On random subgraphs of a Kneser graph.

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Our talk is devoted to the study of a Kneser graph $KG_{n,k}$. The vertices of the graph are the k-subsets of n-element set. Two k-sets are joined by an edge if they are disjoint. These graphs were first investigated by Martin Kneser [3]. He showed that $\chi(KG_{n,k}) \le n - 2k + 2$ and conjectured that this bound is tight. The conjecture was proved by László Lovász [4] over 20 years later. He used tools from algebraic topology, giving birth to the field of topological combinatorics. Later, a very nice and short proof was given by Joshua E. Greene [2].

Several papers we devoted to the study of the chromatic number of Kneser graphs of set systems. For any system of k-sets $\mathcal{A} \subset \binom{[n]}{k}$ we can define the Kneser graph $KG(\mathcal{A})$ in the following natural way. The vertices of $KG(\mathcal{S})$ are the elements of $\mathcal{A}$, while two of them are joined if and only if they are disjoint. In particular, there were results by Dolnikov [1] and Schrijver [5].

Such graphs are induced subgraphs of $KG_{n,k}$. We, in turn, study spanning subgraphs of $KG_{n,k}$. Namely, we study the chromatic number of a random graph $KG_{n,k}(p)$. This graph has the same set of vertices as $KG_{n,k}$, and each edge from $KG_{n,k}$ is included in $KG_{n,k}(p)$ with probability p. Informally, for a large range of values of parameters we show that the chromatic number of the random subgraph is w.h.p. close to the chromatic number of the original graph. In particular, we have the following

Theorem. *1. If p is fixed, $0 < p \le 1$, and $k \gg n^{\frac{3}{2l}}$, then w.h.p. $\chi(KG_{n,k}(p)) \ge \chi(KG_{n,k}) - 2l$.*
2. If for some $p = p(n)$, $0 < p \le 1$, we have $k \gg n^{3/4}p^{-1/4} + (n^{1/2}\ln n)p^{-1/2}$, then w.h.p. $\chi(KG_{n,k}(p)) \ge \chi(KG_{n,k}) - 4$.
3. Let p is fixed, $0 < p \le 1$, and $n - 2k = \bar{o}(\sqrt{n})$, then w.h.p. $\chi(KG_{n,k}(p)) \ge \chi(KG_{n,k}) - 2$.

References

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