POLYNOMIAL-TIME PERFECT MATCHINGS IN DENSE HYPERGRAPHS

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In this talk we consider the decision problem for the existence of a perfect matching in a k-uniform hypergraph (or k-graph) H on n vertices. Since for $k \geq 3$ this problem was one of Karp's 21 NP-complete problems [1], it is natural to seek conditions on H which render it tractable. For any $A \subseteq V(H)$, the *degree* $d(A) = d_H(A)$ of A is the number of edges of H containing A. The *minimum* $(k-1)$*-degree* $\delta_{k-1}(H)$ of H is the minimum of $d(A)$ over all subsets A of $V(H)$ of size $k-1$.

Let $\mathbf{PM}(k, \delta)$ be the decision problem of determining whether a k-graph H with $\delta_{k-1}(H) \geq \delta n$ contains a perfect matching. Szymańska [3] proved that for $\delta < 1/k$ the problem $\mathbf{PM}(k, 0)$ admits a polynomial-time reduction to $\mathbf{PM}(k, \delta)$ and hence $\mathbf{PM}(k, \delta)$ is also NP-complete. We describe an algorithm which shows that the opposite is true for $\delta > 1/k$:

Theorem 1. *Fix $k \geq 3$ and $\gamma > 0$. Then there is an algorithm with running time $O(n^{3k^2-7k+1})$, which given any k-graph H on n vertices with $\delta_{k-1}(H) \geq (1/k + \gamma)n$, finds either a perfect matching or a certificate that no perfect matching exists.*

Previously, Karpiński, Ruciński and Szymańska [2] showed that there exists $\varepsilon > 0$ such that $\mathbf{PM}(k, 1/2 - \varepsilon)$ is in P.

To prove Theorem 1 we establish a strong stability result which states that if H is a k-graph on n vertices, and $\delta_{k-1}(H) \geq n/k + o(n)$, then H either contains a perfect matching or is close to one of a family of lattice-based constructions termed 'divisibility barriers'. While the precise statement of this result for general k requires significant preliminaries, which we cover in the talk, the special case $k = 3$ may be stated as follows:

Theorem 2. *For any $\gamma > 0$ there exists $n_0 = n_0(\gamma)$ such that the following statement holds. Let H be a 3-graph on $n \geq n_0$ vertices, such that 3 divides n and $\delta_2(H) \geq (1/3 + \gamma)n$, and suppose that H does not contain a perfect matching. Then there is a subset $A \subseteq V(H)$ such that $|A|$ is odd but every edge of H intersects A in an even number of vertices.*

References

[1] R. M. Karp, Reducibility among combinatorial problems, *Complexity of Computer Computations* (1972), 85–103.

[2] M. Karpiński, A. Ruciński and E. Szymańska, Computational complexity of the perfect matching problem in hypergraphs with subcritical density, *International Journal of Foundations of Computer Science* **21** (2010), 905–924.

[3] E. Szymańska, The complexity of almost perfect matchings and other packing problems in uniform hypergraphs with high codegree, *European Journal of Combinatorics* **34** (2013), 632–646.

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