Sándor Vályi

On the axiomatizability of some first-order spatio-temporal theories

Abstract. Spatio-temporal logic is a variant of branching temporal logic where one of the so-called causal relations on spacetime plays the role of a time flow. Allowing only rational numbers as space and time co-ordinates, we prove that a first-order spatio-temporal theory over this flow is recursively enumerable if and only if the dimension of spacetime does not exceed 2. The situation is somewhat different compared to the case of real co-ordinates, because we establish that even dimension 2 does not permit recursive enumerability in this case. The proof of the result on rational spacetime involves a more deeper portion of spacetime geometry than the corresponding, more evident result for the real co-ordinates.

Keywords: Relativity and logic, first-order temporal logic, spatio-temporal logic, spacetime, causality

1. Introduction

1.1. Spatio-temporal logic

Both linear and branching temporal logic is widely used to model time-dependent and non-deterministic phenomena, such as future tenses of natural language or random choice among parallel threads of computationV. Its propositional version is exploited successfully in designing reliable finite-state computing devices (see e.g. [11], [13], [30]). Its full first-order version can express properties of arbitrary computing paradigms ([23], [8]). Automatic decision and/ or proof-searching algorithms support automatic specification and verification of such systems (see [4], [1], [24]).

What can temporal logic offer to designers of mobile distributed computing systems? Apart from having dynamics in time, these systems have dynamics in space, too. To cover this area, an analogue of temporal logic has been developed, which is usually called spatio-temporal logic. The need for appropriate knowledge representation systems has generated a big boom of investigations into this direction in the past ten years. One way to follow this is intercrossing a spatial language with a temporal language in such a

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way that in the hybrid language there are separate modalities for time and space ([34], [9], [45], [6], [16]). This idea originated in research on multi-dimensional modal logics ([39], [41], [25], [37]). In this formalization, there are separate modalities for space directions and time. We mention in advance that our recent non-axiomatizability result (Theorem 2.12) is not a consequence of non-axiomatizability of multimodal logics over $(\mathbb{R}, <)$.

1.2. Logic and relativity

There is another tradition to deal with time and space, namely to speak jointly about spacetime and use its geometrical relations and objects to express various properties of the dynamics of processes in spacetime. Assuming that these processes have no synchronised time one comes to consider hyperbolic geometry of Minkowski spacetime, as in the works of F. Mattern ([26], [12]). He proposed investigating so-called causal connectability relations of spacetime from the viewpoint of specification and verification of distributed computing. In the present introduction we will distinguish five relations related to causality: $(x \blacktriangleleft y)$ for pure material causal connectability, while $(x \lhd y)$ for optical connectability, $(x \ll y)$ for the disjuntion of the previous two, $(x =\!\ll y)$ for $(x \ll y \vee x = y)$ and finally $(x =\!\blacktriangleleft y)$ stands for $x \blacktriangleleft y \vee x = y$. Exact definitions for $\blacktriangleleft$ and $\lhd$ will be given when our theorems and proofs will be developed.

A theory of the causal relation $\ll$ of spacetime was axiomatized as early as in 1914 by A. Robb [38] and later on similar results were obtained among others by B. Mundy and J. P. Ax ([28], [29], [5]). R. Goldblatt elaborated the first-order theory of some spacetime relations – including causal relations – in [18] and [19]. V. Pambuccian reinterpreted the Alexandrov–Zeeman theorem concerning causality-preserving mappings from the definitional viewpoint of these relations ([31]).

A relevant new approach of logic to causality and relativity is to axiomatize the whole of the physical theory, including facts about observers, co-ordinates or even co-ordinate transformations, and not only the geometrical core. This approach may be called the analytic version of formalized relativity theory ([3]). Moreover, since this formalization does not utilize second-order or set-theoretical notions, for instance the set of real numbers, only their first-order approximations, we can call it the non-standard analytic version of formalized relativity theory in the sense of non-standard analysis. Our recent results will give a support for this approach in the sense that they imply that no complete and sound axiomatization can be given if we only allow spacetime structures built on a concrete co-ordinatizing field such as $\mathbb{R}$ or $\mathbb{Q}$. This approach is extremely useful when we formalize real physical statements and experiments, as in [2], or would like to execute conceptual analysis, as in [22].

1.3. Modal logic and relativity

Each causal connectability relation of spacetime can be considered to be a generalization of time flows in temporal logic when it serves as the alternativity relation of a Kripke frame for propositional modal logic as it was done first by V. Shehtman and R. Goldblatt, independently. In [42] and [17], modal logics of both $(\mathbb{R}^n, =\ll)$ and $(\mathbb{R}^n, =\blacktriangleleft)$ were proved to be decidable. The more then twenty years long open problem of decidability of modal logics of the frames $(\mathbb{R}^n, \blacktriangleleft)$ and $(\mathbb{R}^n, \ll)$ were propounded in Goldblatt's paper and solved by Shapirovsky and Shehtman ([40]). Modal logics of other spacetime relations on $\mathbb{R}^n$ and more abstract structures concerning spacetime geometry were analysed in [7] and [36]. Modal and temporal logics of frames $(\mathbb{Z}^n, \ll)$ and $(\mathbb{Z}^n, =\ll)$ were investigated in [32], [33].

J. van Benthem drew attention to the spacetime flow $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 1)$ in [10]. At least for $n = 2$, in this book he proved that its first-order theory is ω-categorical (countably categorical), finitely axiomatizable and consequently, complete and decidable. Further, that $(\mathbb{Q}^2, \blacktriangleleft)$ is an elementary substructure of $(\mathbb{R}^2, \blacktriangleleft)$ so their first-order theories coincide. Anyway, the first-order theory of $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 1)$ is decidable through semantic interpretation into the first-order theory of $(\mathbb{R}, +, *, <)$, which is known to be decidable by a well-known result of A. Tarski. To the best of our knowledge, for $n > 2$, the decidability of the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ has neither been proved nor disproved. The previous method does not work, since for $n > 2$, $(\mathbb{Q}^n, \blacktriangleleft)$ is not an elementary substructure of $(\mathbb{R}^n, \blacktriangleleft)$.

1.4. Contributions to first-order spatio-temporal logic

It is a rare and remarkable thing, that a first-order spatio-temporal logic is axiomatizable (we understand this notion simply as recursive enumerability). Over the reals, there is only a little hope to find one. Even the linear first-order temporal theory over $(\mathbb{R}, <)$ is not axiomatizable ([15]). The published proofs for non-axiomatizability of first-order temporal theories use a ternary base first-order signature or not valid for $(\mathbb{R}, <)$. Only the so-called monOdic fragment remains decidable ([20]). In our Theorem 2.14 we establish non-axiomatizability of the first-order temporal theory over $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 2)$ with a monadic signature, what is more, for a signature consisting of a sole unary predicate symbol without the equality symbol. With some modifications, our proof idea can also work for $(\mathbb{R}, <)$ to establish non-axiomatizability with the same simple signature.

We have more chance to find axiomatizable theories over the rational space-time $(\mathbb{Q}^n)$ $(n > 1)$. In [35], M. Reynolds axiomatized the first-order temporal theory over $(\mathbb{Q}, <)$. We observed that a possible reason for the axiomatizability of the first-order temporal theories of a structure is the ω-categoricity and recursive enumerability of the pure first-order theory of this structure itself ([43]). The first-order theory of $(\mathbb{Q}^2, \blacktriangleleft)$ has these properties (see in J. van Benthem's book [10]), this allows us to establish axiomatizability of arbitrary first-order spatio-temporal theory over this spatio-temporal flow (arbitrary base first-order signature, arbitrary temporal operators), see in [43], also cited in Theorem 2.13. This axiomatizability allows to describe interesting spatio-temporal properties of distributed mobile systems, where distributedness is considered on a 1-dimensional space line ([43]).

What is the situation when we step to higher dimensions in rational space-time? The main result of this paper is that a first-order spatio-temporal theory over $(\mathbb{Q}^n, \blacktriangleleft)$ $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$ is not axiomatizable (Theorem 2.12). (This implies also, that $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$ has no such nice metamathematical properties as its 2-dimensional sibling. Either not ω-categorical and/or not recursively enumerable. One may hardly imagine that it is ω-categorical but not recursively enumerable.)

In the next section the results of this paper are presented together with the definitions of the notions needed. In the first subsection of the third section the ideas and difficulties of the proofs are detailed, in the rest of the third section the proofs are given, while in the final section we will discuss what further possibilities we have to turn the situation (non-axiomatizability) to be positive.

2. Definitions and results

We assume the reader to be familiar with the basic semantic and syntactic notions of first-order logic. The style of the definitions concerning first-order temporal logic follows [15].

Definition 2.1. A *temporal operator* is a triple $(\odot, k, \tau)$ where $\odot$ is a symbol, k is a positive integer and τ is a first-order formula in the signature $\mathcal{S}_n$ having a denumerably infinite set $\{t_0, t_1, \ldots\}$ of variables, a binary predicate symbol $\prec$, a finite set $\{P_1, P_2, \dots P_n\}$ of unary predicate symbols and nothing else. Further requirement on τ is to contain exactly the only parameter t_0.

$\odot$ is the visual form of the operator, k is its arity while the role of τ is to describe the intended semantics of the operator. We will name the operators just by their first component, to avoid unneccessary complication of notations. Two examples of temporal operators are presented here:

$(\boxminus\!\!\!\rightarrow, 1, \forall t_1(t_0 \prec t_1 \rightarrow P_1(t_1))$ and
$(\mathit{Until}, 2, \exists t_1\{t_0 \prec t_1 \wedge P_2(t_1) \wedge \forall t_2[t_0 \prec t_2 \wedge t_2 \prec t_1 \rightarrow P_1(t_2)]\}\ )$.

The intuitive reading of the first is P_1 *holds always in the future*, while of the second is *from now on,* P_1 *holds until a timepoint where* P_2 *will come true.*

Definition 2.2. A *temporal language* TL_L^{Op}, based on a first-order signature L and a finite set Op of temporal operators, is the smallest set of formulae (on the appropriate alphabet) satisfying the following requirements:

- any atomic formula of L is an atomic temporal formula of TL_L^{Op},
- $(A \wedge B)$ and $\neg A$ are formulæ of TL_L^{Op} , if A and B belong to that set,
- $\forall x A$ is formula of TL_L^{Op} if A is a formula of TL_L^{Op} and x is a variable of L,
- $\odot(A_1, \dots A_k)$ is a formula of TL_L^{Op} if $(\odot, k, \tau) \in Op$ for some table τ and $A_1, \dots A_k$ are formulæ of TL_L^{Op}.

The set of terms of TL_L^{Op} coincides with the set of terms of pure L.

We assume the usual syntactic notions – as subformula, free and quantified variable, term substitution etc., modified in the adequate way – to be understood. We accept the usual abbreviations of first-order logic, as $(A \vee B)$, $(A \rightarrow B)$, $\exists x A$ etc., and use their well-known semantic properties without any extra remark. We provide here an example formula in TL_L^{Op}, where $Op = \{\boxminus\!\!\!\rightarrow, \mathit{Until}\}$ and signature L contains a unary predicate symbol p and a binary q: $\boxminus\!\!\!\rightarrow \forall x(p(x) \rightarrow \forall y\ \mathit{Until}(q(y, x), p(y)))$.

Definition 2.3. A *time flow* is a non-empty partially ordered set $(T, \ll)$.
$(T, \ll)$ is the intended notion of time.

The essence of semantics of temporal logic is to have time-dynamical interpretations. There are many variations on what part of signature is interpreted dynamically – all of them may find an own application area. The most simple case is if the interpretation of all the terms including the interpreting domain and variable valuations are time-independent, only the predicate interpretations vary on time. While we investigate only axiomatizability questions of theories of temporal logic, the chosen variation of temporal interpretation is indifferent– our following results proving or refuting axiomatizability are insensitive to this variations. So we employ the following simple formalization of first-order temporal semantics.

DEFINITION 2.4. Let L be a first-order signature and let Op be a finite set of temporal operators. A *temporal interpretation* $\mathcal{I}$ for TL_L^{Op} on the time flow $(T, \ll)$ consists of a triple $(D_{\mathcal{I}}, \mathcal{I}^f, \mathcal{I}^p)$ where $D_{\mathcal{I}}$ is a non-empty set (the time-independent domain of $\mathcal{I}$), $\mathcal{I}^f$ is a usual first-order interpretation for the terms of L while $\mathcal{I}^p$ is a function mapping a usual first-order interpretation $\mathcal{I}_t^p$ of the predicate symbols of L to each $t \in T$, where the domain of each $\mathcal{I}_t^p$ is $D_{\mathcal{I}}$.

DEFINITION 2.5. The definition for a *valuation* Θ of the variables of TL_L^{Op} in interpretation $\mathcal{I}$ is a finite partial fuction mapping from the variables of L to $D_{\mathcal{I}}$. We denote the valuation $\{(x_1, v_1), \ldots, (x_m, v_m)\}$, as usual, by $\binom{x_1 \ldots x_m}{v_1 \ldots v_m}$. This implies, that () denotes the empty valuation. Further, $\Theta @ \binom{x}{d}$ stands for a valuation Π whose domain is *dom* $\Theta \cup \{x\}$, Π and Θ agree on *dom* $\Theta \setminus \{x\}$ but $\Pi(x) = d$.

We remind that the interpreting domain and the interpretation of the terms is constant in time.

DEFINITION 2.6. The value $|t\Theta|_{\mathcal{I}}$ of the term t in the interpretation $\mathcal{I}$ after the variable valuation Θ can be defined just as in first-order case.

The temporal satisfaction relates more objects than its classical counterpart. It involves, besides an interpretation, a variable valuation and a formula, also a time flow and an evaluation time point.

DEFINITION 2.7. Let L be a first-order signature and let Op be a finite set of temporal operators. For any time flow $(T, \ll)$, any temporal interpretation $\mathcal{I}$ for TL_L^{Op}, any variable valuation Θ on $\mathcal{I}$, any time point $t (\in T)$ and any temporal formula A of the temporal language just mentioned, the *satisfaction relation* $(T, \ll), \mathcal{I}, \Theta, t \Vdash A$ is defined as follows:

- if A is an atomic formula then $(T,\ll),\mathcal{I},\Theta,t \Vdash A$ iff $\mathcal{I}^p_t \models A\Theta$, where $\models$ denotes the classical first-order satisfaction relation,

- if $A = (B \wedge C)$ then $(T,\ll),\mathcal{I},\Theta,t \Vdash A$ iff $(T,\ll),\mathcal{I},\Theta,t \Vdash B$ and $(T,\ll),\mathcal{I},\Theta,t \Vdash C$,

- if $A = \neg B$ then $(T,\ll),\mathcal{I},\Theta,t \Vdash A$ iff $(T,\ll),\mathcal{I},\Theta,t \Vdash B$ does not hold,

- if $A = \forall x B$ then $(T,\ll),\mathcal{I},\Theta,t \Vdash A$ iff for all $d \in D_{\mathcal{I}}$, $(T,\ll),\mathcal{I},\Theta@\binom{x}{d},\ t \Vdash B$,

- if $A = \odot(B_1,\dots B_n)$ for a temporal operator $(\odot,n,\tau) \in Op$ then $(T,\ll),\mathcal{I},\Theta,t \Vdash A$ iff $\mathcal{B} \models \tau\binom{t_0}{t}$ where $\mathcal{B}$ is an interpretation for signature $\mathcal{S}_n$ (c.f. Def. 2.1) whose domain is T, further, $\prec^{\mathcal{B}} = \ll$ and the interpretation of P_i in $\mathcal{B}$ can be given as the subset of T consisting of time points where B_i holds, that is, $(P_i)^{\mathcal{B}} = \{s \in T : (T,\ll),\mathcal{I},\Theta,s \Vdash B_i\}$ for any integer i between 1 and n.

Consequently, in the sense of the last definition, if $A = \mathit{Until}(B,C)$ then $(T,\ll),\mathcal{I},\Theta,t \Vdash A$ iff there exists an $s \in T$ such that $t \ll s$, $(T,\ll),\mathcal{I},\Theta,s \Vdash B$ and for all $r \in T$ such that $t \ll r \ll s$, $(T,\ll),\mathcal{I},\Theta,r \Vdash C$. Further, if $A = \boxminus\!\!\rightarrow B$ then $(T,\ll),\mathcal{I},\Theta,t \Vdash A$ iff for all $s \in T$ such that $t \ll s$, $(T,\ll),\mathcal{I},\Theta,s \Vdash B$.

Definition 2.8. The *Op-temporal theory* $\mathtt{Th}^{Op}_L(T,\ll)$ of time flow $(T,\ll)$ on signature L is the set of such closed TL^{Op}_L-formulæ A, that for any temporal interpretation $\mathcal{I}$, any $t \in T$ and any variable valuation Θ, $(T,\ll),\mathcal{I},\Theta,t \Vdash A$ holds.

Definition 2.9. To be concise, we say a set S of formulæ *axiomatizable* iff it is recursively enumerable.

Definition 2.10. $GN := \{\boxminus\!\!\rightarrow, \boxdot\}$, where $\boxminus\!\!\rightarrow$ is given after Definition 2.1 and the second operator is $(\boxdot, 1, \forall t_1(\neg\forall t_2(t_2 \ll t_0 \leftrightarrow t_2 \ll t_1) \rightarrow P_1(t_1)))$.

$\boxdot$ will have a special intuitive reading in our spacetime flow which is to be specified later. Let $n > 1$. We recall the definition for the function *Minkowskian distance* $\mu : \mathbb{R}^n \to \mathbb{R}$. It is defined by $\mu((x_1,\dots,x_n),(y_1,\dots,y_n)) = (x_1-y_1)^2-(x_2-y_2)^2-\dots-(x_n-y_n)^2$. Further, for $x = (x_1,\dots,x_n)$, $y = (y_1,\dots,y_n) \in \mathbb{R}^n$, we write $(x \blacktriangleleft y)$ for $\mu(x,y) > 0 \wedge x_1 < y_1$.

In special relativity theory, this relation is also known as *directed material or timelike causal connectability* because it holds iff there is a possibility an event occuring in y to take a material (below-lightspeed) effect from an event in x. In this case we say also that y is inside of the upper lightcone of x.

The time has come to fix which first-order signature we prove our theorem for.

Definition 2.11. Signature L_0 includes no equality symbol just one unary predicate symbol, namely, r. We postulate also that the set of variables of L_0 includes $\{\alpha, \gamma, \delta, \varepsilon\}$ and $\{x, y, z, u, v, w\}$.

Our main result is the following.

Theorem 2.12. Let $n > 2$. $\mathtt{Th}^{GN}_{L_0}(\mathbb{Q}^n, \blacktriangleleft)$ is not axiomatizable.

This is in interesting contrast with the following.

Theorem 2.13. [43] For any first-order signature L and arbitrary finite set of temporal operators Op, $\mathtt{Th}^{Op}_{L}(\mathbb{Q}^2, \blacktriangleleft)$ is axiomatizable.

Theorem 2.14. Let $n > 2$. $\mathtt{Th}^{GN}_{L_0}(\mathbb{R}^n, \blacktriangleleft)$ is not axiomatizable.

Theorem 2.15. [43] $\mathtt{Th}^{GN}_{L_0}(\mathbb{R}^2, \blacktriangleleft)$ is not axiomatizable.

3. Proofs

3.1. Ideas of the proofs

Theorem 2.14 (on spatio-temporal logic over $(\mathbb{R}^n, \blacktriangleleft)$ $(n > 2)$) can be proven according to the non-axiomatizability proofs in first-order temporal logics, except for the absence of binary or ternary relations. Our chosen first-order signature is limited to only one unary predicate symbol without equality. A representation of predicates of more than one argument can be employed to work out this problem. This solution will throw the proof to a more technical level, but this is the strongest result we can prove. The method of the existing proofs of non-axiomatizability of *monadic* first-order temporal logic (see [21] – it proves only non-decidability of some first-order modal logic –, [20] and [27]) are not followed directly but it is unneccessary to deny their motivation to our work.

Although the last mentioned theorem is not a trivial consequence of non-axiomatizability results on first-order temporal theories over $(\mathbb{R}, <)$, especially for our monadic base first-order signature, it is a much more difficult task, to prove Theorem 2.12. The extra difficulty we faced is the following. In the first-order theory of $(\mathbb{R}^n, \blacktriangleleft)$ one can express an equidistance formula, as R. Goldblatt pointed it out in the Appendix of [18]. A crucial point in this process is to express the relation of spacelike betweenness by a first-order formula. This way does not apply to $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$. We have constructed a new definition for spacelike betweenness which is also valid for $(\mathbb{Q}^n, \blacktriangleleft)$, but we cannot continue this work to give definition for equidistance in $(\mathbb{Q}^n, \blacktriangleleft)$. Instead of this, we proceed with describing a situation in $(\mathbb{Q}, \blacktriangleleft)$, by means of the just defined spacelike betweenness, which also makes possible the representation of the first-order theory of $(\mathbb{N}, +, *, =)$.

3.2. Representation of the time flow structure in the temporal interpretation

We begin with proof of Theorem 2.12. We give some definitions in the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$.

Definition 3.1.

- (i) $(x = y) \rightleftharpoons \forall z(z \blacktriangleleft x \leftrightarrow z \blacktriangleleft y)$,
- (ii) $(x \lhd y) \rightleftharpoons \forall z(y \blacktriangleleft z \to x \blacktriangleleft z) \land \neg x \blacktriangleleft y \land \neg x = y$,
- (iii) $\sigma(x, y) \rightleftharpoons x \not\blacktriangleleft y \ \land \ x \ntriangleleft y \ \land \ y \not\blacktriangleleft x \ \land \ y \ntriangleleft x \ \land \ y \neq x$,
- (iv) $\underline{\beta_\sigma}(x, z, y) \rightleftharpoons$
  $\sigma(x, y) \land \forall u(x \blacktriangleleft u \land y \blacktriangleleft u \to z \blacktriangleleft u) \land \forall u(u \blacktriangleleft x \land u \blacktriangleleft y \to u \blacktriangleleft z)$,
- (v) $\beta_\sigma(x, z, y) \rightleftharpoons \underline{\beta_\sigma}(x, z, y) \land x \neq z \land z \neq y$.

In what follows, for the sake of easier readability, we write – for example – $(\mathbb{Q}^n, \blacktriangleleft) \models p \lhd q$ or simply $p \lhd q$ instead of $(\mathbb{Q}^n, \blacktriangleleft) \models (x \lhd y)\binom{x,y}{p,q}$, if $p, q \in \mathbb{Q}^n$. Similar notation applies to the other formulae.

Statement 3.2. The following items hold.

For each $n \geq 2$ and for each $p, q, r, s \in \mathbb{Q}^n$:

- (i) $(\mathbb{Q}^n, \blacktriangleleft) \models p = q$ if and only if p and q coincide,
- (ii) $(\mathbb{Q}^n, \blacktriangleleft) \models p \lhd q$ if and only if $\mu(p, q) = 0$ and $p_1 < q_1$, that is, q is on the boundary of the upper lightcone of p (directed optical connectability).

- (iii) $(\mathbb{Q}^n, \blacktriangleleft) \models \sigma(p,q)$ if and only if the line joining p to q is spacelike, that is, $\mu(p,q) < 0$.

For each $n > 2$ and for each $p, q, r, s \in \mathbb{Q}^n$:

- (iv) $(\mathbb{Q}^n, \blacktriangleleft) \models \underline{\beta_\sigma}(p,q,r)$ if and only if p, q, r lie on a common spacelike straight line and q is between the other two points,

- (v) $(\mathbb{Q}^n, \blacktriangleleft) \models \beta_\sigma(p,q,r)$ if and only if p, q, r lie on a common spacelike straight line and q is between the other two points and p,q and r are pairwise distinct.

Only the definition for $\underline{\beta_\sigma}$ [(iv)] differs from the way of defining the corresponding definitions in $(\mathbb{R}^n, \blacktriangleleft)$. If a definition is a Boolean combination of expressions involving only already defined notions [(iii), (v)], then the argumentation remains the same as for $(\mathbb{R}^n, \blacktriangleleft)$. Some definitions involve quantifiers on spacetime points, namely (i),(ii), and (iv). In case (i) and (ii), it is not too difficult to check the conditions, while case (iv) needs a rather lengthy, but elementary calculation. For the details, consult [44].

The rest of this subsection is a rather standard part of non-axiomatizability proofs in first-order modal and temporal logic. First we introduce some defined temporal operators. If A is an arbitrary temporal formula then $\Box A$ stands for $A \vee \boxdot A$ and $\Diamond A$ for $\neg\Box\neg A$. Further we will write $\Diamond\!\!\!\cdot\;A$ instead of $\neg\boxdot\neg A$, respectively. It turns to be clear, that $\Box A$ expresses that A is true at all the points of spacetime and $\Diamond A$ is its existential counterpart, if we realize that, in a spatio-temporal setting, $\boxdot\ A$ holds in a spacetime point q if and only if A in all spacetime points maybe except for q itself. As usual, $\Diamond\!\!\!\rightarrow$ denotes the existential counterpart of $\boxminus\!\!\!\rightarrow$.

We fix now some formulæ of $TL^{GN}_{L_0}$, as follows. We remind the reader that r is the only predicate symbol in L_0.

Definition 3.3. $Id := \Diamond(r(x) \wedge \boxdot\neg r(x))$, $\nu_1 := \Box\exists x(r(x) \wedge \boxdot\neg r(x))$.

Let us fix a temporal interpretation $\mathcal{I}$ for $TL^{GN}_{L_0}$ on the flow $(\mathbb{Q}^n, \blacktriangleleft)$ and a rational point $q \in \mathbb{Q}^n$ satisfying $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, (), q \Vdash \nu_1$. It is clear that then the same holds for all $q' \in \mathbb{Q}^n$. We will see later that such an interpretation exists, it follows from the stronger result of Theorem 3.24. Throughout this and the next two subsections, these $\mathcal{I}$ and q remain fixed. (Till Statement 3.23.)

Definition 3.4. We define the relation $\varphi_0 \subseteq D_{\mathcal{I}} \times \mathbb{Q}^n$ by the condition $(d,q) \in \varphi_0 \Leftrightarrow (\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, q \Vdash r(x)$. The set $\{d \in D_{\mathcal{I}} : (\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, q \Vdash Id\}$ will be denoted by ID_1. Further, φ_1 stands for the restriction of φ_0 to $ID_1 \times \mathbb{Q}^n$.

Statement 3.5. With the notations of the previous definition, φ_1 is a surjective function taking ID_1 onto $\mathbb{Q}^n$.

This follows obviously from the way we have fixed $\mathcal{I}$ and makes right when we use a function denotation for the elements of φ_1, that is, we write $\varphi_1(d) = q$ instead of $(d,q) \in \varphi_1$.

Definition 3.6. The binary relation ρ on ID_1 is defined by the condition $(d_1, d_2) \in \rho \Leftrightarrow \varphi_1(d_1) = \varphi_1(d_2)$.

Statement 3.7. By standard algebraic arguments, ρ is an equivalence on ID_1.

Definition 3.8. With the notations of the previous items, the set of the equivalence classes of ρ is denoted by ID. The function on ID corresponding to φ_1 is denoted by φ, that is, if the ρ-equivalence class of d is denoted by $[d]_\rho$, then $\varphi([d]_\rho)$ is just $\varphi_1(d)$.

All the following statements of this subsection can be verified by standard algebraic arguments.

Statement 3.9. φ is a bijection from ID onto $\mathbb{Q}^n$.

Definition 3.10. Let Ord denote the formula $\Diamond(r(x) \wedge \Leftrightarrow r(y)) \wedge Id \wedge Id^x_y$, where A^x_y is the result of substituting y into the free occurences of x in formula A. If $d_1, d_2 \in ID_1$ then we write $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x\ y}{d_1 d_2}, q \Vdash Ord$ also in form $Ord_1(d_1, d_2)$ – in this way we define a binary relation Ord_1 on ID_1.

Statement 3.11. The relation Ord_1 on ID_1 is compatible with ρ. That is, $Ord_1(d_1, d_2) \Leftrightarrow Ord_1(d_3, d_4)$ whenever $\{(d_1, d_2), (d_3, d_4)\} \subseteq \rho$.

Definition 3.12. Let $Ord^{\mathcal{I}}$ denote the inherited relation of ρ-equivalence classes, that is, $Ord^{\mathcal{I}}$ is a binary relation on ID and for any $d_1, d_2 \in ID_1$, $Ord_1(d_1, d_2)$ if and only if $Ord^{\mathcal{I}}([d_1]_\rho, [d_2]_\rho)$.

Statement 3.13. φ is an isomorphism from (ID, Ord) onto $(\mathbb{Q}^n, \blacktriangleleft)$.

Definition 3.14. Writing Ord into the place of $\blacktriangleleft$ in defining formulæ (i)-(v) of Definition 3.1, we obtain temporal formulæ $Eq(x,y)$, $Opt(x,y)$, $Sim(x,y)$, $\underline{Betw}(x,y,z)$ and $Betw(x,y,z)$ of $TL^{GN}_{L_0}$, respectively. For example, $Eq(x,y)$ is just $\forall z(Ord(z,x) \leftrightarrow Ord(z,y))$, and $Opt(x,y)$ is $\forall z(Ord(y,z) \rightarrow Ord(x,z) \wedge \neg Ord(x,y) \wedge \neg Eq(x,y))$. The corresponding relations on ID are denoted by $Eq^{\mathcal{I}}$, $Opt^{\mathcal{I}}$, $Sim^{\mathcal{I}}$, $\underline{Betw}^{\mathcal{I}}$ and $Betw^{\mathcal{I}}$, respectively.

For example, for each $d_1, d_2 \in ID_1$, $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x\ y}{d_1 d_2}, q \Vdash Opt(x,y)$ is also denoted by $Opt^{\mathcal{I}}([d_1]_\rho, [d_2]_\rho)$. This is reasonable, because all these relations are ρ-compatible on ID_1, since they are defined from $\blacktriangleleft$.

Corollary 3.15. φ is an isomorphism from $(ID, Ord^{\mathcal{I}}, Eq^{\mathcal{I}}, Opt^{\mathcal{I}}, Sim^{\mathcal{I}}, Betw^{\mathcal{I}})$ onto $(\mathbb{Q}^n, \blacktriangleleft, =, \lhd, \sigma, \beta_\sigma)$.

Now we have an isomorphism from a separable subset of $D_{\mathcal{I}}$ onto the time flow structure. Separable means here that we can write a temporal formula whose extension is this subset.

3.3. Isomorphism to the ordering of $\mathbb{N}$

Definition 3.16. Formulæ $\Diamond(r(\delta) \wedge r(x))$, $Betw(\alpha, x, y)$ (in $TL^{GN}_{L_0}$) will be abbreviated as $N(x)$ and $O(x,y)$, respectively. Further, we write $O(x,y) \wedge \neg\exists z(N(z) \wedge O(x,z) \wedge O(z,y))$ also in form $S(x,y)$.

Parameter δ in the first formula is used for separating a subset of ID without adding a new predicate symbol into L. This is natural enough. Dealing with predicates with more than one argument requires a more involved representation, as the next subsection will show.

Definition 3.17. Let ν_2 be defined as conjunction of ν_1 and the following formulæ :
(1) $Sim(\alpha,\varepsilon) \wedge Sim(\alpha,\gamma) \wedge Id^x_\alpha \wedge Id^x_\varepsilon \wedge Id^x_\gamma$,
(2) $\forall x(N(x) \rightarrow Id \wedge (Betw(\alpha,\varepsilon,x) \vee Eq(x,\varepsilon)))$,
(3) $\forall x(N(x) \rightarrow \exists! y(N(y) \wedge S(x,y))$,
where $\exists!$ is to understand with respect to the defined Eq handled as equality symbol (c.f. Def. 3.14),
(4) $\forall xy(N(x) \wedge N(y) \wedge S(x,y) \rightarrow$
$\exists z(N(z) \wedge Betw(\alpha,\gamma,z) \wedge Opt(x,z) \wedge Opt(y,z))$,
(5) $\forall xyzw(N(x) \wedge N(y) \wedge N(z) \wedge N(w) \wedge S(x,y) \wedge Opt(x,z) \wedge Opt(y,z) \wedge$
$\neg Opt(x,w) \wedge Opt(y,w) \wedge Betw(\alpha,\gamma,z) \wedge Betw(\alpha,\gamma,w) \rightarrow Betw(\alpha,z,w))$.

DEFINITION 3.18. $N^{\mathcal{I}} := \{[d]_\rho : (\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x}{d}, q \Vdash N(x)\}$,
$O^{\mathcal{I}} := \{([d_1]_\rho, [d_2]_\rho) : (\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x\ y}{d_1 d_2}, q \Vdash O(x, y)\}$, and $S^{\mathcal{I}}$ can be defined in an analogous way.

Let Θ denote a fixed valuation satisfying that $dom\ \Theta \supseteq \{\alpha, \delta, \varepsilon, \gamma\}$, till the end of the next subsection.

LEMMA 3.19. (Main lemma) If $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Theta, q \Vdash \nu_2$ then $(N^{\mathcal{I}}, O^{\mathcal{I}}, S^{\mathcal{I}})$ is isomorphic to $(\mathbb{N}, <, succ^r)$, where $succ^r = \{(n, n+1) : n \in \mathbb{N}\}$.

PROOF. The way of defining ν_2 results in that $(N^{\mathcal{I}}, O^{\mathcal{I}}, S^{\mathcal{I}})$ is a discrete linear ordering whose minimal element is $\Theta(\varepsilon)$ but without any maximal element. We only have to take into account the properties of the betweenness and that the existing isomorphism φ allows us to use the mentioned spacetime geometrical notions also for the elements of ID. By the way, if admitting of the above statement would be an overloading task, then we simply attach to ν_2 the extra condition that O is a linear ordering on the elements satisfying N.

Thus, the only point is to show that $(N^{\mathcal{I}}, O^{\mathcal{I}}, S^{\mathcal{I}})$ is isomorphic to $(\mathbb{N}, <, succ^r)$. For this, it is enough to prove that $N^{\mathcal{I}}$ is exhausted by the set $\{\Theta(\varepsilon), s_{\mathcal{I}}(\Theta(\varepsilon)), s^2_{\mathcal{I}}(\Theta(\varepsilon)), \ldots\}$, where $s_{\mathcal{I}}$ is the function denotation of relation $S^{\mathcal{I}}$ which is actually a function by $\nu_2(3)$ and by the fact that Eq coincides with the real equality on ID. In this proof, we write simply s instead of $s_{\mathcal{I}}$.

The fulfilment of the above exhaust can be verified by means of the statement that for all non-negative integer m,
$\delta(s^{m+2}\Theta(\varepsilon), s^{m+1}\Theta(\varepsilon)) > \delta(s^{m+1}\Theta(\varepsilon), s^m\Theta(\varepsilon))$ holds, where δ is the Euclidean distance and s^0 is, as usual, the identity function. (Please remind that usage of spacetime geometrical notions on the elements of ID is reasonable through the isomorphism φ. For example, for $d_1, d_2 \in ID_1$, $\delta([d_1]_\rho, [d_1]_\rho)$ is just the Euclidean distance between $\varphi([d_1]_\rho)$ and $\varphi([d_1]_\rho)$.) This inequality can be shown by properties of spacelike betweenness, parallelity of $\triangleleft$-linear straight lines and similar triangles, as follows.

We interject a remark, namely, that this condition cannot be attached directly to ν_2, as a first-order condition in terms of $\blacktriangleleft$, because we are not able to define equidistance in the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ – it is an important difference to $(\mathbb{R}^n, \blacktriangleleft)$ which makes this proof more complicated.

Let us fix an integer $m \geq 0$ and $a_0 := s^m(\Theta(\varepsilon))$, $a_1 := s^{m+1}(\Theta(\varepsilon))$, $a_2 := s^{m+2}(\Theta(\varepsilon))$. Then $Betw^{\mathcal{I}}(\Theta(\alpha), a_0, a_1)$ and $Betw^{\mathcal{I}}(\Theta(\alpha), a_1, a_2)$ follow from $\nu_2(2)$ and from the definition for S. This implies $\delta(\Theta(\alpha), a_0) < \delta(\Theta(\alpha), a_1) <$

$\delta(\Theta(\alpha), a_2)$ because of the properties of betweenness. By $\nu_2(4)$–(5), there exist $b, c \in ID$ that the following properties hold:

$Opt^{\mathcal{I}}(a_0, b)$, $Opt^{\mathcal{I}}(a_1, b)$, $Opt^{\mathcal{I}}(a_1, c)$, $Opt^{\mathcal{I}}(a_2, c)$,$\neg Opt^{\mathcal{I}}(a_0, c)$, $\neg Opt^{\mathcal{I}}(a_2, b)$, $Betw^{\mathcal{I}}(\Theta(\alpha), \Theta(\gamma), b)$, $Betw^{\mathcal{I}}(\Theta(\alpha), \Theta(\gamma), c)$ and $Betw^{\mathcal{I}}(\Theta(\alpha), b, c)$.

One can conclude from them on $\delta(\Theta(\alpha), b) < \delta(\Theta(\alpha), c)$, furthermore, on that the lightlike line $a_1 b$ is parallel to the lightlike line $a_2 b$ and the lightlike line $a_0 b$ is parallel to the lightlike line $a_1 c$.

Using the properties of similar triangles $\Theta(\alpha)a_0 b$ and $\Theta(\alpha)a_1 c$, $\Theta(\alpha)a_1 b$ and $\Theta(\alpha)a_2 c$, respectively, and applying the inequality on the geometrical and arithmetic mean, we can derive now the desired inequality in the form $\delta(\Theta(\alpha), a_2) - \delta(\Theta(\alpha), a_1) > \delta(\Theta(\alpha), a_1) - \delta(\Theta(\alpha), a_0)$ regarding that the mentioned elements of this inequality are collinear.

Finally, the set $\{\Theta(\varepsilon), s(\Theta(\varepsilon)), s^2(\Theta(\varepsilon)), \ldots\}$ exhausts $N^{\mathcal{I}}$ because any $r \in N^{\mathcal{I}} \setminus \{\Theta(\varepsilon)\}$ satisfies also $Betw^{\mathcal{I}}(\Theta(\alpha), \Theta(\varepsilon), r)$ by $\nu_2(2)$, and $\Theta(\varepsilon), s(\Theta(\varepsilon)), s^2(\Theta(\varepsilon)), \ldots$ form a growing distance ω-sequence on the half-line $\{d \in ID : Betw^{\mathcal{I}}(\Theta(\alpha), \Theta(\varepsilon), d) \vee Eq^{\mathcal{I}}(d, \Theta(\varepsilon))\}$, without any accumulation point. Since there is no element of $N^{\mathcal{I}}$ between two consecutive points of the sequence or outside of the half-line mentioned, the above exhaust and consequently, Lemma 3.19 is proved.

3.4. Representation of predicates of more than one argument

We introduce the following abbreviations in $TL^{GN}_{L_0}$. They allow to represent predicates with more than one argument.

DEFINITION 3.20. $A(x, y, z) :=$

$$
\begin{aligned}
&\{Eq(x,\varepsilon) \wedge Eq(y,z)\} \vee \{Eq(y,\varepsilon) \wedge Eq(x,z)\} \vee \\
&\{Eq(x,y) \wedge \neg Eq(x,\varepsilon) \wedge \\
&\quad \exists uv\ [Id^x_u \wedge Opt(u,x) \wedge Sim(u,z) \wedge \Box(r(v) \leftrightarrow r(x) \vee r(z) \vee r(u))]\} \vee \\
&\{\neg Eq(x,y) \wedge \neg Eq(x,\varepsilon) \wedge \neg Eq(y,\varepsilon) \wedge \\
&\quad \exists uv\ [Id^x_u \wedge Opt(u,x) \wedge Ord(u,y) \wedge Sim(u,z) \wedge \\
&\qquad\quad \Box(r(v) \leftrightarrow r(x) \vee r(y) \vee r(z) \vee r(u))]\},
\end{aligned}
$$

$$
\begin{aligned}
&M(x,y,z) := \\
&\{Eq(x,\varepsilon) \wedge Eq(z,\varepsilon)\} \vee \{Eq(y,\varepsilon) \wedge Eq(z,\varepsilon)\} \vee \\
&\{S(\varepsilon,x) \wedge Eq(y,z)\} \vee \{S(\varepsilon,y) \wedge Eq(x,z)\} \vee \\
&\{Eq(x,y) \wedge \neg Eq(x,\varepsilon) \wedge \neg S(\varepsilon,x) \wedge \\
&\quad \exists uv\ [Id^x_u \wedge Opt(x,u) \wedge Sim(z,u) \wedge \Box(r(v) \leftrightarrow r(x) \vee r(z) \vee r(u))]\} \vee \\
&\{\neg Eq(x,y) \wedge \neg Eq(x,\varepsilon) \wedge \neg Eq(y,\varepsilon) \wedge \neg S(\varepsilon,x) \wedge \neg S(\varepsilon,y) \wedge \\
&\quad \exists uv\ [Id^x_u \wedge Opt(x,u) \wedge Ord(y,u) \wedge Sim(z,u) \wedge
\end{aligned}
$$

$\Box(r(v) \leftrightarrow r(x) \vee r(y) \vee r(z) \vee r(u))]\}$.

Once we have represented two predicates of three arguments by the means of our sole unary predicate symbol r (it was the harder to provide than what follows), we can endow these formulæ to represent addition and multiplication, in the expected way, by postulating the following ν_3 on them.

DEFINITION 3.21. Let ν_3 is conjuntion of ν_2 and the following conditions (the usual primitive recursive definitions for addition and multiplication in our representation):

(1) $\forall xy(N(x) \wedge N(y) \rightarrow \exists!^{Eq} z(N(z) \wedge A(x,y,z)) \wedge \exists!^{Eq} w(N(w) \wedge M(x,y,w)))$, where $\exists!^{Eq}$ is to understand regarding Eq as equality,
(2) $\forall x(N(x) \rightarrow A(\varepsilon, x, x))$,
(3) $\forall xyzvw(N(x) \wedge N(y) \wedge N(z) \wedge N(v) \wedge N(w) \wedge$
$S(x,y) \wedge A(x,z,v) \wedge S(v,w) \rightarrow A(y,z,w))$,
(4) $\forall x(N(x) \rightarrow M(\varepsilon, x, \varepsilon))$,
(5) $\forall xyzvw(N(x) \wedge N(y) \wedge N(z) \wedge N(v) \wedge N(w) \wedge$
$S(x,y) \wedge M(x,z,v) \wedge A(v,z,w) \rightarrow M(y,z,w))$.

DEFINITION 3.22. Let $A^{\mathcal{I}}$ and $M^{\mathcal{I}}$ express the meaning of A and M, resp., on ID.

For example, for $d_1, d_2, d_3 \in ID_1$, we write also $A^{\mathcal{I}}([d_1]_\rho, [d_2]_\rho, [d_3]_\rho)$ for $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \binom{x\ y\ z}{d_1 d_2 d_3}, q \Vdash A(x,y,z)$, and similar applies to $M^{\mathcal{I}}$.

STATEMENT 3.23. If $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Theta, q \Vdash \nu_3$ then $(N^{\mathcal{I}}, O^{\mathcal{I}}, S^{\mathcal{I}}, A^{\mathcal{I}}, M^{\mathcal{I}})$ is isomorphic to $(\mathbb{N}, <, succ^r, +^r, *^r)$, where $succ^r = \{(n, n+1) : n \in \mathbb{N}\}$, $+^r = \{(k,l,m) \subseteq \mathbb{N}^3 : k + l = m\}$, and $*^r$ is $\{(k,l,m) \subseteq \mathbb{N}^3 : k \cdot l = m\}$. The isomorphism can be given by $\psi : \mathbb{N} \rightarrow ID$, where $\psi(k) = s^k_{\mathcal{I}}(\Theta(\varepsilon))$.

This follows from the following and similar facts. For any $k, l, m \in \mathbb{N}$, the conditions $k + l = m$ and $A^{\mathcal{I}}(s^k_{\mathcal{I}}(\Theta(\varepsilon)), s^l_{\mathcal{I}}(\Theta(\varepsilon)), s^m_{\mathcal{I}}(\Theta(\varepsilon))$ are equivalent. The last fact can be verified by Lemma 3.19 and induction on k and l, taking into account that on $(\mathbb{N}, <, succ^r)$ only the functions of addition and multiplication satisfy their defining primitive recursive equations. We omit the routine details.

3.5. Translation of true arithmetics into our theory

In this subsection, the proof is finished by the standard way of non-axiomatizability proofs of first-order temporal theories. The only difference is that consistency is not straightforward because of the rather complex way of representing the three-argument predicate symbols.

THEOREM 3.24. There exist a temporal interpretation for TL_L^{GN} on the time flow $(\mathbb{Q}^n, \blacktriangleleft)$, a rational point $q \in \mathbb{Q}^n$ and a valuation Θ in $\mathcal{I}$ such that $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Theta, q \Vdash \nu_3$.

PROOF. We supply only the asked $\mathcal{I}$, q and Θ and leave checking for satisfaction of ν_3 to the reader. q will be specified as $(0,\dots,0)$. Let $D_{\mathcal{I}}$ be the set

$$\mathbb{Q}^n \cup \{D\} \cup$$
$$\{A_{k,l,m} : k+l=m, k\neq 0, l\neq 0, k\neq l\} \cup \{A^{=}_{k,m} : k+k=m, k\neq 0\} \cup$$
$$\{M_{k,l,m} : k\cdot l=m, k>1, l>1, k\neq l\} \cup \{M^{=}_{k,m} : k\cdot k=m, k>1\},$$

where D and the other objects are just formal symbols.

The interpretation $\mathcal{I}_t^p$ is defined via its value on predicate symbol r (no other symbol in L). We write shortly r_t for $\mathcal{I}_t^p(r)$. r_t can be defined via the definition for the truth values $r_t(d)$, for arbitrary $d \in D_{\mathcal{I}}$.
if $d = q \in \mathbb{Q}^n$ then $r_t(q) = (t = q)$,
if $d = D$ then $r_t(D) = (\exists m \geq 1)\ t = (0,\dots,0,2^m)$,
if $d = A_{k,l,m}$ for $k,l,m \in \mathbb{N}$ satisfying $k+l=m, k\neq 0, l\neq 0, k\neq l$ then
$$r_t(A_{k,l,m}) = t \in \{(0,\dots,0,k),(0,\dots,0,l),(0,\dots,0,m),$$
$$\left(-\left|\tfrac{k}{2}-\tfrac{l}{2}\right|+\tfrac{1}{4},0,\dots,0,\tfrac{k}{2}+\tfrac{l}{2}+\tfrac{1}{4}\right)\},$$
if $d = A^{=}_{k,m}$ for $k,m \in \mathbb{N}$ satisfying $k+k=m, k\neq 0$, then
$$r_t(A^{=}_{k,m}) = t \in \{(0,\dots,0,k),(0,\dots,0,m),\left(-\tfrac{1}{4},0,\dots,0,k+\tfrac{1}{4}\right)\},$$
if $d = M_{k,l,m}$ for $k,l,m \in \mathbb{N}$ satisfying $k\cdot l=m, k\geq 2, l\geq 2, k\neq l$ then
$$r_t(M_{k,l,m}) = t \in \{(0,\dots,0,k),(0,\dots,0,l),(0,\dots,0,m),$$
$$\left(\left|\tfrac{l}{2}-\tfrac{k}{2}\right|+\tfrac{1}{4},0,\dots,0,\tfrac{l}{2}+\tfrac{k}{2}+\tfrac{1}{4}\right)\},$$
if $d = M^{=}_{k,m}$ for $k,m \in \mathbb{N}$ satisfying $k\cdot k=m, k\neq 0$, then
$$r_t(M^{=}_{k,m}) = t \in \{(0,\dots,0,k),(0,\dots,0,m),\left(\tfrac{1}{4},0,\dots,0,k+\tfrac{1}{4}\right)\}.$$

The valuation Θ can be determined by setting $\Theta(\alpha)$, $\Theta(\varepsilon)$, $\Theta(\gamma)$, $\Theta(\delta)$ to $(0,\dots,0)$, $(0,\dots,0,1)$, $(\frac{1}{2},0,\dots,0,\frac{3}{2})$ and $\Theta(\delta) = D$, respectively.

DEFINITION 3.25. For any first-order formula A in the signature of $(\mathbb{N}, <, succ^r, +^r, *^r)$ (somewhat loosely, we does not differ the predicate symbol from the corresponding interpreting relation), we give a translation A^t into $TL_{L_0}^{GN}$, by structural induction, as follows. We assume that the variables of the arithmetical language are that of L excluding $\{\alpha, \gamma, \delta, \varepsilon\}$.

$(x<y)^t = O(x,y)\ (= Betw(\alpha,x,y))$,
$(succ^r(x,y))^t = S(x,y)$, where S is defined in 3.16,
$(+^r(x,y,z))^t = A(x,y,z)$, where A is defined in 3.20,
$(*^r(x,y,z))^t = M(x,y,z)$, where M is defined in 3.20,

$(A \wedge B)^t = (A^t \wedge B^t)$, $(\neg A)^t = \neg A^t$ and
$(\forall x A)^t = \forall x(N(x) \to A^t)$.

DEFINITION 3.26. Assume that $\mathcal{I}$ is a temporal interpretation for TL_L^{GN} on the time flow $(\mathbb{Q}^n, \blacktriangleleft)$ which also satisfies ν_3, and Θ is a valuation in $\mathcal{I}$. We associate a valuation $\Sigma \oplus \Theta$ of the variables of that temporal language in $\mathcal{I}$, to every valuation Σ of the variables of the arithmetical language into $\mathbb{N}$.

Values for $\alpha, \gamma, \varepsilon, \delta$ come from Θ, that is, for example, $(\Sigma \oplus \Theta)(\alpha) = \Theta(\alpha)$, while the other variables gets value $(\Sigma \oplus \Theta)(x) = s_{\mathcal{I}}^{\Sigma(x)}(\Theta(\varepsilon))$, where $s_{\mathcal{I}}$ is described in the proof of 3.19.

LEMMA 3.27. Let us assume that $\mathcal{I}$ is a temporal interpretation for TL_L^{GN} on the time flow $(\mathbb{Q}^n, \blacktriangleleft)$, $q \in \mathbb{Q}^n$, Θ is a valuation in $\mathcal{I}$ such that $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Theta, q \Vdash \nu_3$, further, that A is a first-order formula in the language of $(\mathbb{N}, <, succ^r, +^r, *^r)$ and Σ is a valuation of the variables of this language into $\mathbb{N}$. Then we have

$(\mathbb{N}, <, succ^r, +^r, *^r) \models A\Sigma$ if and only if $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Sigma \oplus \Theta, q \Vdash A^t$.

PROOF. By induction on the complexity of the arithmetical formula A. For atomic formulæ , this follows from Statement 3.23. For $\wedge$-formulae and $\neg$-formulae this is a trivial consequence of the induction hypotheses, while, for $\forall x B$, it is enough to consider that ID is exhausted by $\{s_{\mathcal{I}}^k(\Theta(\varepsilon)) : k \geq 0\}$ (Statement 3.19).

LEMMA 3.28. If A is a closed first-order formula in the language of $(\mathbb{N}, <, succ^r, +^r, *^r)$ then we have $A \in Th\ (\mathbb{N}, <, succ^r, +^r, *^r)$ if and only if $\forall \alpha\delta\gamma\varepsilon(\nu_3 \to A^t) \in \mathtt{Th}_{L_0}^{GN}(\mathbb{Q}^n, \blacktriangleleft)$, where $Th\ K$ denotes the first-order theory of structure K.

PROOF. We can prove this by specializing the previous lemma to $\Sigma = ()$, remembering that there exist $\mathcal{I}$, Θ, $q \in \mathbb{Q}^n$ such that $(\mathbb{Q}^n, \blacktriangleleft), \mathcal{I}, \Theta, q \Vdash \nu_3$, and observing that the left side of equivalence in the previous lemma is independent of $\mathcal{I}$, Θ and q.

FINISHING THE PROOF OF THEOREM 2.12. If $\mathtt{Th}_{L_0}^{GN}(\mathbb{Q}^n, \blacktriangleleft)$ would be recursively enumerable then
$\mathtt{Th}_{L_0}^{GN}(\mathbb{Q}^n, \blacktriangleleft) \cap \{\forall \alpha\delta\gamma\varepsilon\ (\nu_3 \to A^t) | A \textit{ is an arithmetical formula}\}$ would be recursively enumerable, too. This is impossible by the previous lemma.

PROOF OF THEOREM 2.14. The proof of Theorem 2.12 goes through also for $(\mathbb{R}^n, \blacktriangleleft)$, even with the following simplification. In [18] an equidistance formula was presented in the first-order theory of $(\mathbb{R}^n, \blacktriangleleft)$. In our main lemma 3.19, we could simply require that for all three neighbour elements a, b, c of $N^{\mathcal{I}}$ satisfies that the distance between a and b is equal to the distance of b and c. All the other parts of that proof can be taken directly.

The proof of Theorem 2.13 in [43] depends on the observation that over a time flow whose first-order theory is ω-categorical and recursively enumerable, arbitrary first-order temporal theory is axiomatizable. The proof of Theorem 2.13 in the same report is a modification of our proof of Theorem 2.12 without a possible definition for equidistance or even betweenness in $(\mathbb{R}^2, \blacktriangleleft)$ but employ another method (which also applies for higher dimensional cases of $\mathbb{R}^n$ but not for $\mathbb{Q}^n$.

4. Final suggestions

We have proved non-axiomatizability of a first-order spatio-temporal theory over $(\mathbb{Q}^n, \blacktriangleleft)$ $(n > 2)$. What is the situation, if we restrict our temporal operator set to a more simple one, for example, to a single *future* operator or allowing its past counterpart, too.

Over $(\mathbb{R}, <)$, no non-axiomatizability proof for monadic signature first-order temporal logic is present in the literature. One could produce a similar representation over the reals, to prove non-axiomatizability of $\mathtt{Th}^{FP}_{L_0}(\mathbb{R}, <)$, for our signature L_0 having only one unary predicate symbol.

Further, the first-order theory of $(\mathbb{Q}^n, \blacktriangleleft)$ is probably not ω-categorical. Is it yet decidable? Concerning the first-order theory of $(\mathbb{R}^n, \blacktriangleleft)$, we know that it is decidable, through translating into real arithmetics. Can we do it faster? For example, is the first-order theory of $(\mathbb{R}^n, \blacktriangleleft)$ quantifier eliminable?

References

[1] Andréka, H., Goranko, V., Mikulás, Sz., Németi, I., Sain, I., *Effective temporal logics of programs*, in: Time and Logic, ed: Bolc and Szalas, UCL Press, 1995.

[2] Andréka, H., Madarász, J. X., Németi, I., *On the logical structure of relativity theories*, internet book, http://www.renyi.hu/pub/algebraic-logic/olsort.html.

[3] Andréka, H., Madarász, J. X., Németi, I., *Logical analysis of relativity theories*, in: *First-order logic revisited*, Hendricks et all. eds, Logos Verlag, Berlin, 2004.

[4] Andréka, H., Németi, I., Sain, I., *On the strength of temporal proofs*, Theo. Comp. Sci. 80(1991):125–15.1

[5] Ax, J., P., *The elementary foundations of spacetime*, Found. Phys. 8/7-8(1978):507–546.

[6] Balbiani, P., Condotta, J.-F., *Computational complexity of propositional linear temporal logics based on qualitative spatial or temporal reasoning*, in: A. Armando(ed): Proc. of Frontiers of Combining Systems (ProCoS 2002) vol. 2309 of LNCS.

[7] Balbiani, P., Goranko, V., *Modal logics of parallelism, orthogonality and affine geometries*, J. of Applied Non-Classical Logics 12/3-4(2002):365–398.

[8] Bacchus, F., Kabanza, F., *Using temporal logic to express search and control knowledge for planning*, Artif. Intelligence 116/1-2(2000):123–191.

[9] Bennett, B., Cohn, A., *Multi-dimensional modal logic as a framework for spatio-temporal reasoning*, Proc. of the Hot Topics in Spatio-Temporal Reasoning Workshop, IJCAI'99, Stockholm, 1999.

[10] van Benthem, J., *The logic of time*,, Reidel, Synthese Library 156, 1983.

[11] Clarke, E., Browne, M., Emerson, E., Sistla, A., *Using temporal logic for formal verification of finite state systems*, in: Logics and models of concurrent systems, 1985, NATO ASI series, vol. F13.

[12] Charron-Bost, B., Mattern, F., *Synchronous, asynchronous and causally ordered communication*, Distributed Computing 9(1996):173–191.

[13] Clarke, E., Emerson, E., Sistla, A., *Automatic verification of finite-state concurrent systems using temporal logic specifications*, ACM Toplas 8(1986):244–263.

[14] Dragalin, A. G., *Lecture Notes on Logic and Automata*, Debrecen, 1997.

[15] gabbay, D., Hodkinson, I., Reynolds, M., *Temporal logic*, Clarendon Press, 1994.

[16] Gerevini, A., Nevel, B., *Qualitative spatio-temporal reasoning with rcc-8 and Allen's interval calculus: computational complexity*, Proc. of the 15th European Conf. on Artif. Int. (ECAI'02) pp. 312–316, IOS Press, 2002.

[17] Goldblatt, R., *Diodorean modality in Minkowski space*, Studia Logica, 39(1980):219–236.

[18] Goldblatt, R., *Orthogonality and spacetime geometry*, Springer, 1987.

[19] Goldblatt, R., *First-order spacetime geometry*, in: Logic, methodology and philosophy of science, VIII (Moscow, 1987) pp.303–316, Stud. Logic Found. Math. 126, North-Holland, 1989.

[20] Hodkinson, I., Wolter, F. and Zakaryashev, M. *On decidable fragments of first-order temporal logics*, Annals of Pure and Applied Logic 106:85-134(2000)

[21] Hughes,G., E. and Creswell, M., *An Introduction to Modal Logic*, London, Methuen, 1968.

[22] Madarász, J. X., Németi, I., Székely, G.,

Twin paradox and the logical foundation of relativity theory, to appear in Foundations of Physics, http://www.arxiv.org/abs/gr-qc/0504118.

[23] Manna, Z., Pnueli, A., *Verification of concurrent programs — the temporal framework*, in: Boyer – Moore (ed.): The correctness problem in computer science pp. 215–273, 1981, Academic Press, Int. Lecture Series in Comp. Sci.

[24] Manna, Z., Pnueli, A., *The temporal logic of reactive and concurrent systems*, Springer, 1992.

[25] Marx, M., Venema, Y., *Multi-dimensional modal logic*, Kluwer, 1997.

[26] Mattern, F., F. Mattern: *On the relativistic structure of logical time in distributed systems*, in: Informatik pp. 309–331, Teubner Texte zur Informatik, Stuttgart, 1992.

[27] Merz, S., *Decidability and incompleteness results for first-order temporal logics of linear time*, Journal of Applied Non-Classical Logics 2/2(1992).

[28] Mundy, B., *The physical content of Minkowski geometry*, British J. of Philos. Sci. 37/1(1986):25–54.

[29] Mundy, B., *Optical axiomatization of Minkowski space-time geometry*, Philos. Sci. 53/1(1986):1–30.

[30] *Formal methods specification and analysis guidebook for the verification of software and computer systems*, http://eisl.jpl.nasa.gov/quality/Formal_Methods.

[31] Pambuccian, V., *Theorems of Alexandrov-Zeeman type as definability statements and the axiomatics of hyperbolic geometry*, Conf. Abstracts of Logic in Hungary 2005, Budapest, 2005.

[32] Phillips, J., *A note on the modal and temporal logics of n-dimensional spacetime*, Notre Dame J of Formal Logic 39/4(1998):545–553.

[33] Phillips, J. F., *Modal logic of succession for 2-dimensional integral spacetime*, J. Phil. Logic 30(2001):1–25.

[34] Reif, J., Sistla, A., *A multiprocess network logic with temporal and spatial modalities*, J. of Comp. and System Sci. 30(1985):41–53.

[35] Reynolds, M., *Axiomatising first-order temporal logic: Until and since over linear time*, Studia Logica, 57(1996):279–302.

[36] Reynolds, M., *A decidable temporal logic of parallelism*, Notre Dame J of Formal Logic 38/3(1997): 419–436.

[37] Reynolds, M., zakaryashev, M., *On the products of linear modal logics*, J. of Logic and Comput. 11(2001):909–931.

[38] Robb, A. A., *A theory of time and space*, Cambridge Uni. Press, 1914.

[39] Segerberg K., *Two-dimensional modal logic*, J. of Phil. Logic 2(1973):77–96.

[40] Shapirovsky, I. and Shehtman, V., *Chronological future modality in Minkowski spacetime.* Advances in modal logic 4(2002):437–459.

[41] Shehtman, V., *Two-dimensional modal logics*, Math. Notices of the USSR Academy of Sciences 23(1978):417–424.

[42] Shehtman, V., *Modal logics of domains on the real plane*, Studia Logica 42/1(1983):63–80.

[43] Vályi, S., *A note on the axiomatizability of some first-order spatio-temporal logics*, technical report, 2006.

[44] Vályi, S., *Non-classical logic – Axiomatizability of spatio-temporal theories and interval-valued computations*, dissertation for PhD, 2006.

[45] Wolter, F., Zakaryashev, M., *Spatio-temporal representation and reasoning based on RCC-8*, in: A. Cohn et all.(ed): Proc. of the 7th Conf. on Principles of Knowledge Representation and Reasoning KR2000, pp. 3–14, Morgan Kaufmann, 2000.

Sándor Vályi
Faculty of Natural Sciences and Informatics
College of Nyíregyháza

valyis@nyf.hu