Relativity and modal logic meet Hausdorff

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The Hausdorff property, a rather abstract topological separation property, has been a hotly debated topic in some foundational quarters in general relativity as well as in the agency theory (a field of modal logic), there having been virtually no connection between the two disciplines. The property says that for any two distinct points in a topological space there is a pair of non-overlapping open neighborhoods of these points. Nowadays a GR spacetime is typically identified with a Hausdorff manifold, yet, in the 1970's discoveries of various causal anomalies of GR spacetimes prompted some physicists to investigate non-Hausdorff spacetimes—see e.g. Hajicek (1971). As it turned out subsequently, non-Hausdorff GR spacetimes have some non-desirable properties (for an overview, see Earman (2008)), which helped to coin the current standard "GR space-time = Hausdorff manifold". The dynamics in the agency theory has gone in a different direction, partly because this enterprise aims at modeling actions in *non-deterministic* contexts. After Belnap (1992) extended the theory to capture spatial and special-relativistic aspects, a question emerged what an object's life-line look like if a chancy event (say, a coin toss) occurs at a remote location: is there (1) the last moment before a result of tossing, or (2) the first moment after each result? Belnap's (1992) axioms imply answer (2), which in turn entails a failure of the Hausdorff property in an associated topology.

The aim of this paper is to develop a theory of generalized manifolds, which is, on the one hand, friendly towards indeterminism and, on the other, sensitive to the topological constraints of GR. For a similar approach, see Muller (2011).

In contrast to the standard construction of differential manifolds, our point of departure is not a "naked" set, but a set, call it W, equipped with a pri-order $\preccurlyeq$ (to allow for "loops") and satisfying some further constraints, which imply two important features: (1) each $e \in W$ is in at least one "patch" $O \subseteq W$, in which the ordering $\preccurlyeq_{|O}$ is partial, and (2) if a chain C in some patch O is upper bounded in O by some $e \in O$, then there is a unique minimum $m = \min\{z \in O \mid C \preccurlyeq_{|O} z \wedge z \preccurlyeq_{|O} e\}$ of C's upper bounds below e. This ultimately leads to the notion of a "splitting pair", i.e., two distinct minima of two (different) sets of upper bounds of some chain in some patch; the idea is to capture an intuitive concept of a bifurcating path. We use splitting pairs to define global consistency: $e, e' \in W$ are said to be g-consistent iff there is no splitting pair $\{x, x'\}$ such that $x \preccurlyeq e$ and $x' \preccurlyeq e'$. Finally, histories are defined as maximal pairwise g-consistent subsets of W. Note that any two histories branch.

With the notion of history at hand, we proceed to construct a (generalized) smooth differential manifold on W, modifying the Geroch-Malament approach (see Malament (2012)) to a context with many histories (spacetimes). The modification concerns charts: as expected, a chart is a pair $\langle O, \varphi \rangle$, where O is a patch and $\varphi : O \to \mathbb{R}^n$ but the usual requirement that φ is injective and that $\varphi[O]$ is an open subset of $\mathbb{R}^n$ is restricted to nonempty intersections $O \cap H$ (for each history H). That is, $\varphi_{|O \cap H}$ should be injective and $\varphi[O \cap H]$ should be an open subset of $\mathbb{R}^n$. The notion of consistency of charts is similarly modified. A generalized manifold is then identified with set W paired with a maximal set of consistent charts (= atlas). We follow Malament in the definition of a manifold topology $\mathcal{T}(W)$: $S \in \mathcal{T}(W)$ iff $\forall p \in S\ \exists \langle O, \varphi \rangle \in \mathcal{C}\ (p \in O\ \wedge\ O \subseteq S)$, where $\mathcal{C}$ is the atlas of charts.

This machinery allows one to prove the following:
(1) $\mathcal{T}(W)$ is generically non-Hausdorff;
(2) each history $H \subseteq W$ is downward closed (wrt $\preccurlyeq$) and Hausdorff, and maximally so;
(3) each subset $A \subseteq W$ that is maximal wrt the Hausdorff property and being downward closed is identical to some history $H \subseteq W$.

These results permit the identification of our histories and GR spacetimes, we believe. With this identification, our generalized manifold is a collection of branching GR spacetimes, each spacetime representing a maximal possible course of events.

References

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