Maximum density of exact copies of a graph in the n-cube and a Turán surprise.

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Let G be an induced subgraph of the d-cube Q_d. We define $f(d,G)$, the d-cube density of G, to be the limit as n goes to infinity of the maximum fraction, over all subsets J of the vertex set of the n-cube Q_n, of sub-d-cubes of Q_n whose intersection with J induces an exact copy of G (isomorphic to G, with the same embedding in Q_d). In general, it is difficult to determine $f(d,G)$. We show that if C is a perfect 8-cycle (4 pairs of vertices at distance 4) then $f(4,C) = 3/32$. Amazingly, to establish the upper bound we needed to determine the Turán density of $\{F, H\}$, where $F = \{1234, 1235, 1245\}$ and $G = \{1234, 1235, 1456\}$ and where the only 4-graphs allowed are those where there is a bipartition of the vertex set such that each edge has two vertices in each part. We note that the link graphs of the vertex 1 in F and G are the two forbidden 3-graphs in Bollobas well-known theorem on the maximum number of edges in a 3-graph where no edge contains the symmetric difference of two others.