Zero-one k-laws for small k

M.E. Zhukovskii
MIPT, MSU, Yandex

We study asymptotical behavior of the probabilities of first-order properties for Erdős–Rényi random graphs $G(n, p(n))$ with $p(n) = n^{-\alpha}$, $\alpha \in (0, 1)$. The following zero-one law was proved in 1988 by S. Shelah and J.H. Spencer [1]: if α is irrational then for any first-order property L either the random graph satisfies the property L asymptotically almost surely or it doesn't satisfy (in such cases the random graph is said to *obey zero-one law*). When $\alpha \in (0, 1)$ is rational the zero-one law for these graphs doesn't hold.

Let k be a positive integer. Denote by $\mathcal{L}_k$ the class of the first-order properties of graphs defined by formulae with quantifier depth bounded by the number k (the sentences are of a finite length). Let us say that the random graph obeys *zero-one k-law*, if for any first-order property $L \in \mathcal{L}_k$ either the random graph satisfies the property L almost surely or it doesn't satisfy. Since 2010 we prove several zero-one laws for rational α from $I_k = (0, \frac{1}{k-2}] \cup [1 - \frac{1}{2^{k-1}}, 1)$. For some points from I_k we disprove the law. In particular, for $\alpha \in (0, \frac{1}{k-2}) \cup (1 - \frac{1}{2^k-2}, 1)$ zero-one k-law holds. If $\alpha \in \{\frac{1}{k-2}, 1 - \frac{1}{2^k-2}\}$, then zero-one law does not hold (in such cases we call the number α k-critical).

From our results it follows that zero-one 3-law holds for any $\alpha \in (0, 1)$. Therefore, there are no 3-critical points in $(0, 1)$. Zero-one 4-law holds when $\alpha \in (0, 1/2) \cup (13/14, 1)$. Numbers 1/2 and 13/14 are 4-critical. Moreover, we know some rational 4-critical and not 4-critical numbers in $[7/8, 13/14)$. Recently we obtain new results concerning zero-one 4-laws for $\alpha \in (1/2, 7/8)$ and, thereby, narrow the gap.

References

[1] S. Shelah, J.H. Spencer, *Zero-one laws for sparse random graphs*, J. Amer. Math. Soc. **1**: 97–115, 1988.