Füredi–Hajnal constants are typically subexponential

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Abstract

A *binary matrix* is a matrix with entries from the set $\{0, 1\}$. We say that a binary matrix A *contains* a binary matrix B if B can be obtained from A by removal of some rows, some columns, and changing some 1-entries to 0-entries. If A does not contain B, we say that A *avoids* B. A *permutation matrix* P is a binary square matrix with exactly one 1-entry in every row and one 1-entry in every column.

The Füredi–Hajnal conjecture, proved by Marcus and Tardos in 2004, states that for every permutation matrix P, there is a constant c_P such that for every $n \in \mathbb{N}$, every $n \times n$ binary matrix A with at least $c_P n$ 1-entries contains P. Klazar proved that the Füredi–Hajnal conjecture implies the Stanley–Wilf conjecture, which states the following. For every permutation matrix P, there is a constant s_P such that for every $n \in \mathbb{N}$, the number of $n \times n$ permutation matrices avoiding P is at most s_P^n.

Fox recently found a randomized construction showing that for every k, there are $k \times k$ permutation matrices P with $c_P \geq 2^{\Omega(k^{1/4})}$. He additionally showed that as k goes to infinity, almost all $k \times k$ permutation matrices satisfy $c_P \geq 2^{\Omega((k/\log(k))^{1/4})}$. Fox also improved the original Marcus–Tardos upper bound $c_P \leq 2k^4\binom{k^2}{k}$ to $c_P \leq 2^{8k}$ (which can be easily lowered to $c_P \leq 2^{6k}$), for all $k \times k$ permutation matrices P.

A 1-entry in a matrix is identified by the pair (i, j) of the row index i and the column index j. The *distance vector* between the entries (i_1, j_1) and (i_2, j_2) is $(i_2 - i_1, j_2 - j_1)$. We say that a $k \times k$ permutation matrix P is *scattered* if every pair (d, d') is the distance vector of at most $\log_2(k)$ pairs of 1-entries of P. As k goes to infinity, almost all $k \times k$ permutation matrices are scattered.

We show that $c_P \leq 2^{O(k^{2/3}\log^{7/3}(k))}$ for every scattered $k \times k$ permutation matrix P. The main part of the proof is showing that every $4k \times 4k$ binary matrix with at most $O(k^{4/3}/\log^{1/3}(k))$ 0-entries contains every $k \times k$ scattered permutation matrix.

We also further improve the upper bound on c_P to $c_P \leq 2^{(4+o(1))k}$, for all $k \times k$ permutation matrices P.

All the bounds mentioned here imply similar bounds on the Stanley-Wilf limit, s_P, since it is known that $c_P \leq O(s_P^{4.5})$ and $s_P \leq O(c_P^2)$ for every permutation matrix P.