EDGE-COLORINGS OF GRAPHS AVOIDING COMPLETE GRAPHS WITH A PRESCRIBED COLORING PATTERN

FABRÍCIO SIQUEIRA BENEVIDES, CARLOS HOPPEN, AND RUDINI MENEZES SAMPAIO

Abstract. For any fixed graph F, we say that a graph G is F-free if it does not contain F as a subgraph. We denote by $\mathrm{ex}(n, F)$ the maximum number of edges in a n-vertex graph which is F-free, known as the Turán number of F.

In 1974, Erdős and Rothschild considered a related question where we count the number of certain colorings. Given an integer r, by an r-coloring of a graph G we mean any r-edge-coloring of G. In particular, it does not have to be proper and does not have to use all r colors. Let $c_{r,F}(G)$ be the number of r-colorings of G such that *every color class is F-free.* They considered the problem of finding $c_{r,F}(n) = \max\{c_{r,F}(G)\}$ where the maximum is over all n-vertex graphs G. Let us say that G is extremal for $c_{n,F}(n)$ if it realizes the above maximum. Clearly, $c_{r,F}(n) \geq r^{\mathrm{ex}(n,F)}$, as we take G to be the Turán graph and color it arbitrarily. The problem of determining $c_{r,F}(n)$ was investigates by several authors, for various classes of graphs such as: complete graphs [1, 8, 9], odd cycles [1], matchings [4], paths and stars [5]. And for hypergraphs [3, 6, 7]. One common concern is to determine when the Turán Graph is extremal for $c_{r,F}(n)$ (with r fixed and n large).

Here we consider a natural generalization of the above. Given an r-colored k-vertex graph $\hat{F}$, we consider the number of r-edge-colorings of a larger graph G that avoids the 'color pattern' of $\hat{F}$. More formally, $c_{r,\hat{F}}(G)$ denote the number or r-colorings of G such there are no k vertices of G that induce a colored graph isomorphic to $\hat{F}$. For example, the above problem consists of the case where $\hat{F}$ is a colouring of F that uses only one of the r colors. We define $c_{r,\hat{F}}(n)$ and extremal graphs as before.

We note that Balogh [2] had also considered a related but not analogous "colored version" of the problem. He considered the number $C_{r,\hat{F}}(G)$ of colorings of G which do not have a set of k-vertices colored exactly as in $\hat{F}$. In this case, for example, if $\hat{F}$ has only one color, $C_{r,H}(G)$ is the number of coloring of G which does not contains $\hat{F}$ in this particular color class.So $c_{r,\hat{F}}(G) \leq C_{r,\hat{F}}(G)$. Balogh proved that in the case where $r = 2$ and $\hat{F}$ is a 2-coloring of a clique that uses both colors then $C_{2,\hat{F}}(n) = 2^{\mathrm{ex}(n,\hat{F})}$ for n large enough.

Here, we focus on the case where $r = 3$. Let $\hat{F}_3$ be a 3-colored K_3. We proved that if the three colors are used in $\hat{F}_3$ then the complete graph on n vertices is the extremal graph for $c_{3,\hat{F}_3}(n)$. And if only two colors are used in F_3 then the Turán Graph is extremal for $c_{3,\hat{F}_3}(n)$ (whereas this is trivially not true for $C_{3,\hat{F}_3}(n)$). Much more generally we prove the following: with $r = 3$, let $\hat{F}_k$ be a coloring of K_k that uses only two colors one of which induces a graph H whose Ramsey Number is smaller than k, then the Turán Graph is extremal for $c_{3,\hat{F}_k}(n)$.

References

1. N. Alon, J. Balogh, P. Keevash, and B. Sudakov, *The number of edge colorings with no monochromatic cliques*, J. London Math. Soc. (2) 70, 2004, 273–288.
2. J. Balogh, *A remark on the number of edge colorings of graphs*, European J. Combin. 77, 2006, 565–573.
3. C. Hoppen, Y. Kohayakawa, and H. Lefmann, *Hypergraphs with many Kneser colorings*, European Journal of Combinatorics 33, 2012, 816–843.
4. C. Hoppen, Y. Kohayakawa, and H. Lefmann, *Edge colorings of graphs avoiding monochromatic matchings of a given size*, Combinatorics, Probability & Computing 21, 2012, 203–218.
5. C. Hoppen, Y. Kohayakawa, and H. Lefmann, *Edge colorings of graphs avoiding some fixed monochromatic subgraph with linear Turán number*, European Journal of Combinatorics 35, 2014, 354–373.
6. H. Lefmann and Y. Person, *Exact results on the number of restricted edge colorings for some families of linear hypergraphs*, Journal of Graph Theory 73, 2013, 1–31.

7. H. Lefmann, M. Schacht, and Y. Person, *A structural result for hypergraphs with many restricted edge colorings*, Journal of Combinatorics 1, 2010, 441–475.
8. O. Pikhurko, and Z. B. Yilma, *The maximum number of K_3-free and K_4-free edge 4-colorings*, J. London Math. Soc. 85, 2012, 593–615.
9. R. Yuster, *The number of edge colorings with no monochromatic triangle*, J. Graph Theory 21, 1996, 441–452.

Departamento de Computação, Centro de Ciências, UFC – Campus do Pici, Bloco 910, 60451–760 Fortaleza, CE, Brazil
E-mail address: fabricio@mat.ufc.br

Instituto de Matemática, Universidade Federal do Rio Grande do Sul, Avenida Bento Gonçalves, 9500, 91509-900, Porto Alegre, Brazil
E-mail address: choppen@ufrgs.br

Departamento de Computação, Centro de Ciências, UFC – Campus do Pici, Bloco 910, 60451–760 Fortaleza, CE, Brazil
E-mail address: rudini@lia.ufc.br