RESIDUATED ALGEBRAS OF BINARY RELATIONS AND POSITIVE FRAGMENTS OF RELEVANCE LOGIC

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ABSTRACT. The aim of this paper is to apply some results obtained jointly with Hajnal Andréka and István Németi about finite axiomatizability of Tarski's representable relation algebras in the context of completeness of fragments of relevance logic.
KEYWORDS: finite axiomatizability, relation algebras, relevance logic, residuation

1. FAMILIES OF RELATIONS

Let Λ be a collection of operation (function) symbols. By a Λ-*family of relations* we mean a collection $\mathfrak{C}$ of binary relations over a base set $U_{\mathfrak{C}}$ such that such that $\mathfrak{C}$ is closed under the operations in Λ. Let $\mathsf{R}(\Lambda)$ denote the class of all Λ-families of relations.

In this paper we will focus on the following operations.

Binary operations: join $+$, meet $\cdot$, relation composition ;, right $\setminus$ and left $/$ residuals of composition.
Unary operations: converse $\smile$, converse negation $\sim$.
Constants: identity $1'$, zero 0, unit 1.

The interpretations of the elements of Λ in a Λ-family of relations $\mathfrak{C}$ are as follows. Join $+$ is union, meet $\cdot$ is intersection, zero 0 is the empty set, unit 1 is the universal relation $U_{\mathfrak{C}} \times U_{\mathfrak{C}}$ and

$$x \,;\, y = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : (u,w) \in x \text{ and } (w,v) \in y \text{ for some } w\}$$
$$x \setminus y = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : \text{for every } w, (w,u) \in x \text{ implies } (w,v) \in y\}$$
$$x \,/\, y = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : \text{for every } w, (v,w) \in y \text{ implies } (u,w) \in x\}$$
$$x^{\smile} = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : (v,u) \in x\}$$
$$\sim x = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : (v,u) \notin x\}$$
$$1' = \{(u,v) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}} : u = v\}$$

Terms are built from variables using the operations in the usual way. The interpretation of a Λ-term in $\mathfrak{C} \in \mathsf{R}(\Lambda)$ is standard. That is, we assume a valuation v of the variables into $\mathfrak{C}$ and extend it to compound terms according to the interpretations of the elements of Λ in $\mathfrak{C} \in \mathsf{R}(\Lambda)$. We write $\mathfrak{C} \models \tau = \sigma$ iff the interpretation of τ equals the interpretation of σ, for every valuation v into $\mathfrak{C}$. $\mathfrak{C} \models \tau \leq \sigma$ is defined analogously by interpreting $\leq$ as the subset relation $\subseteq$. Validity will be denoted by $\models$.

An important feature of the (right) residual is the following. We have $x \leq y$ iff $x \setminus y$ contains the identity relation:

$$\mathfrak{C} \models x \leq y \text{ iff } \mathfrak{C} \models 1' \leq x \setminus y \tag{1}$$

for every $\mathfrak{C} \in \mathsf{R}(\Lambda)$. Note that this makes sense even when $1'$ is not in Λ, since it is meaningful whether $\{(u,u) \in U_{\mathfrak{C}} \times U_{\mathfrak{C}}\}$ is a subset of the interpretation of a Λ-term.

Assume that ; is definable by Λ and let $\mathfrak{C}$ be a Λ-family of relations. We say that $\mathfrak{C}$ is *commutative* if

$$\mathfrak{C} \models x \,;\, y = y \,;\, x$$

and *dense* if

$$\mathfrak{C} \models x \leq x \,;\, x$$

for all elements x, y of $\mathfrak{C}$, respectively. The class of commutative and dense Λ-families of relations is denoted by $\mathsf{R}^{cd}(\Lambda)$. Note that the interpretations of the two residuals coincide in commutative families of relations $\mathfrak{C}$: $\mathfrak{C} \models x \setminus y$ iff $\mathfrak{C} \models y \,/\, x$.

2. State-semantics

Next we define an alternative, *state-semantics* for terms. Let $\mathfrak{C} \in \mathsf{R}(\Lambda)$ for some signature Λ. We define, for every Λ-term τ,

$$\mathfrak{C} \models_s \tau \text{ iff } \mathfrak{C} \models 1' \leq \tau \tag{2}$$

We say that τ is *state-valid* in $\mathsf{R}(\Lambda)$ (in symbols, $\models_s \tau$) iff $\mathfrak{C} \models_s \tau$ for every $\mathfrak{C} \in \mathsf{R}(\Lambda)$.[1]

State-semantics restricted to commutative and dense $\{\cdot, \setminus, \sim\}$-families of relations provides sound semantics for relevance logic [ABD92, RM73]. Relevance logic $\vdash_R$ is a Hilbert-style derivation system and has the logical connectives conjunction, implication and negation. The logical connectives are interpreted in a $\mathfrak{C} \in \mathsf{R}^{cd}(\cdot, \setminus, \sim)$ as meet $\cdot$, (right) residual $\setminus$ and converse negation $\sim$, respectively. State-validity w.r.t. commutative and dense families of relations will be denoted by $\models_s^{cd}$. While this semantics is sound[2]

$$\vdash_R \varphi \text{ implies } \models_s^{cd} \varphi$$

completeness does not hold, since state-validity for $\mathsf{R}^{cd}(\cdot, \setminus, \sim)$ cannot be finitely axiomatized [Mik09].

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[1] The terminology 'state-semantics' refers to the fact that truth is restricted to pairs of the form (u,u). Note that the concept of truth uses the more general concept of interpretation, thus whether a term is true at (u,u) in general depends on whether pairs of the form (v,w) are in the interpretations of some other terms. For instance, $x \setminus y$ is true at (u,u) iff, for every v, (v,u) is in the interpretation of y whenever it is in the interpretation of x.

[2] For the sake of simplicity we will not distinguish between a relevance logic formula and the corresponding algebraic term where the logical connectives are replaced by the corresponding algebraic operations.

We note that, using $\cdot$, $\backslash$ and $\sim$, additional connectives can be defined. For instance, $x + y$ as $\sim(\sim x \cdot \sim y)$ and $x \,;\, y$ as $\sim(y \backslash \sim x)$.

It would be interesting to see for precisely which signatures Λ one can give a complete calculus capturing state-validity in $\mathsf{R}^{cd}(\Lambda)$, since that would show (in)completeness of other fragments of relevance logic with respect to state-semantics. For instance, [HM11] shows that finite axiomatization of state-validities for $\mathsf{R}^{cd}(\cdot, +, \backslash)$ is not possible, hence establishing incompleteness of the positive fragment of relevance logic.[3] The same non-finite axiomatizability results hold for $\mathsf{R}^{cd}(\cdot, +, ;, \backslash)$ and $\mathsf{R}(\cdot, +, ;, \backslash, /)$, see [HM11] and [Mik11]. Below we will look at certain signatures where we do not assume that both lattice operations meet and join are present.

3. Ordered residuated semigroups

First let us look at the basic signature $\Lambda = \{;, \backslash, /\}$.

The $\leq$-theory of the class $\mathsf{R}(;, \backslash, /)$ was investigated in [AM94] in connection with the completeness of the Lambek calculus. We showed [AM94, Theorem 3.3] that for *sequents*, validity in $\mathsf{R}(;, \backslash)$ coincides with derivability from the Lambek calculus with empty terms LC_0. Let us recall that a sequent has the form $A_0, \ldots, A_{n-1} \Rightarrow A_n$ where every A_i is a $\{;, \backslash, /\}$-term. A sequent $A_0, \ldots, A_{n-1} \Rightarrow A_n$ is valid iff $\mathfrak{C} \models A_0 \,;\, \ldots \,;\, A_{n-1} \leq A_n$ for every $\mathfrak{C} \in \mathsf{R}(;, \backslash, /)$. Note that in the case of $n = 0$, we have that $\Rightarrow A_0$ is valid iff $\mathsf{R}(;, \backslash, /) \models 1' \leq A_0$.

The proof of [AM94, Theorem 3.3] goes through with straightforward modifications if we define the semantics by $\mathsf{R}^{cd}(;, \backslash)$ instead of $\mathsf{R}(;, \backslash, /)$ and add the following two axioms (corresponding to commutativity and density) to LC_0:

$$A \,;\, B \Rightarrow B \,;\, A \qquad A \Rightarrow A \,;\, A$$

By (1) and (2), any sequent $A_0, \ldots, A_{n-1} \Rightarrow A_n$ can be rewritten as $(A_0 \,;\, \ldots \,;\, A_{n-1}) \backslash A_n$ such that

$$\begin{aligned} \models^{(cd)} A_0, \ldots, A_{n-1} \Rightarrow A_n \text{ iff } &\models^{(cd)} \Rightarrow (A_0 \,;\, \ldots \,;\, A_{n-1}) \backslash A_n \\ \text{iff } &\models_s^{(cd)} (A_0 \,;\, \ldots \,;\, A_{n-1}) \backslash A_n \end{aligned}$$

Using this observation, we can equivalently rewrite the sequent calculus LC_0 (with commutativity and density) in Hilbert-style. Putting together we get the following.

Theorem 3.1. *There are (strongly) complete and sound Hilbert-style calculi w.r.t. the state-semantics for* $\mathsf{R}^{cd}(;, \backslash)$ *and* $\mathsf{R}(;, \backslash, /)$.

In the next section we work out the details when we include meet as well in the signature.

[3]Analogues of commutativity and density can be defined even when composition is not in the similarity type.

4. Lower semilattice-ordered residuated semigroups

In this section we look at $\Lambda = \{\cdot\,, ;, \backslash, /\}$.

We define the following Hilbert-style calculus $\vdash_s$. We use the convention that x, y, z may denote empty formulas, while A, B, C must be non-empty formulas. If x is the empty formula, then $x \setminus A$ and $x ; A$ denote A. We have the axioms

$$
\begin{array}{rl}
\text{(Refl)} & A \setminus A \\
\text{(Ass1)} & ((A ; B) ; C) \setminus (A ; (B ; C)) \\
\text{(Ass2)} & (A ; (B ; C)) \setminus ((A ; B) ; C) \\
\text{(ResR)} & ((A ; x) \setminus B) \setminus (x \setminus (A \setminus B)) \\
\text{(Meet1)} & (A \cdot B) \setminus A \\
\text{(Meet2)} & (A \cdot B) \setminus B \\
\text{(Meet3)} & ((A \setminus B) \cdot (A \setminus C)) \setminus (A \setminus (B \cdot C))
\end{array}
$$

and derivation rules

$$
\begin{array}{rl}
\text{(MP)} & \dfrac{x \setminus A \qquad A \setminus B}{x \setminus B} \\
\text{(ResL)} & \dfrac{x \setminus A \qquad (y ; B ; z) \setminus C}{(y ; x ; (A \setminus B) ; z) \setminus C} \\
\text{(Mon1)} & \dfrac{x \setminus A \qquad y \setminus B}{(x ; y) \setminus (A ; B)} \\
\text{(Mon2)} & \dfrac{x \setminus A \qquad y \setminus B}{(x \cdot y) \setminus (A \cdot B)} \\
\text{(Ide1)} & \dfrac{A}{B \setminus (A ; B)} \\
\text{(Ide2)} & \dfrac{A}{B \setminus (B ; A)}
\end{array}
$$

If we include the additional axioms

$$
\begin{array}{rl}
\text{(Comm)} & (A ; B) \setminus (B ; A) \\
\text{(Dens)} & A \setminus (A ; A)
\end{array}
$$

then the calculus is denoted by $\vdash_s^{cd}$.

It is easy to check that these axioms and derivation rules are valid w.r.t. the state-semantics for $\mathsf{R}^{cd}(\cdot\,, ;, \setminus)$. We just note that (Ide1) says that if $1' \leq A$, then $B \leq A ; B$.

Theorem 4.1. *The calculus $\vdash_s^{cd}$ is strongly sound and complete w.r.t. state-semantics for $\mathsf{R}^{cd}(\cdot\,, ;, \setminus)$:*

$$\Gamma \vdash_s^{cd} \varphi \textit{ iff } \Gamma \models_s^{cd} \varphi$$

for any set Γ *of formulas and formula* φ.

Proof. The proof is similar to that of [AM94, Theorem 3.2]. We take the Lindenbaum–Tarski algebra $\mathfrak{F}_\Gamma$ of $\vdash_s^{cd}$ and show that $\mathfrak{F}_\Gamma \in \mathsf{R}^{cd}(\cdot\,, ;, \backslash)$. The representation of $\mathfrak{F}_\Gamma$ is done step-by-step as in the proof of [AM94, Theorem 3.2]. Here we will just indicate the differences. First of all, we do not have to deal with the left residual $/$.

By a filter of $\mathfrak{F}_\Gamma$ we mean a subset of elements closed upward (w.r.t. the ordering defined by meet $\cdot$) and under meet. For an element a, let $F(a)$ denote the principal filter generated by a. We will need E, the filter of (the equivalence classes of) Γ-theorems of $\vdash_s^{cd}$, as well. Indeed, E is closed upward by (Meet1) and (Meet2), and it is closed under meet by (Mon2) (applied to empty x, y).

In the 0th step of the step-by-step construction, we choose distinct u_a, v_a for distinct elements a, and let

$$\begin{aligned} \ell_0(u_a, u_a) &= \ell_0(v_a, v_a) = E \\ \ell_0(u_a, v_a) &= F(a) \end{aligned}$$

Note that the labels are coherent, e.g., for every $e \in \ell_0(u_a, u_a)$ and $a' \in \ell_0(u_a, v_a)$, we have $e\,; a' \in \ell_0(u_a, v_a)$ by (Ide1).

In the $(\alpha+1)$th step we have two subcases. To deal with the residual $\backslash$ we choose a fresh point z, for every point x and element a, and define

$$\begin{aligned} \ell_{\alpha+1}(z, z) &= E \\ \ell_{\alpha+1}(z, x) &= F(a) \\ \ell_{\alpha+1}(z, p) &= F(a\,; \ell_\alpha(x, p)) \qquad p \neq x, z \end{aligned}$$

See Figure 1. Coherence is easy to check.

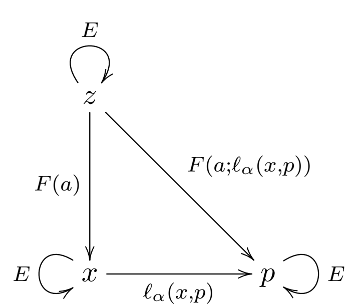

FIGURE 1. Step for the residual

To deal with composition ; we choose a fresh point z, for every $a \in \ell_\alpha(x, y)$ and b, c such that $a \leq b\,;c$, and define

$$
\begin{aligned}
\ell_{\alpha+1}(z, z) &= E \\
\ell_{\alpha+1}(x, z) &= F(b) \\
\ell_{\alpha+1}(z, y) &= F(c) \\
\ell_{\alpha+1}(r, z) &= F(\ell_\alpha(r, x)\,; b) && r \neq x, z \\
\ell_{\alpha+1}(z, s) &= F(c\,; \ell_\alpha(y, s)) && s \neq y, z
\end{aligned}
$$

See Figure 2 for the case when $x \neq y$. Checking coherence is a bit more

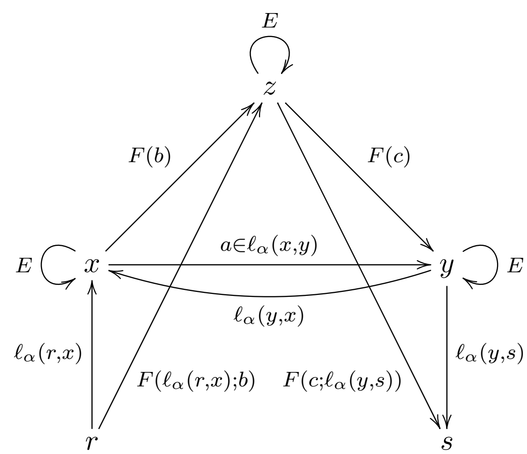

FIGURE 2. Step for composition

involved in this case. For instance, we need $c\,; d\,; b \in \ell_{\alpha+1}(z, z)$ for every $d \in \ell_\alpha(y, x)$ (in case $\ell_\alpha(y, x) \neq \emptyset$). By induction, we have that $a;d \in \ell_\alpha(x, x)$, i.e., $e \leq a;d$ for some $e \in E$. Thus $e \leq (b;c);d$ by $a \leq b;c$. By commutativity (Comm), we get $e \leq c\,; d\,; b$, whence $c\,; d\,; b \in E = \ell_{\alpha+1}(z, z)$, as desired.

Limit step of the construction: take the union of the constructed labelled structures.

After the construction terminates we end up with a labelled structure $(U \times U, \ell)$. We can define a representation of $\mathfrak{F}_\Gamma$ by

$$\mathrm{rep}(a) = \{(u, v) \in U \times U : a \in \ell(u, v)\}$$

Since we used filters as labels, rep respects meet. Injectivity is guaranteed by the 0th step (and the fact that we do not alter labels in later steps). Checking that rep preserves composition and the residual can be done as in the proof of [AM94, Theorem 3.2]. This finishes the proof of Theorem 4.1. □

The commutativity axiom (Comm) is essential in the above proof of Theorem 4.1 (to show that E is the appropriate label for reflexive edges). In the absence of commutativity we still get weak completeness.

Theorem 4.2. *The calculus $\vdash_s$ is weakly complete and sound w.r.t. state-semantics for $\mathsf{R}(\cdot\,, ;, \backslash)$:*

$$\vdash_s \varphi \textit{ iff } \models_s \varphi$$

for any formula φ.

Proof. The proof goes along the same lines as above, but for $\Gamma = \emptyset$. In this case, however, the constructed labelled graph will be antisymmetric:

$$\ell(x, y) \neq \emptyset \neq \ell(y, x) \text{ implies } x = y.$$

The reason is that $\vdash_s A\,; B$ implies $\vdash_s A$ and $\vdash_s B$, as an easy induction on the length of the derivation $\vdash_s A\,; B$ shows. Thus, in the composition case of the construction, we can assume that $x \neq y$: for if $b\,; c \in \ell_\alpha(x, x) = E$, then $b \in \ell_\alpha(x, x)$ and $c \in \ell_\alpha(x, x)$. Hence we do not have to find witnesses for labels on reflexive edges, and thus we can assume that $\ell_\alpha(y, x) = \emptyset$ when $b\,; c \in \ell_\alpha(x, y) \neq \emptyset$, cf. Figure 2.

Because of the simpler structure of the labelled graph, coherence is easier to check, and in fact we do not have to consider the case where we used the commutativity axiom (Comm). □

Remark 4.3. The analogous completeness result holds for $\mathsf{R}(\cdot\,, ;, \backslash, /)$ if we include the corresponding axioms and inference rules for $/$.

In the next section we look at the situation when meet is replaced by join.

5. Upper semilattice-ordered residuated semigroups

Now we look at the signature $\Lambda = \{+, ;, \backslash, /\}$.

Theorem 5.1. *Let $\{+, ;, \backslash, /\} \subseteq \Lambda \subseteq \{+, ;, \backslash, /, ^\smile, 0, 1', 1\}$. Then state-validities for $\mathsf{R}^{cd}(\Lambda)$ and $\mathsf{R}(\Lambda)$ are not finitely axiomatizable.*

Proof. The heart of the proof is the following [AMN12, Theorem 3.2].

Theorem 5.2. *Let $\{+, ;, \backslash, /\} \subseteq \Lambda \subseteq \{+, ;, \backslash, /, ^\smile, 0, 1', 1\}$. The equational theory of $\mathsf{R}(\Lambda)$ is not finitely axiomatizable.*

Moreover, there is no first-order logic formula valid in $\mathsf{R}(+, ;, \backslash, /, ^\smile, 0, 1', 1)$ which implies all the equations valid in $\mathsf{R}(+, ;, \backslash, /)$.

In the proof of Theorem 5.2, for every $n \in \omega$, we had a $\{+, ;, \backslash, /, ^\smile, 0, 1', 1\}$-algebra $\mathfrak{A}_n$ and $\{+, ;, \backslash, /\}$-terms τ_n and σ_n such that

(1) $\mathfrak{A}_n$ is not representable, i.e., it is not isomorphic to a family of relations;
(2) any non-trivial ultraproduct $\mathfrak{A}$ of $(\mathfrak{A}_n : n \in \omega)$ is representable;
(3) $\tau_n \leq \sigma_n$ fails in $\mathfrak{A}_n$;
(4) $\tau_n \leq \sigma_n$ is valid in representable algebras.

From these facts Theorem 5.2 easily follows.[4]

Relation composition is defined so that commutativity and density hold in $\mathfrak{A}_n$ (hence the two residuals coincide), and $\mathfrak{A} \in \mathsf{R}^{cd}(+, ;, \backslash, /, ^{\smile}, 0, 1', 1)$. Thus the equational theory of $\mathsf{R}^{cd}(\Lambda)$ is not finitely axiomatizable.

Finally, using the displayed formulas (1) and (2), for every $\mathfrak{C} \in \mathsf{R}^{(cd)}(\Lambda)$, we have $\mathfrak{C} \models \tau_n \leq \sigma_n$ iff $\mathfrak{C} \models_{\mathrm{s}} \tau_n \setminus \sigma_n$. Thus the theory

$$\{\rho : \models_{\mathrm{s}}^{(cd)} \rho\}$$

is not finitely axiomatizable, finishing the proof of Theorem 5.1. □

6. Conclusion

We have seen that state-validities are finitely axiomatizable for $\mathsf{R}^{cd}(\cdot\,, ;, \backslash)$ and $\mathsf{R}(\cdot\,, ;, \backslash, /)$. We could ask the same question for (standard) validities.

Problem 6.1. *Are the validities finitely axiomatizable for* $\mathsf{R}^{cd}(\cdot\,, ;, \backslash)$ *and* $\mathsf{R}(\cdot\,, ;, \backslash, /)$*?*

Note that the following equations are valid: for any $x_0, \ldots, x_n$,

$$((x_0 \setminus x_0) \cdot \ldots \cdot (x_{n-1} \setminus x_{n-1})) \,;\, x_n \geq x_n \tag{3}$$

In the case of state-semantics the axiom (Refl) and the derivation rule (Ide1) did the trick for deriving the corresponding state-validities. But is there a finite (quasi-)equational base for all the above equations (3) in case we do not have state-semantics (thus the implicit use of the identity constant)?

A related problem is the following.

Problem 6.2. *Are the validities for* $\mathsf{R}^{cd}(\cdot\,, ;, \backslash, 1')$ *and* $\mathsf{R}(\cdot\,, ;, \backslash, /, 1')$ *finitely axiomatizable?*

Note that in this case we can have an explicit use of the identity constant $1'$. Then we can use the quasi-equation

$$y \geq 1' \text{ implies } y \,;\, x \geq x \tag{4}$$

to derive the equations (3). But if we have ordered monoids instead of semigroups, then additional problems arise in the quest for axiomatization, see [HM07].

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[4] In passing we note that the sequence of algebras $(\mathfrak{A}_n : n \in \omega)$ were used in [AM11]. We proved that the quasi-variety $\mathsf{R}(+, ;)$ is not finitely axiomatizable by establishing items (1)–(2) above and showing items (3)–(4) for quasi-equations instead of equations $\tau_n \leq \sigma_n$. The novelty in [AMN12] is that the quasi-equations can be replaced by equations provided that we include the residuals into the signature.

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