A proof of the stability of extremal graphs.

Simonovits' stability from Szemerédi's regularity

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ABSTRACT:

We present a concise, contemporary proof (i.e., one using Szemerédi's regularity lemma) for the following classical stability result of Simonovits 1968:

If an n-vertex F-free graph G is almost extremal, $\chi(F) = p + 1$, then the structure of G is close to a p-partite Turán graph.

More precisely, for $\forall\ F$ and $\varepsilon > 0$, $\exists\ \delta > 0$ and n_0 (depending on F and ε) such that if $n > n_0$ and

$$e(G) > (1 - \frac{1}{p})\binom{n}{2} - \delta n^2$$

then one can **change** (add and delete) at most εn^2 edges of G and obtain a complete p-partite graph.

Notations

$[n] := \{1, 2, \ldots, n\}$

G_n graph on n vertices

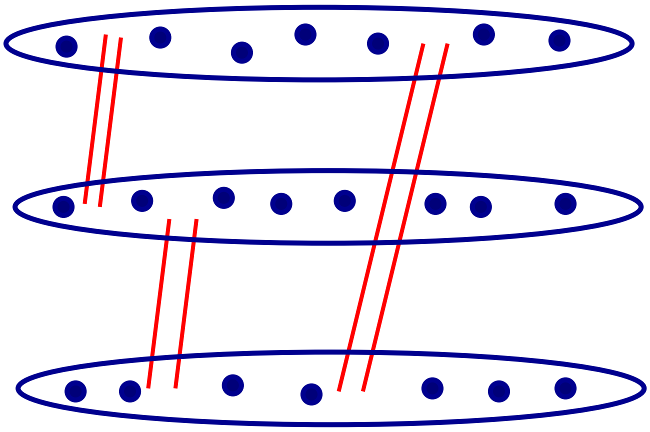
$\chi(G)$:= chromatic number,

$e(G)$:= number of edges,

$\deg_G(x)$ degree of vertex x of graph G

$N_G(x) \subset V$, neighborhood $\quad (x \notin N(x))$

$T_{n,p}$:= the Turán graph,
the p-chromatic graph having the most edges.



The Turán Graph

Turán's theorem
Turán type graph problems

Theorem. Mantel (1903) (for K_3)

Turán (1940)

$$e(G_n) > e(T_{n,p}) \quad \Longrightarrow \quad K_{p+1} \subseteq G_n.$$

Unique extremal graph for K_{p+1}.

E.g.: the largest triangle-free graph is the complete bipartite one with $\lfloor n^2/4 \rfloor$ edges.

General question: Given a family $\mathcal{F}$ of forbidden graphs, what is the maximum of $e(G_n)$ if G_n does not contain subgraphs $F \in \mathcal{F}$?

Notation: $\mathbf{ext}(n, \mathcal{F}) := \max e(G)$

$$\mathbf{ext}(n, K_{p+1}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + O(n).$$

Degree majorization

Theorem (Erdős, 1970)
Suppose G *is* K_{p+1}*-free. Then*
there is a p*-chromatic* H *on the same vertex set,*

$$V(H) = V(G)$$

with

$$\deg_H(x) \geq \deg_G(x).$$

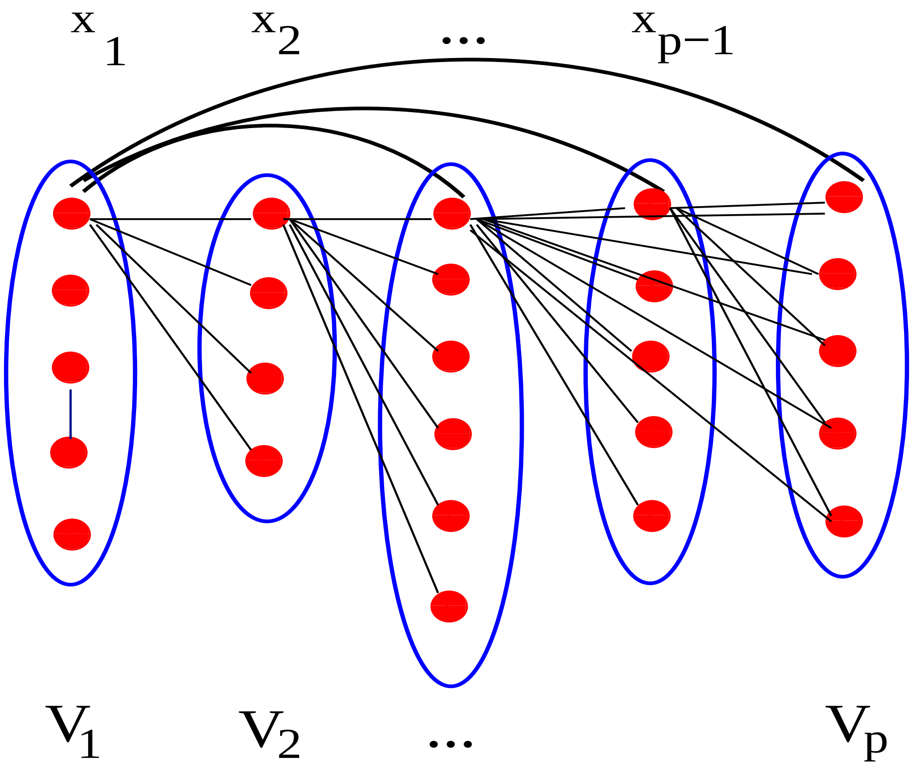
H majorizes the degrees of G for every $x \in V$.

Proof of Erdős' degree majorization: (An Algorithm.)

Input: G (with no K_{p+1})
Output: $V_1, V_2, \ldots, V_p$ a p-partition of $V(G)$.
$H := K(V_1, \ldots, V_p)$ a p-partite complete graph

Let $x_1 :=$ a vertex with max degree.
Let $V_1 := V \setminus N(x_1)$.

Let $x_i :=$ a vertex with max degree on the graph of the rest of the vertices, of $G - (V_1 \cup \ldots V_{i-1})$.
Procedure stops in p steps, $\{x_1, x_2, \ldots, x_p\}$ spans a complete graph. Q.e.d.



$$\forall y \in V_1 \quad \mathsf{deg}(y) \leq \mathsf{deg}(x_1) =: d$$

$$e(V_1) + e(V_1, \overline{V_1}) \leq d(n-d).$$

Toward a general theory: The Erdős-Stone theorem (1946)

$$\mathbf{ext}(n, K_{p+1}(t, \ldots, t)) = \mathbf{ext}(n, K_{p+1}) + o(n^2).$$

Here $K_{p+1}(t, \ldots, t)$ is the blow up of the K_{p+1}.
It is a $T_{t(p+1),p+1}$.

General asymptotics

Erdős-Stone-Simonovits (1946), (1966)

If

$$\min_{F \in \mathcal{F}} \chi(F) = p + 1$$

then

$$\mathbf{ext}(n, \mathcal{F}) = \left(1 - \frac{1}{p}\right)\binom{n}{2} + o(n^2).$$

The asymptotics depends only on the **minimum chromatic number**.

How to prove the asymptotic from Erdős-Stone?

- pick $F \in \mathcal{F}$ with $\chi(F) = p + 1$.
- pick t with $F \subseteq K_{p+1}(t, \ldots, t)$.
- apply Erdős-Stone:

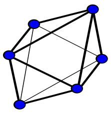
$$\begin{aligned} \mathbf{ext}(n, \mathcal{F}) \le \mathbf{ext}(n, F) &\le \mathbf{ext}(n, K_{p+1}(t, \ldots, t)) \\ &\le e(T_{n,p}) + \varepsilon n^2. \end{aligned}$$

On the other hand

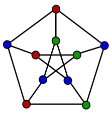
$$e(T_{n,p}) \le \mathbf{ext}(n, \mathcal{F}).$$

Q.e.d.

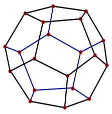
Corollary $O_6 =$ *Octahedron graph*, $P_{10} =$ *Petersen graph*, $D_{20} =$ *Dodecahedron graph*. $(\chi = 3.)$



$$\mathbf{ext}(n, O_6) = \frac{1}{4}n^2 + o(n^2)$$

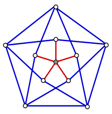


$$\mathbf{ext}(n, P_{10}) = \frac{1}{4}n^2 + o(n^2)$$

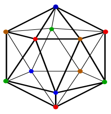


$$\mathbf{ext}(n, D_{20}) = \frac{1}{4}n^2 + o(n^2).$$

Corollary $G_{11} =$ *Groetsch graph*, $I_{12} =$ *Icosahedron graph*. (Here $\chi = 4$.)

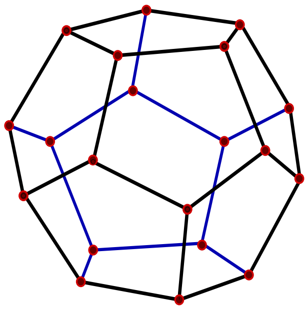


$$\mathbf{ext}(n, G_{11}) = \frac{1}{3}n^2 + o(n^2)$$

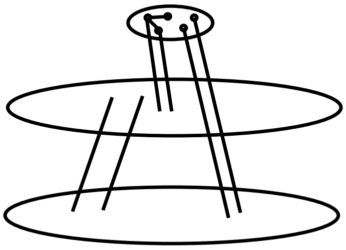


$$\mathbf{ext}(n, I_{12}) = \frac{1}{3}n^2 + o(n^2).$$

Exact Extrema: Simonovits' Dodecahedron Theorem



Dodecahedron: D_{20} $\qquad H(n,2,5) := T_{n-5,2} \otimes K_5.$



$H(n,2,5)$

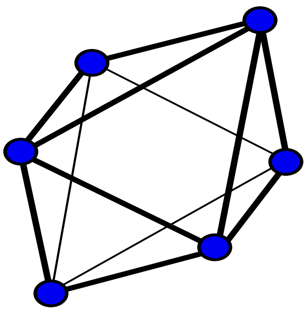
For D_{20}, $H(n,2,5)$ is the (only) extremal graph for $n > n_0$.

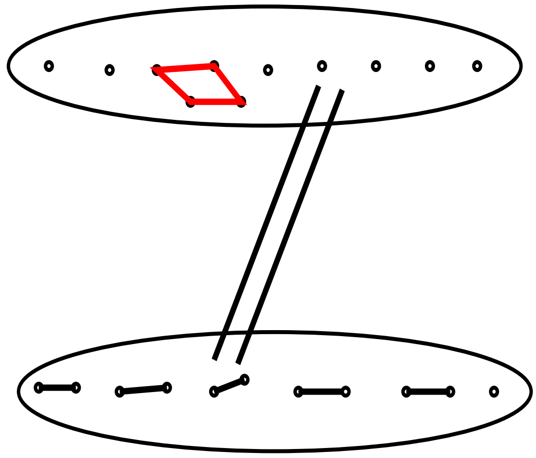
$H(n,2,5)$ cannot contain a D_{20} since one can delete 5 points of $H(n,2,5)$ to get a bipartite graph but one cannot delete 5 points from D_{20} to make it bipartite.

Octahedron Theorem

Erdős-Simonovits (Bollobás, Erdős, Simonovits, Szemerédi)

For $n > n_0$, the extremal O_6-free graph is
a complete bipartite graph +
on one side an extremal C_4-free +
on the other side a matching.





Excluded: octahedron, O_6

extremal graph

STRUCTURAL STABILITY

Each extremal graph was close to some $T_{n,p}$. More is true:

If $p+1=\chi(F)$ then the extremal or almost extremal graphs are very similar to $T_{n,p}$.

Erdős and Simonovits

For every $\varepsilon > 0$ and F there is a $\delta > 0$, and n_0 such that if $F \not\subseteq G_n$, $n > n_0$ and

$$e(G_n) \geq \left(1 - \frac{1}{p}\right)\binom{n}{2} - \delta n^2,$$

then

$$E(G_n) \triangle E(T_{n,p}) \leq \varepsilon n^2.$$

Almost extremal F-free graphs are almost p-colorable.

I.e., one can change (add and delete) at most εn^2 edges of G and obtain a complete p-partite graph.

(Same result for the class $\mathcal{F}$, instead of F).

AIM of talk:
to present a new proof for the Stability Thm.

Stability of $\mathbf{ext}(n, K_{p+1})$

First: stability for K_{p+1}.
Very first: **Large p-chromatic subgraphs**

Theorem 1 *(ZF 2010, new and simple)*
Suppose $K_{p+1} \not\subset G$, $|V(G)| = n$ *and*

$$e(G) \geq e(T_{n,p}) - t.$$

Then there exists a p-chromatic subgraph H_0, $E(H_0) \subset E(G)$ such that

$$e(H_0) \geq e(G) - t.$$

Large p-chromatic subgraphs:

There are other (more exact) stability results, but!
Advantage of this one:
No $\varepsilon, \delta, n_0, \ldots$, it is true for every n and t.

If $t < n/(2p) - O(1)$ then G itself is p-chromatic
(Hanson, Toft 1991), there is no need to delete.

E. Győri 1987, 1991

$$e(H_0) \geq e(G) - O(t^2/n^2).$$

One can delete at most $e/2$ edges to make G bipartite
(Erdős). (at most e/p to make it p-chromatic)
Gen's: Alon 1996, Tuza et al., Bollobás & Scott, . . .

Proof of Thm. 1 ($\exists$ large p-partite $H_0 \subset G$)

Algorithm. Input: G

Output: $V_1, V_2, \ldots, V_p$, a partition of $V(G)$ such that

$$\sum_i e(G|V_i) \le t.$$

Consider the previous partition $V_1, V_2, \ldots, V_p$, $d_i = \deg(x_i)$.
We have $\deg_{G|V_i \cup V_{i+1} \cdots \cup V_p}(y) \le d_i$ for $y \in V_i$. Then

$$e(G) \le \sum |V_i| \times d_i = e(K(V_1, V_2, \ldots, V_p)) \le e(T_{n,p}).$$

However! edges inside V_i are counted twice:

$$e(G) + \sum_i e(G|V_i) \le e(T_{n,p}) \quad \Longrightarrow \quad \sum \le t. \quad \Box$$

Stability of $\mathbf{ext}(n, K_{p+1})$

Theorem 2 *(ZF 2010)*
Suppose G_n *is* K_{p+1}*-free with* $e(G) \geq e(T_{n,p}) - t$*.*
Then $\exists$ *a complete* p*-chromatic graph* H*,*
$V(H) = V(G)$*, such that*

$$|E(G) \triangle E(H)| \leq 3t.$$

Proof: Delete t edges to make it p-partite,
add at most $2t$ to make it complete p-partite.
Q.e.d.

Proof of stability of F

TOOLS:
Theorem 2. i.e., the stability for K_{p+1}.
(We suppose $\chi(F) = p + 1$.)

&

Szemerédi's regularity lemma.
Usually we use 'Counting lemma' + 'Removal lemma' + 'Blow-up lemma.'
Here we will use a corollary we call: Subgraph lemma.

$\boxed{\textbf{Szemerédi's Lemma asserts:}}$

that **every** graph can be approximated by quasi-random graphs.

Basic notion of **quasi-randomness:**

Def: $G(A,B)$ bipartite graph is α-**regular**, (α-quasi random) if for all $A' \subset A$, $|A'| \geq \alpha|A|$ and $B' \subset B$, $|B'| \geq \alpha|B|$ one has

$$|d_G(A',B') - d_G(A,B)| \leq \alpha,$$

where

$$d_G(A',B') = \frac{e(G[A',B'])}{|A'||B'|}$$

is the **density** of the induced bipartite subgraph $G[A',B']$.

The Regularity Lemma

Theorem (Szemerédi 1978)
For **every** $\alpha > 0$ *and integer* ℓ_0, *there exist integers* $L_0 = L_0(\alpha, \ell_0)$ *and* $n_0 = n_0(\alpha, \ell_0)$ *so that for* **every graph** $G = (V, E)$, $|V| \geq n_0$, V *admits a partition* $V = V_1 \cup \ldots \cup V_L$, $\ell_0 \leq L \leq L_0$, *satisfying*
(i) $|V_1| \leq |V_2| \leq \ldots \leq |V_L| \leq |V_1| + 1$ *and*
(ii) *all but at most* $\alpha\binom{L}{2}$ *pairs* (V_i, V_j), $1 \leq i < j \leq L$, *are* α*-regular*.

In many versions: we also have a set $V_0 \subset V$, $|V_0| < \alpha|V|$, a small set of 'exceptional' vertices.

The Cluster Graph

Given G, having a Szemerédi-partition
$V = V_0 \cup V_1 \cup \ldots \cup V_L$, most applications use the Cluster Graph (= reduced graph, skeleton graph).

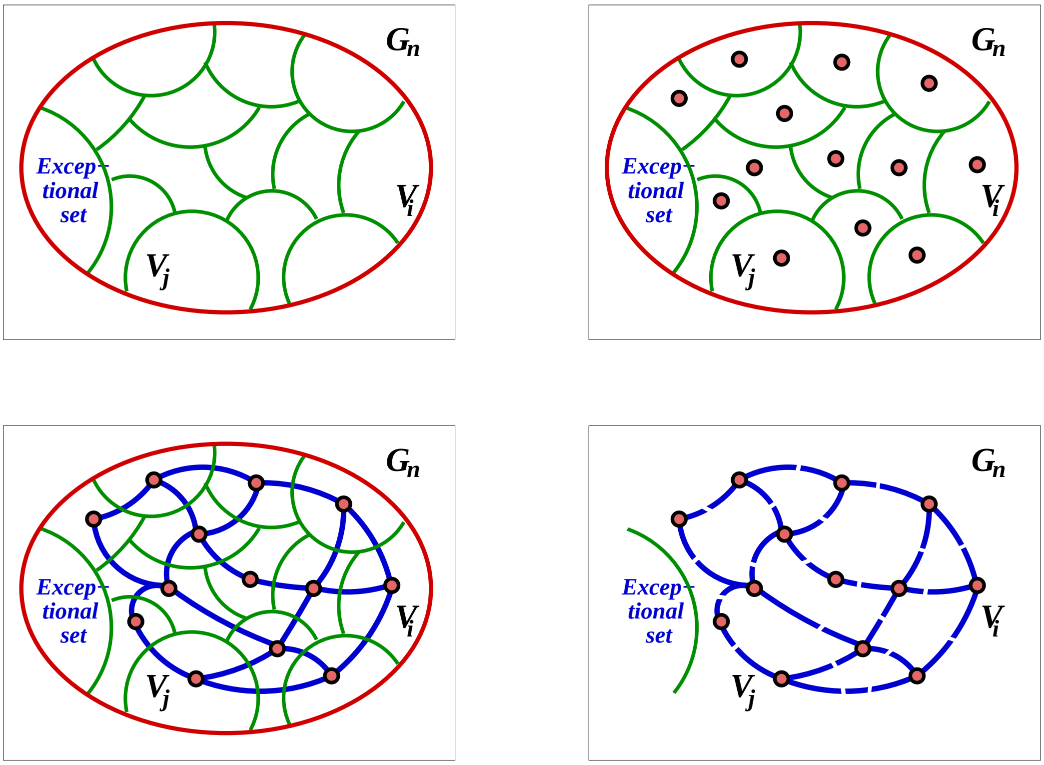
$G \to R$ β-**reduced cluster graph**
where R has L vertices $\{1, 2, \ldots, L\}$ and
$(i, j) \in E(R)$ if (V_i, V_j) is α-regular with density $\geq \beta$.

Here we will have $\beta >> \alpha$, but still small, defined later.

How to obtain the cluster graph R

Start with G, α. Add new vertices $\{1, 2, \ldots, L\}$

Identify β-dense α-reg pairs. Obtain R.



The Subgraph Lemma

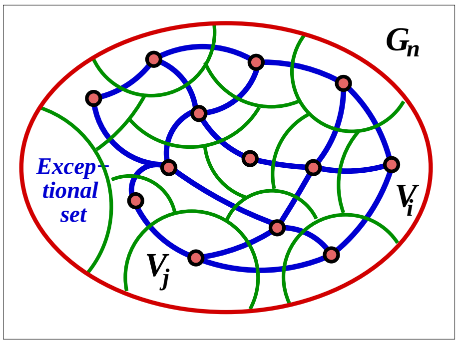
Many properties of G are inherited by R,
one can count embeddings, homomorphisms.

Subgraph Lemma. (Folklore, easy corollary.)
If G is F-free, $\beta > 2\alpha^{1/p^2}$, then R is K_{p+1}-free.

Early forms: Szemerédi, Ruzsa & Szemerédi.
Contemporary forms: Lovász, Szegedy, Elek, et al.
Surveys: Komlós-Simonovits 1996,
Komlós-Shokoufandeh-Simonovits-Szemerédi 2002.
For hypergraphs: Frankl, Rödl, Nagle, Skokan, Solymosi, Tao, Gowers etc.

How to prove Erdős-Stone?

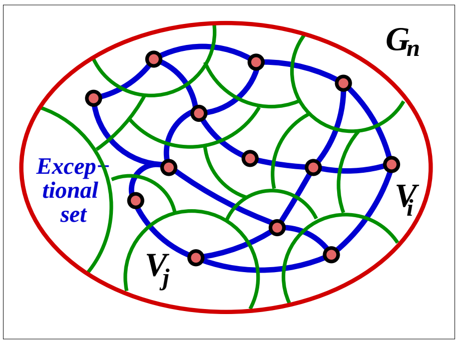
Start with an F-free graph G_n, $\chi(F) = p + 1$.



- No K_{p+1} in the Cluster graph R
- Apply Turán's theorem, $e(R) \leq (1 - \frac{1}{p})\binom{L}{2} + L$
- Estimate the edges of the original graph:

$$e(G_n) \leq e(R) \left(\frac{n}{L}\right)^2 + O(\alpha + \beta)n^2.$$

How to prove Stability?



- No K_{p+1} in the Cluster graph R
- Apply Turán's theorem with stability
- Estimate the edges of the original graph

Sketch of the proof of stability for F

Given G with $n > n_0$, G is F-free, $\chi(F) = p + 1$ and $e(G) > (1 - \frac{1}{p})\binom{n}{2} - \delta n^2$. ($\delta$ will be defined later).

Apply Szemerédi's regularity lemma to G, with $\alpha > 0$.
Obtain regular partition $V_1, \ldots, V_L$.
Leave out edges of
– irregular pairs
– inner edges (inside V_i's)
– low density pairs (less than density β), i.e.,
consider the β-reduced cluster graph R on $\{1, 2, \ldots, L\}$.

By the subgraph lemma: R is K_{p+1}-free.
So $e(R) \leq (1-\frac{1}{p})\binom{L}{2} + L$ by Turán.
On the other hand

$$e(R)(\frac{n}{L})^2 + O(\alpha+\beta)n^2 > e(G) > \frac{p-1}{2p}n^2 - \delta n^2.$$

Hence $e(R) > e(T_{L,p}) - (\alpha+\beta+\delta)L^2$.
Remainder term: t, it is 'small'.

Use stability for K_{p+1}: One can change $3t$ edges of R to get a complete p-partite one.
This corresponds changing $O(tn^2/L^2)$ edges of G to make it complete p-partite. Q.e.d.