On cliques in diameter graphs.

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Our talk is devoted to the study of the properties of cliques in diameter graphs.

Let us remind the definition of a diameter graph.

Definition. *A graph $G = (V, E)$ is a* **diameter graph** *in $\mathbb{R}^d$ (on S^d_r), if $V \subset \mathbb{R}^d$ (S^d_r) is a finite set of diameter 1 and edges of G are formed by vertices that are at unit distance apart.*

Note that we assume of the sphere being embedded in $\mathbb{R}^{d+1}$, and the unit distance included from the ambient space.

Diameter graphs arise naturally in the context of Borsuk's problem. In 1933 Borsuk [3] asked whether any set of diameter 1 in $\mathbb{R}^d$ can be partitioned into $(d+1)$ parts of strictly smaller diameter. The positive answer to this question is called Borsuk's conjecture. This was shown to be true in dimensions up to 3. In 1993 Kahn and Kalai [6] constructed a finite set of points in dimensions 1325 that does not admit a partition into 1326 parts of smaller diameter. The minimal dimension in which the counterexample is known is 64 (see [2], [5]).

We focus on one conjecture, posed by Morić and Pach [12].

Conjecture 1. *Any two d-cliques in a diameter graph in $\mathbb{R}^d$ must share at least $(d-2)$ vertices.*

This was proved for $d = 2$ by Hopf and Pannwitz in [7] and for $d = 3$ by V. Dol'nikov in [4]. As it is shown in [12], the following conjecture (Schur et al., [13]) reduces to Conjecture 1: in any diameter graph G in $\mathbb{R}^d$ the number of d-cliques is not greater than the number of vertices.

In [9] we proved Conjecture 1, which is the main topic of our talk. In the papers [1], [8], [10] we studied similar properties of cliques in diameter graphs in the space $\mathbb{R}^d$ and on the sphere S^d_r of radius $r > 1/\sqrt{2}$.

We note that some questions related to Conjecture 1 were studied in different terms by Maehara in [11]. In that paper he studies sphericity of complete bipartite graphs, where sphericity of a graph is the smallest dimension d such that the vertices of a graph can be represented by closed unit balls in $\mathbb{R}^d$ with two balls intersecting exactly if two corresponding vertices are adjacent. The result he obtained almost gives the following weaker version of Conjecture 1: any two d-cliques in a diameter graph in $\mathbb{R}^d$ must share at least one vertex.

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