Improved bounds for Pach's selection theorem

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March 15, 2014

Abstract

We improve the estimates on the selection constant in the following geometric selection theorem by Pach: For every positive integer d there is a constant $c_d > 0$ such that whenever $X_1, \dots, X_{d+1}$ are n-element subsets of $\mathbb{R}^d$, then we can find a point $\mathbf{p} \in \mathbb{R}^d$ and subsets $Y_i \subseteq X_i$ for every $i \in [d+1]$, each of size at least $c_d n$, such that $\mathbf{p}$ belongs to all *rainbow* d-simplices determined by $Y_1, \dots, Y_{d+1}$, that is, simplices with one vertex in each Y_i.

We provide a lower bound $c_d > 2^{-2^{d^2+o(d)}}$, which is doubly exponentially decreasing in d (up to a polynomial in the exponent). For comparison, Pach's original approach yields a triply exponentially decreasing lower bound. We also show an exponentially decreasing upper bound $c_d \leq \kappa^d$ for a suitable constant $\kappa < 1$.

For the lower bound, we improve the 'separation' part of the argument by showing that in one of the key steps only $d+1$ separations are necessary, compared to 2^d separations in the original proof. In our construction for the upper bound, we use the fact that the minimum solid angle of every d-simplex is exponentially small. This fact was previously unknown and might be of independent interest.