Matchings in balanced hypergraphs

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We investigate d-matchings and d-vertex covers in balanced hypergraphs $H = (V, E)$ where a weight function $d : E \to \mathbb{N}$ is given. The d-matching number $\Upsilon_d(H)$ is the maximum value of $\sum_{m \in M} d(m)$ where M is a matching in H. Some function $x : V \to \mathbb{N}$. is called a d-vertex cover if the inequality $\sum_{v \in e} x(v) \geq d(e)$ holds for every edge $e \in E$. The d-vertex cover number $\tau_d(H)$ is the minimal value of $\sum_{v \in V} x(v)$ where x is a d-vertex cover in H.

Berge and Las Vergnas (Annals of the New York Academy of Science, **175**, 1970, 32-40) proved what may be called Kőnig's Theorem for balanced hypergraphs, namely $\Upsilon_d(H) = \tau_d(H)$ for all weight functions $d : E \to \mathbb{N}$ Conforti, Cornuéjols Kapoor and Vušković (Combinatorica, **16**, 1996, 325-329) proved that the existence of a perfect matching is equivalent to the following analogue of Hall's condition: If some vertices are colored red and blue, and if there are more blue than red vertices in total, then there is an edge containing more blue than red vertices. This generalizes Hall's Theorem for bipartite graphs.

We prove a Min-Max Theorem which generalizes both results. In particular, we obtain a defect version of the generalized Hall Theorem. The proof is purely combinatorial.

Let $H = (V, E)$ denote a balanced hypergraph and assume that a second weight function $b : V \to \mathbb{N}$ is given. Define the weight of a partial hypergraph H' of H as

$$w(H') := \sum_{e \in E(H')} d(e) - \sum_{v \in V(H')} (\deg_{H'}(v) - 1) b(v).$$

Consider the optimization problem of maximizing $w(H')$ over all partial hypergraphs $H' \subseteq H$.

As a dual notion, let

$$X := X(H, d, b) := \{x | x \text{ is a } d-\text{ vertex cover and } 0 \leq x(v) \leq b(v) \text{ for all } v \in V\}.$$

Main result:

Theorem 1. *Let $H = (V, E)$ be a balanced hypergraph and $d : E \to \mathbb{N}$ and $b : V \to \mathbb{N}$, such that for all $e \in E : \sum_{v \in e} b(v) \geq d(e)$. Then the following minimax-relation holds:*

$$\max_{H' \subseteq H} w(H') = \min_{x \in X} \sum_{v \in V} x(v)$$