A hypergraph Turán theorem via a generalised notion of hypergraph Lagrangian.

Matthew Jenssen, London School of Economics

The theory of hypergraph Lagrangians, developed by Frankl and Füredi [Bull. Inst. Math. Acad. Sin. **16** (1988), 305–313] and Sidorenko [Mat. Zametki **41** (1987), 433–455], is a valuable tool in the field of hypergraph Turán problems. Here we present a generalised notion of the hypergraph Lagrangian and use the Karush-Kuhn-Tucker conditions from the theory of non-linear programming to exploit some of it's properties. As an application we show that the maximum Lagrangian of an r-graph H with the property that for all $e, f \in E(H), e \cap f \neq r-2$ is attained by $K_{r+1}^{(r)}$, the complete r-graph on $r+1$ vertices in the cases $r = 3, 4, 5, 6, 7$and 8. As a consequence we determine the Turán density of what we shall call the 'r-uniform generalised K_4' for these values of r. More precisely, the r-uniform generalised K_4, denoted by $\mathcal{K}_4^{(r)}$, is the r-graph on the $5r-6$ vertices $\{x_i, y_j, z_{ijk} : i = 1, \ldots, r, j = 1, 2, k = 1, \ldots r-2\}$ and with the 6 edges

$$\{x_1, \ldots, x_r\}, \{y_1, y_2, x_3, \ldots, x_r\} \text{ and } \{x_i, y_j, z_{ij1}, \ldots, z_{ij(r-2)}\} \text{ for } i, j \in \{1, 2\}.$$

We note that $\mathcal{K}_4^{(2)} = K_4$, the complete graph on 4 vertices, so that the above results may be viewed as hypergraph extensions of known Turán results on K_4. The generalised K_4 is naturally related to the generalised triangle, whose Turán density is considered (either implicitly or explicitly) in the works of Frankl and Füredi [J. Combin. Theory Ser. A **52** (1989), 129–147] and Pikhurko [Combinatorica **28** (2008) 187–208] amongst others.