Three-monotone interpolation*

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Abstract

A function $f\colon \mathbb{R} \to \mathbb{R}$ is called *k-monotone* if it is $(k-2)$-times differentiable and its $(k-2)$nd derivative is convex. A point set $P \subset \mathbb{R}^2$ is *k-monotone interpolable* if it lies on a graph of a k-monotone function. These notions have been studied in analysis, approximation theory etc. since the 1940s.

We show that 3-monotone interpolability is very non-local: we exhibit an arbitrarily large finite P for which every proper subset is 3-monotone interpolable but P itself is not. On the other hand, we prove a Ramsey-type result: for every n there exists N such that every N-point P with distinct x-coordinates contains an n-point Q such that Q or its vertical mirror reflection are 3-monotone interpolable. The analogs for k-monotone interpolability with $k = 1$ and $k = 2$ are classical theorems of Erdős and Szekeres, while the cases with $k \geq 4$ remains open.

We also investigate the computational complexity of deciding 3-monotone interpolability of a given point set. Using a known characterization, this decision problem can be stated as an instance of polynomial optimization and reformulated as a semidefinite program. We exhibit an example for which this semidefinite program has only doubly exponentially large feasible solutions, and thus known algorithms cannot solve it in polynomial time. While such phenomena have been well known for semidefinite programming in general, ours seems to be the first such example in polynomial optimization, and it involves only univariate quadratic polynomials.

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