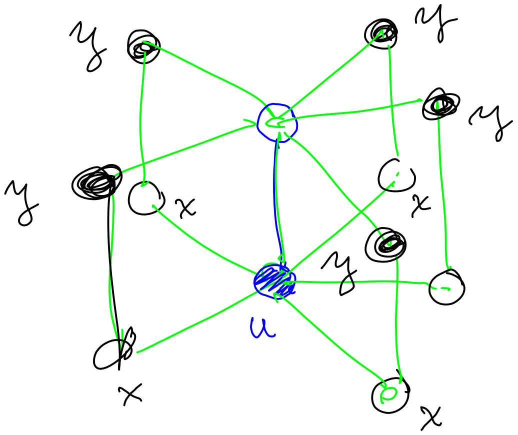
— Mr. Bingley]

Thm (Ilinca-K) G d-regular on N $\Rightarrow$

$$\log \text{m.i.s.}(G) < (1+o_d(1)) \begin{cases} N/4 & \text{bipartite} \\ \frac{\log 3}{6} N & \text{gen'l} \end{cases}$$

[HW: conjecture worst examples]

[d-reg, N vertices, bipartite]



← canon: (IDEA)

$I \cap N(u)$ typ. large

step 1: $J \sim \mathrm{Bin}(I, d^{-\varepsilon}) \to$

(a) $|J| \lesssim d^{-\varepsilon} N$ (small)

(b) $|\{x : |I \cap N(x)| > d^{2\varepsilon},\ x \nsim J\}|$ small

→ cheap: specify J & these

step 2 $X = \{x :$ still don't know$\}$

$\subseteq \{x : |I \cap N(x)| < d^{2\varepsilon}\}$ $[X \supseteq I \setminus J]$

step 2 $X = \{x : \text{still don't know}\}$:

$|I \cap N(x)| \leq d^{2\varepsilon} \quad \forall x \in X \quad (\circledast)$

$Y = \{x \in X : d_x(y) > d^{3\varepsilon}\}$

$\mathbb{I}$ unif from $\{I\text{'s } \bar{\omega}\ (\circledast)\} \rightarrow$

$\log(\#\text{ of } I \cap Y\text{'s}) \leq H(\mathbb{I} \cap Y)$

$$\leq d^{-3\varepsilon} \sum_{x \in X} H(\mathbb{I} \cap N_Y(x))$$

Shearer

$\lesssim d^{2\varepsilon} \log d$ by $(\circledast)$

$\lesssim (d^{-\varepsilon} \log d) N$ (cheap) step 3 $Z = X \setminus Y$ (need $I \cap Z$)

(Im) all degrees in Z $< d^{2\varepsilon}$

$\xRightarrow{\text{EX}}$ $|Z| \lesssim N/2$

$\Longrightarrow$ # of $I \cap Z$'s $\lesssim 2^{N/4}$

TRIV: G bipartite on $A \cup B$ $\Longrightarrow$

m.i.s.$(G) \leq \min\{2^{|A|}, 2^{|B|}\}$

☐

Non-example (?): # of k-SAT fns*

more or less

$\mathcal{C} \subseteq \{$ k-clauses from $x_1, \ldots, x_n \}$

e.g. $x_3 \bar{x}_{22} x_{91}$ $(k=3)$

with witnesses: $\forall c \in \mathcal{C}$ $\exists w_c \in \{0,1\}^n$:

$c \models w_c$, $c' \not\models w_c$ $\forall c' \in \mathcal{C} \setminus \{c\}$

e.g. $x_1 x_2 \longleftrightarrow (1 1 0 \ldots, 0)$

$x_2 x_5 \longleftrightarrow (0 1 0 0 1 0 \ldots 0)$

$x_1 x_3 \longleftrightarrow (1 0 1 0 \ldots 0)$ ...

$\rightarrow$ $2^{\binom{n}{k}}$ $\mathcal{C}$'s

better: $2^{n + \binom{n}{k}}$ $\mathcal{C}$'s (choose literals)

---

* Bollobás-Brightwell-Leader, P. Allen, Ilinca-K.

Conj 1 $\log \sim \binom{n}{k}$ OPEN $k > 3$

$\cap \mid$

Conj 2 $\# \sim 2^{n + \binom{n}{k}}$ TRUE $k = 2, 3$

⇒ **simple** ("sampling") proof of Conj 1 for $k = 2$?